Subject – Math AA(Higher Level)
Topic - Calculus
Year - May 2021 – Nov 2024
Paper -1
Answers

Question 1

(a) attempting to use the chain rule to find the first derivative **_M1_**

$f'(x) = (\cos x)e^{\sin x}$ **_A1_**

attempting to use the product rule to find the second derivative **_M1_**

$f''(x) = e^{\sin x}\left(\cos^2 x - \sin x\right)$ (or equivalent) **_A1_**

attempting to find $f(0)$, $f'(0)$ and $f''(0)$ **_M1_**

$f(0) = 1;\ f'(0) = (\cos 0)e^{\sin 0} = 1;\ f''(0) = e^{\sin 0}\left(\cos^2 0 - \sin 0\right) = 1$ **_A1_**

substitution into the Maclaurin formula $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$ **_M1_**

so the Maclaurin series for $f(x)$ up to and including the x^2 term is $1 + x + \frac{x^2}{2}$ **_A1_**

[8 marks]

(b) ***METHOD 1***

attempting to differentiate $f''(x)$ **_M1_**

$f'''(x) = (\cos x)e^{\sin x}\left(\cos^2 x - \sin x\right) - (\cos x)e^{\sin x}(2\sin x + 1)$ (or equivalent) **_A2_**

substituting $x = 0$ into **their** $f'''(x)$ **_M1_**

$f'''(0) = 1(1-0) - 1(0+1) = 0$

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero **_AG_**

METHOD 2

substituting $\sin x$ into the Maclaurin series for e^x **_(M1)_**

$$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2!} + \frac{\sin^3 x}{3!} + \ldots$$

substituting Maclaurin series for $\sin x$ **_M1_**

$$e^{\sin x} = 1 + \left(x - \frac{x^3}{3!} + \ldots\right) + \frac{\left(x - \frac{x^3}{3!} + \ldots\right)^2}{2!} + \frac{\left(x - \frac{x^3}{3!} + \ldots\right)^3}{3!} + \ldots$$ **_A1_**

coefficient of x^3 is $-\frac{1}{3!} + \frac{1}{3!} = 0$ **_A1_**

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero **_AG_**

[4 marks]

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(c) substituting $3x$ into the Maclaurin series for e^x ***M1***

$$\mathrm{e}^{3x}=1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\ldots$$ ***A1***

substituting $\left(\mathrm{e}^{3x}-1\right)$ into the Maclaurin series for $\arctan x$ ***M1***

$$\arctan\left(\mathrm{e}^{3x}-1\right)=\left(\mathrm{e}^{3x}-1\right)-\frac{\left(\mathrm{e}^{3x}-1\right)^3}{3}+\frac{\left(\mathrm{e}^{3x}-1\right)^5}{5}-\ldots$$

$$=\left(3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\ldots\right)-\frac{\left(3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\ldots\right)^3}{3}+\ldots$$ ***A1***

selecting correct terms from above ***M1***

$$=\left(3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}\right)-\frac{(3x)^3}{3}$$

$$=3x+\frac{9x^2}{2}-\frac{9x^3}{2}$$ ***A1***

[6 marks]

(d) ***METHOD 1***

substitution of **their** series ***M1***

$$\lim_{x\to 0}\frac{x+\frac{x^2}{2}+\ldots}{3x+\frac{9x^2}{2}+\ldots}$$ ***A1***

$$=\lim_{x\to 0}\frac{1+\frac{x}{2}+\ldots}{3+\frac{9x}{2}+\ldots}$$

$$=\frac{1}{3}$$ ***A1***

METHOD 2

use of l'Hôpital's rule ***M1***

$$\lim_{x\to 0}\frac{(\cos x)\mathrm{e}^{\sin x}}{\frac{3\mathrm{e}^{3x}}{1+\left(\mathrm{e}^{3x}-1\right)^2}}$$ (or equivalent) ***A1***

$$=\frac{1}{3}$$ ***A1***

[3 marks]

Total [21 marks]

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Question 2

(a) attempt to use quotient rule *(M1)*

correct substitution into quotient rule

$$f'(x)=\frac{5kx\left(\frac{1}{5x}\right)-k\ln 5x}{(kx)^2}$$ (or equivalent) ***A1***

$$=\frac{k-k\ln 5x}{k^2x^2},\left(k\in\mathbb{R}^+\right)$$ ***A1***

$$=\frac{1-\ln 5x}{kx^2}$$ ***AG***

[3 marks]

(b) $f'(x)=0$ ***M1***

$$\frac{1-\ln 5x}{kx^2}=0$$

$\ln 5x=1$ ***(A1)***

$$x=\frac{\mathrm{e}}{5}$$ ***A1***

[3 marks]

(c) $f''(x)=0$ ***M1***

$$\frac{2\ln 5x-3}{kx^3}=0$$

$$\ln 5x=\frac{3}{2}$$ ***A1***

$$5x=\mathrm{e}^{\frac{3}{2}}$$ ***A1***

so the point of inflexion occurs at $x=\frac{1}{5}\mathrm{e}^{\frac{3}{2}}$ ***AG***

[3 marks]

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(d) attempt to integrate **(M1)**

$u = \ln 5x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$\int \frac{\ln 5x}{kx}\,\mathrm{d}x = \frac{1}{k}\int u\ \mathrm{d}u$ **(A1)**

EITHER

$= \frac{u^2}{2k}$ **A1**

so $\frac{1}{k}\int_{1}^{\frac{3}{2}} u\ \mathrm{d}u = \left[\frac{u^2}{2k}\right]_{1}^{\frac{3}{2}}$ **A1**

OR

$= \frac{(\ln 5x)^2}{2k}$ **A1**

so $\int_{\frac{e}{5}}^{\frac{1}{5}e^{\frac{3}{2}}} \frac{\ln 5x}{kx}\,\mathrm{d}x = \left[\frac{(\ln 5x)^2}{2k}\right]_{\frac{e}{5}}^{\frac{1}{5}e^{\frac{3}{2}}}$ **A1**

THEN

$= \frac{1}{2k}\left(\frac{9}{4} - 1\right)$

$= \frac{5}{8k}$ **A1**

setting **their** expression for area equal to 3 **M1**

$\frac{5}{8k} = 3$

$k = \frac{5}{24}$ **A1**

[7 marks]

Total [16 marks]

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Question 3

(a) attempt to differentiate and set equal to zero ***M1***

$f'(x) = 2e^{2x} - 6e^{x} = 2e^{x}(e^{x} - 3) = 0$ ***A1***

minimum at $x = \ln 3$

$a = \ln 3$ ***A1***

[3 marks]

(b) **Note:** Interchanging x and y can be done at any stage.

$y = \left(e^{x} - 3\right)^{2} - 4$ ***(M1)***

$e^{x} - 3 = \pm\sqrt{y+4}$ ***A1***

as $x \le \ln 3$, $x = \ln\left(3 - \sqrt{y+4}\right)$ ***R1***

so $f^{-1}(x) = \ln\left(3 - \sqrt{x+4}\right)$ ***A1***

domain of f^{-1} is $x \in \mathbb{R}, -4 \le x < 5$ ***A1***

[5 marks]

Total [8 marks]

Question 4

attempt to integrate ***(M1)***

$u = 2x^{2} + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 4x$

$\int \frac{8x}{\sqrt{2x^{2}+1}}\mathrm{d}x = \int \frac{2}{\sqrt{u}}\mathrm{d}u$ ***(A1)***

EITHER

$= 4\sqrt{u}\,(+C)$ ***A1***

OR

$= 4\sqrt{2x^{2}+1}\,(+C)$ ***A1***

THEN

correct substitution into **their** integrated function (must have C) ***(M1)***

$5 = 4 + C \Rightarrow C = 1$

$f(x) = 4\sqrt{2x^{2}+1} + 1$ ***A1***

Total [5 marks]

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Question 5

(a) attempt to use the chain rule **M1**

$$f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}}$$ **A1**

$$f''(x)=-\frac{1}{4}(1+x)^{-\frac{3}{2}}$$ **A1**

$$=-\frac{1}{4\sqrt{(1+x)^3}}$$ **AG**

Note: Award ***M1A0A0*** for $f'(x)=\frac{1}{\sqrt{1+x}}$ or equivalent seen

[3 marks]

(b) let $n=2$

$$f''(x)=\left(-\frac{1}{4\sqrt{(1+x)^3}}=\right)\left(-\frac{1}{4}\right)^1\frac{1!}{0!}(1+x)^{\frac{1}{2}-2}$$ **R1**

Note: Award ***R0*** for not starting at $n=2$. Award subsequent marks as appropriate.

assume true for $n=k$, (so $f^{(k)}(x)=\left(-\frac{1}{4}\right)^{k-1}\frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k}$) **M1**

Note: Do not award ***M1*** for statements such as "let $n=k$" or "$n=k$ is true". Subsequent marks can still be awarded.

consider $n=k+1$

$$\text{LHS}=f^{(k+1)}(x)=\frac{d\left(f^{(k)}(x)\right)}{dx}$$ **M1**

$$=\left(-\frac{1}{4}\right)^{k-1}\frac{(2k-3)!}{(k-2)!}\left(\frac{1}{2}-k\right)(1+x)^{\frac{1}{2}-k-1}$$ (or equivalent) **A1**

EITHER

$$\text{RHS}=f^{(k+1)}(x)=\left(-\frac{1}{4}\right)^{k}\frac{(2k-1)!}{(k-1)!}(1+x)^{\frac{1}{2}-k-1}$$ (or equivalent) **A1**

$$=\left(-\frac{1}{4}\right)^{k}\frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ **A1**

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Note: Award ***A1*** for $\frac{(2k-1)!}{(k-1)!} = \frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!} \left(= \frac{2(2k-1)(2k-3)!}{(k-2)!} \right)$

$$= \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

$$\left(= \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1} \right)$$

Note: Award ***A1*** for leading coefficient of $-\frac{1}{4}$.

$$= \left(\frac{1}{2}-k\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

OR

Note: The following ***A*** marks can be awarded in any order.

$$= \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}\left(\frac{1-2k}{2}\right)(1+x)^{\frac{1}{2}-k-1}$$

$$= \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

Note: Award ***A1*** for isolating $(2k-1)$ correctly.

$$= \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)!}{(2k-2)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

Note: Award ***A1*** for multiplying top and bottom by $(k-1)$ or $2(k-1)$.

$$= \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

Note: Award ***A1*** for leading coefficient of $-\frac{1}{4}$.

$$= \left(-\frac{1}{4}\right)^{k} \frac{(2k-1)!}{(k-1)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

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$$= \left(-\frac{1}{4}\right)^{(k+1)-1} \frac{(2(k+1)-3)!}{((k+1)-2)!}(1+x)^{\frac{1}{2}-(k+1)} = \text{RHS}$$

THEN

since true for $n = 2$, and true for $n = k+1$ if true for $n = k$, the statement is true for all $n \in \mathbb{Z}, n \geq 2$ by mathematical induction ***R1***

Note: To obtain the final ***R1,*** at least four of the previous marks must have been awarded.

[9 marks]

(c) **METHOD 1**

$h(x) = \sqrt{1+x}\,\mathrm{e}^{mx}$

using product rule to find $h'(x)$ ***(M1)***

$h'(x) = \sqrt{1+x}\,m\mathrm{e}^{mx} + \frac{1}{2\sqrt{1+x}}\mathrm{e}^{mx}$ ***A1***

$h''(x) = m\left(\sqrt{1+x}\,m\mathrm{e}^{mx} + \frac{1}{2\sqrt{1+x}}\mathrm{e}^{mx}\right) + \frac{1}{2\sqrt{1+x}}m\mathrm{e}^{mx} - \frac{1}{4\sqrt{(1+x)^3}}\mathrm{e}^{mx}$ ***A1***

substituting $x = 0$ into $h''(x)$ ***M1***

$h''(0) = m^2 + \frac{1}{2}m + \frac{1}{2}m - \frac{1}{4}\left(= m^2 + m - \frac{1}{4}\right)$ ***A1***

$h(x) = h(0) + xh'(0) + \frac{x^2}{2!}h''(0) + \ldots$

equating x^2 coefficient to $\frac{7}{4}$ ***M1***

$\frac{h''(0)}{2!} = \frac{7}{4} \quad \left(\Rightarrow h''(0) = \frac{7}{2}\right)$

$4m^2 + 4m - 15 = 0$ ***A1***

$(2m+5)(2m-3) = 0$

$m = -\frac{5}{2}$ or $m = \frac{3}{2}$ ***A1***

[8 marks]

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METHOD 2

EITHER

attempt to find $f(0)$, $f'(0)$, $f''(0)$ ***(M1)***

$f(x)=(1+x)^{\frac{1}{2}}$ $\quad f(0)=1$

$f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}}$ $\quad f'(0)=\frac{1}{2}$

$f''(x)=-\frac{1}{4}(1+x)^{-\frac{3}{2}}$ $\quad f''(0)=-\frac{1}{4}$

$f(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\dots$ ***A1***

OR

attempt to apply binomial theorem for rational exponents ***(M1)***

$$f(x)=(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2+\dots$$

$f(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\dots$ ***A1***

THEN

$g(x)=1+mx+\frac{m^2}{2}x^2+\dots$ ***(A1)***

$h(x)=\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\dots\right)\left(1+mx+\frac{m^2}{2}x^2+\dots\right)$ ***(M1)***

coefficient of x^2 is $\frac{m^2}{2}+\frac{m}{2}-\frac{1}{8}$ ***A1***

attempt to set equal to $\frac{7}{4}$ and solve ***M1***

$\frac{m^2}{2}+\frac{m}{2}-\frac{1}{8}=\frac{7}{4}$

$4m^2+4m-15=0$ ***A1***

$(2m+5)(2m-3)=0$

$m=-\frac{5}{2}$ or $m=\frac{3}{2}$ ***A1***

[8 marks]

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METHOD 3

$g'(x) = me^{mx}$ and $g''(x) = m^2e^{mx}$ ***(A1)***

$h(x) = h(0) + xh'(0) + \frac{x^2}{2!}h''(0) + \ldots$

equating x^2 coefficient to $\frac{7}{4}$ ***M1***

$\frac{h''(0)}{2!} = \frac{7}{4} \quad \left(\Rightarrow h''(0) = \frac{7}{2}\right)$

using product rule to find $h'(x)$ and $h''(x)$ ***(M1)***

$h'(x) = f(x)g'(x) + f'(x)g(x)$

$h''(x) = f(x)g''(x) + 2f'(x)g'(x) + f''(x)g(x)$ ***A1***

substituting $x = 0$ into $h''(x)$ ***M1***

$h''(0) = f(0)g''(0) + 2g'(0)f'(0) + g(0)f''(0)$

$= 1 \times m^2 + 2m \times \frac{1}{2} + 1 \times \left(-\frac{1}{4}\right) \quad \left(= m^2 + m - \frac{1}{4}\right)$ ***A1***

$4m^2 + 4m - 15 = 0$ ***A1***

$(2m+5)(2m-3) = 0$

$m = -\frac{5}{2}$ or $m = \frac{3}{2}$ ***A1***

[8 marks]

Total [20 marks]

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Question 6

attempt to differentiate numerator and denominator **_M1_**

$$\lim_{x\to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$$

$$=\lim_{x\to 0}\frac{\left(\frac{2}{1+4x^2}\right)}{3\sec^2 3x}$$ **_A1A1_**

Note: ***A1*** for numerator and ***A1*** for denominator. Do not condone absence of limits.

attempt to substitute $x=0$ **_(M1)_**

$$=\frac{2}{3}$$ **_A1_**

Note: Award a maximum of ***M1A1A0M1A1*** for absence of limits.

[5 marks]

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Question 7

(a) $f'(x) = -2(x-h)$ ***A1***

[1 mark]

(b) $g'(x) = \mathrm{e}^{x-2}$ OR $g'(3) = \mathrm{e}^{3-2}$ (may be seen anywhere) ***A1***

Note: The derivative of g must be explicitly seen, either in terms of x or 3.

recognizing $f'(3) = g'(3)$ ***(M1)***

$-2(3-h) = \mathrm{e}^{3-2}\ \ (= \mathrm{e})$

$-6+2h = \mathrm{e}$ OR $3-h = -\frac{\mathrm{e}}{2}$ ***A1***

Note: The final ***A1*** is dependent on one of the previous marks being awarded.

$h = \frac{\mathrm{e}+6}{2}$ ***AG***

[3 marks]

(c) $f(3) = g(3)$ ***(M1)***

$-(3-h)^2 + 2k = \mathrm{e}^{3-2} + k$

correct equation in k

EITHER

$-\left(3 - \frac{\mathrm{e}+6}{2}\right)^2 + 2k = \mathrm{e}^{3-2} + k$ ***A1***

$k = \mathrm{e} + \left(\frac{6-\mathrm{e}-6}{2}\right)^2 \left(= \mathrm{e} + \left(\frac{-\mathrm{e}}{2}\right)^2\right)$ ***A1***

OR

$k = \mathrm{e} + \left(3 - \frac{\mathrm{e}+6}{2}\right)^2$ ***A1***

$k = \mathrm{e} + 9 - 3\mathrm{e} - 18 + \frac{\mathrm{e}^2 + 12\mathrm{e} + 36}{4}$ ***A1***

THEN

$k = \mathrm{e} + \frac{\mathrm{e}^2}{4}$ ***AG***

[3 marks]
Total [7 marks]

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Question 8

(a) $\frac{\mathrm{d}v}{\mathrm{d}t}=-(1+v)$ ***(A1)***

$\int 1\,\mathrm{d}t=\int -\frac{1}{1+v}\,\mathrm{d}v$ (or equivalent / use of integrating factor) ***M1***

$t=-\ln(1+v)(+C)$ ***A1***

EITHER

attempt to find C with initial conditions $t=0, v=v_0$ ***M1***

$C=\ln(1+v_0)$

$t=\ln(1+v_0)-\ln(1+v)$

$t=\ln\left(\frac{1+v_0}{1+v}\right)\Rightarrow \mathrm{e}^t=\frac{1+v_0}{1+v}$ ***A1***

$\mathrm{e}^t(1+v)=1+v_0$

$1+v=(1+v_0)\mathrm{e}^{-t}$ ***A1***

$v(t)=(1+v_0)\mathrm{e}^{-t}-1$ ***AG***

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(b) (i) recognition that when $t = T, v = 0$ ***M1***

$(1+v_0)\mathrm{e}^{-T} - 1 = 0 \Rightarrow \mathrm{e}^{-T} = \dfrac{1}{1+v_0}$ ***A1***

$\mathrm{e}^{T} = 1+v_0$ ***AG***

Note: Award ***M1A0*** for substituting $v_0 = \mathrm{e}^{T} - 1$ into v and showing that $v = 0$.

(ii) $s(t) = \int v(t)\ \mathrm{d}t\ \left(= \int \left((1+v_0)\mathrm{e}^{-t} - 1\right)\mathrm{d}t\right)$ ***(M1)***

$= -(1+v_0)\mathrm{e}^{-t} - t(+D)$ ***A1***

$(t = 0, s = 0$ so) $D = 1+v_0$ ***A1***

$s(t) = -(1+v_0)\mathrm{e}^{-t} - t + 1 + v_0$

at $s_{\max}$, $\mathrm{e}^{T} = 1+v_0 \Rightarrow T = \ln(1+v_0)$

Substituting into $s(t)\left(= -(1+v_0)\mathrm{e}^{-t} - t + 1 + v_0\right)$ ***M1***

$s_{\max} = -(1+v_0)\left(\dfrac{1}{1+v_0}\right) - \ln(1+v_0) + v_0 + 1$ ***A1***

$\left(s_{\max} = v_0 - \ln(1+v_0)\right)$

[7 marks]

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(c) **METHOD 1**

$v(T-k)=(1+v_0)e^{-T}e^{k}-1$ ***(M1)***

$=(1+v_0)\left(\frac{1}{1+v_0}\right)e^{k}-1$ ***A1***

$=e^{k}-1$ ***AG***

[2 marks]

METHOD 2

$v(T-k)=(1+v_0)e^{-(T-k)}-1$

$=e^{T}e^{-(T-k)}-1$ ***M1***

$=e^{T-T+k}-1$ ***A1***

$=e^{k}-1$ ***AG***

[2 marks]

(d) **METHOD 1**

$v(T+k)=(1+v_0)e^{-T}e^{-k}-1$ ***(A1)***

$=e^{-k}-1$ ***A1***

[2 marks]

METHOD 2

$v(T+k)=(1+v_0)e^{-(T+k)}-1$ ***(A1)***

$=e^{T}e^{-(T+k)}-1$

$=e^{T-T-k}-1$

$=e^{-k}-1$ ***A1***

[2 marks]

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(e) **METHOD 1**

$v(T-k)+v(T+k)=\mathrm{e}^{k}+\mathrm{e}^{-k}-2$ ***A1***

attempt to express as a square ***M1***

$=\left(\mathrm{e}^{\frac{k}{2}}-\mathrm{e}^{-\frac{k}{2}}\right)^{2}(\geq 0)$ ***A1***

so $v(T-k)+v(T+k)\geq 0$ ***AG***

[3 marks]

METHOD 2

$v(T-k)+v(T+k)=\mathrm{e}^{k}+\mathrm{e}^{-k}-2$ ***A1***

Attempt to solve $\frac{\mathrm{d}}{\mathrm{d}k}\left(\mathrm{e}^{k}+\mathrm{e}^{-k}\right)=0$ $(\Rightarrow k=0)$ ***M1***

minimum value of 2, (when $k=0$), hence $\mathrm{e}^{k}+\mathrm{e}^{-k}\geq 2$ ***R1***

so $v(T-k)+v(T+k)\geq 0$ ***AG***

[3 marks]
Total [20 marks]

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Question 9

(a) $6+6\cos x=0$ (or setting their $f'(x)=0$) *(M1)*

$\cos x=-1$ (or $\sin x=0$)

$x=\pi,\ x=3\pi$ ***A1A1***

[3 marks]

(b) attempt to integrate $\int_{\pi}^{3\pi}(6+6\cos x)\mathrm{d}x$ *(M1)*

$=\left[6x+6\sin x\right]_{\pi}^{3\pi}$ ***A1A1***

substitute their limits into their integrated expression and subtract *(M1)*

$=(18\pi+6\sin 3\pi)-(6\pi+6\sin\pi)$

$=(6(3\pi)+0)-(6\pi+0)\ \ (=18\pi-6\pi)$ ***A1***

area $=12\pi$ ***AG***

[5 marks]

(c) attempt to substitute into the formula for surface area (including base) *(M1)*

$\pi(2^2)+\pi(2)(l)=12\pi$ *(A1)*

$4\pi+2\pi l=12\pi$

$2\pi l=8\pi$

$l=4$ ***A1***

[3 marks]

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(d) valid attempt to find the height of the cone **(M1)**

e.g. $2^2 + h^2 = (\text{their } l)^2$

$h = \sqrt{12}\ \left(= 2\sqrt{3}\right)$ **(A1)**

attempt to use $V = \dfrac{1}{3}\pi r^2 h$ with their values substituted **M1**

$\left(\dfrac{1}{3}\pi\left(2^2\right)\left(\sqrt{12}\right)\right)$

$\text{volume} = \dfrac{4\pi\sqrt{12}}{3}\left(= \dfrac{8\pi\sqrt{3}}{3} = \dfrac{8\pi}{\sqrt{3}}\right)$ **A1**

[4 marks]

Question 10

$u = \sin x \Rightarrow \mathrm{d}u = \cos x\,\mathrm{d}x$ (or equivalent) **A1**

$= \displaystyle\int \frac{u}{u^2 - u - 2}\,\mathrm{d}u$ **A1**

attempt to use partial fractions **M1**

$\left(\dfrac{u}{(u+1)(u-2)} \equiv \dfrac{A}{u+1} + \dfrac{B}{u-2} \Rightarrow u \equiv A(u-2) + B(u+1)\right)$

Valid attempt to solve for A and B **(M1)**

$A = \dfrac{1}{3}$ and $B = \dfrac{2}{3}$ **A1**

$\dfrac{u}{(u+1)(u-2)} \equiv \dfrac{1}{3(u+1)} + \dfrac{2}{3(u-2)}$

$\displaystyle\int\left(\frac{1}{3(u+1)} + \frac{2}{3(u-2)}\right)\mathrm{d}u = \frac{1}{3}\ln|u+1| + \frac{2}{3}\ln|u-2|\,(+C)$ (or equivalent) **A1**

Note: Condone the absence of $+C$ or lack of moduli here but not in the final answer.

$= \dfrac{1}{3}\ln|\sin x + 1| + \dfrac{2}{3}\ln|\sin x - 2| + C$ **A1**

Note: Condone further simplification of the correct answer.

[7 marks]

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Question 11

(a) $\ln\left(x^2-16\right)=0$ **(M1)**

$e^0 = x^2-16(=1)$

$x^2=17$ OR $x=\pm\sqrt{17}$ **(A1)**

$a=\sqrt{17}$ **A1**

[3 marks]

(b) attempt to differentiate (must include $2x$ and/or $\frac{1}{x^2-16}$) **(M1)**

$f'(x)=\frac{2x}{x^2-16}$ **A1**

setting their derivative $=\frac{1}{3}$ **M1**

$\frac{2x}{x^2-16}=\frac{1}{3}$

$x^2-16=6x$ ***OR*** $x^2-6x-16=0$ (or equivalent) **A1**

valid attempt to solve their quadratic **(M1)**

$x=8$ **A1**

Note: Award ***A0*** if the candidate's final answer includes additional solutions (such as $x=-2,\ 8$).

[6 marks]
Total [9 marks]

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Question 12

(a) For $n=1$

LHS: $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\mathrm{e}^x\right)=x^2\mathrm{e}^x+2x\mathrm{e}^x\left(=\mathrm{e}^x\left(x^2+2x\right)\right)$ ***A1***

RHS: $\left(x^2+2(1)x+1(1-1)\right)\mathrm{e}^x\left(=\mathrm{e}^x\left(x^2+2x\right)\right)$ ***A1***

so true for $n=1$

now assume true for $n=k$; i.e. $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^2\mathrm{e}^x\right)=\left[x^2+2kx+k(k-1)\right]\mathrm{e}^x$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n=k$". Subsequent marks can still be awarded.

attempt to differentiate the RHS ***M1***

$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(x^2\mathrm{e}^x\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\left[x^2+2kx+k(k-1)\right]\mathrm{e}^x\right)$

$=(2x+2k)\mathrm{e}^x+\left(x^2+2kx+k(k-1)\right)\mathrm{e}^x$ ***A1***

$=\left[x^2+2(k+1)x+k(k+1)\right]\mathrm{e}^x$ ***A1***

so true for $n=k$ implies true for $n=k+1$

therefore $n=1$ true and $n=k$ true $\Rightarrow$ $n=k+1$ true

therefore, true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Award **R1** only if three of the previous four marks have been awarded

[7 marks]

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(b) **METHOD 1**

attempt to use $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right)=\left[x^2+2nx+n(n-1)\right]\mathrm{e}^x$ ***(M1)***

> **Note***:* For $x=0$, $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right)_{|x=0}=n(n-1)$ may be seen.

$f(0)=0, f'(0)=0, f''(0)=2, f'''(0)=6, f^{(4)}(0)=12$

use of $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{(4)}(0)+\ldots$ ***(M1)***

$\Rightarrow f(x)\approx x^2+x^3+\frac{1}{2}x^4$ ***A1***

[3 marks]

METHOD 2

'$x^2\times$ Maclaurin series of e^x' ***(M1)***

$x^2\left(1+x+\frac{x^2}{2!}+\ldots\right)$ ***(A1)***

$\Rightarrow f(x)\approx x^2+x^3+\frac{1}{2}x^4$ ***A1***

[3 marks]

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(c)

$$\lim_{x\to 0}\left[\frac{\left(x^2e^x - x^2\right)^3}{x^9}\right] = \lim_{x\to 0}\left(\frac{x^2e^x - x^2}{x^3}\right)^3$$ ***M1***

$$= \lim_{x\to 0}\left(\frac{e^x - 1}{x}\right)^3$$ ***(A1)***

attempt to use L'Hôpital's rule ***M1***

$$= \lim_{x\to 0}\left(\frac{e^x - 0}{1}\right)^3$$

$$= \left[\lim_{x\to 0} e^x\right]^3$$

$$= 1$$ ***A1***

[4 marks]

Total [14 marks]

Question 13

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(a) (i) valid approach to find turning point ($v' = 0$, $-\frac{b}{2a}$, average of roots) ***(M1)***

$4 - 6t = 0$ OR $-\frac{4}{2(-3)}$ OR $\frac{-\frac{2}{3}+2}{2}$

$t = \frac{2}{3}$ (s) ***A1***

(ii) attempt to integrate v ***(M1)***

$\int v \, \mathrm{d}t = \int \left(4 + 4t - 3t^2\right) \mathrm{d}t = 4t + 2t^2 - t^3 (+c)$ ***A1A1***

Note: Award ***A1*** for $4t + 2t^2$, ***A1*** for $-t^3$.

attempt to substitute their t into their solution for the integral ***(M1)***

distance $= 4\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3$

$= \frac{8}{3} + \frac{8}{9} - \frac{8}{27}$ (or equivalent) ***A1***

$= \frac{88}{27}$ (m) ***AG***

[7 marks]

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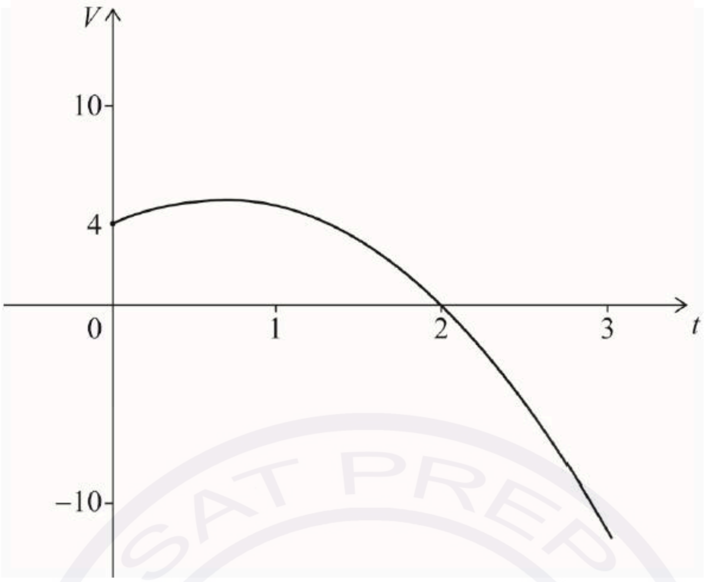
(b)



valid approach to solve $4+4t-3t^2=0$ (may be seen in part (a)) ***(M1)***

$(2-t)(2+3t)$ OR $\frac{-4\pm\sqrt{16+48}}{-6}$

correct x- intercept on the graph at $t=2$ ***A1***

Note: The following two ***A*** marks may only be awarded if the shape is a concave down parabola. These two marks are independent of each other and the ***(M1)***.

correct domain from 0 to 3 starting at $(0,4)$ ***A1***

Note: The 3 must be clearly indicated.

vertex in approximately correct place for $t=\frac{2}{3}$ and $v>4$ ***A1***

[4 marks]

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(c) recognising to integrate between 0 and 2, or 2 and 3 OR $\int_0^3 \left|4+4t-3t^2\right| \mathrm{d}t$ ***(M1)***

$$\int_0^2 \left(4+4t-3t^2\right) \mathrm{d}t$$

$=8$ ***A1***

$$\int_2^3 \left(4+4t-3t^2\right) \mathrm{d}t$$

$=-5$ ***A1***

valid approach to sum the two areas (seen anywhere) ***(M1)***

$\int_0^2 v \mathrm{d}t - \int_2^3 v \mathrm{d}t$ OR $\int_0^2 v \mathrm{d}t + \left|\int_2^3 v \mathrm{d}t\right|$

total distance travelled $=13$ (m) ***A1***

[5 marks]

Total [16 marks]

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Question 14

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2y}{x}=\frac{\ln 2x}{x^2}$$ **(M1)**

attempt to find integrating factor **(M1)**

$$\left(\mathrm{e}^{\int \frac{2}{x}\mathrm{d}x}=\mathrm{e}^{2\ln x}\right)=x^2$$ **(A1)**

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x}+2xy=\ln 2x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2y\right)=\ln 2x$$

$$x^2y=\int \ln 2x\ \mathrm{d}x$$

attempt to use integration by parts **(M1)**

$$x^2y=x\ln 2x-x(+c)$$ **A1**

$$y=\frac{\ln 2x}{x}-\frac{1}{x}+\frac{c}{x^2}$$

substituting $x=\frac{1}{2}, y=4$ into an integrated equation involving c **M1**

$$4=0-2+4c$$

$$\Rightarrow c=\frac{3}{2}$$

$$y=\frac{\ln 2x}{x}-\frac{1}{x}+\frac{3}{2x^2}$$ **A1**

[7 marks]

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Question 15

(a) $f'(4)=6$ **A1**

[1 mark]

(b) $f(4)=6\times4-1=23$ **A1**

[1 mark]

(c) $h(4)=f\left(g(4)\right)$ ***(M1)***

$h(4)=f(4^2-3\times4)=f(4)$

$h(4)=23$ **A1**

[2 marks]

(d) attempt to use chain rule to find h' ***(M1)***

$f'\left(g(x)\right)\times g'(x)$ **OR** $\left(x^2-3x\right)'\times f'\left(x^2-3x\right)$

$h'(4)=\left(2\times4-3\right)f'\left(4^2-3\times4\right)$ **A1**

$=30$

$y-23=30\left(x-4\right)$ OR $y=30x-97$ **A1**

[3 marks]

Total [7 marks]

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Question 16

METHOD 1

recognition that $y=\int \cos\left(x-\frac{\pi}{4}\right)\mathrm{d}x$ ***(M1)***

$y=\sin\left(x-\frac{\pi}{4}\right)(+c)$ ***(A1)***

substitute both x and y values into their integrated expression including c ***(M1)***

$2=\sin\frac{\pi}{2}+c$

$c=1$

$y=\sin\left(x-\frac{\pi}{4}\right)+1$ ***A1***

[4 marks]

METHOD 2

$\int_2^y \mathrm{d}y=\int_{\frac{3\pi}{4}}^{x}\cos\left(x-\frac{\pi}{4}\right)\mathrm{d}x$ ***(M1)(A1)***

$y-2=\sin\left(x-\frac{\pi}{4}\right)-\sin\frac{\pi}{2}$ ***A1***

$y=\sin\left(x-\frac{\pi}{4}\right)+1$ ***A1***

[4 marks]

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Question 17

(a) **METHOD 1**

recognition of both known series ***(M1)***

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots \text{ and } \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

attempt to multiply the two series up to and including x^3 term ***(M1)***

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$$

$$= x - \frac{x^3}{3!} + x^2 + \frac{x^3}{2!} + \dots$$ ***(A1)***

$$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \dots$$ ***A1***

[4 marks]

METHOD 2

$$f(x) = e^x \sin x$$

$$f'(x) = e^x \cos x + e^x \sin x$$ ***A1***

$$f''(x) = e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x \left(= 2e^x \cos x\right)$$

$$f'''(x) = 2e^x \cos x - 2e^x \sin x$$

$f''(x) = 2e^x \cos x$ and $f'''(x) = 2e^x\left(\cos x - \sin x\right)$ ***A1***

substitute $x = 0$ into f or its derivatives to obtain Maclaurin series ***(M1)***

$$e^x \sin x = 0 + \frac{x}{1!} \times 1 + \frac{x^2}{2!} \times 2 + \frac{x^3}{3!} \times 2 + \dots$$

$$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \dots$$ ***A1***

[4 marks]

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(b) $e^{x^2}\sin\left(x^2\right)=x^2+x^4+\frac{1}{3}x^6+\ldots$ ***(A1)***

substituting their expression and attempt to integrate ***M1***

$$\int_0^1 e^{x^2}\sin(x^2)\,dx \approx \int_0^1\left(x^2+x^4+\frac{1}{3}x^6\right)dx$$

Note: Condone absence of limits up to this stage.

$=\left[\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{21}\right]_0^1$ ***A1***

$=\frac{61}{105}$ ***A1***

[4 marks]

(c) (i) attempt to use product rule at least once ***M1***

$g'(x)=e^x\cos x-e^x\sin x$ ***A1***

$g''(x)=e^x\cos x-e^x\sin x-e^x\sin x-e^x\cos x\left(=-2e^x\sin x\right)$ ***A1***

EITHER

$2\left(g'(x)-g(x)\right)=2\left(e^x\cos x-e^x\sin x-e^x\cos x\right)=-2e^x\sin x$ ***A1***

OR

$g''(x)=2\left(e^x\cos x-e^x\sin x-e^x\cos x\right)$ ***A1***

THEN

$g''(x)=2\left(g'(x)-g(x)\right)$ ***AG***

Note: Accept working with each side separately to obtain $-2e^x\sin x$.

(ii) $g'''(x)=2\left(g''(x)-g'(x)\right)$ ***A1***

$g^{(4)}(x)=2\left(g'''(x)-g''(x)\right)$ ***AG***

Note: Accept working with each side separately to obtain $-4e^x\cos x$.

[5 marks]

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(d) attempt to substitute $x=0$ into a derivative **(M1)**

$g(0)=1,\ g'(0)=1,\ g''(0)=0$ **A1**

$g'''(0)=-2,\ g^{(4)}(0)=-4$ **(A1)**

attempt to substitute into Maclaurin formula **(M1)**

$g(x)=1+x-\frac{2}{3!}x^3-\frac{4}{4!}x^4+\ldots\left(=1+x-\frac{1}{3}x^3-\frac{1}{6}x^4+\ldots\right)$ **A1**

Note: Do not award any marks for approaches that do not use the part (c) result.

[5 marks]

(e) **METHOD 1**

$\lim\limits_{x\to 0}\frac{\mathrm{e}^x\cos x-1-x}{x^3}=\lim\limits_{x\to 0}\frac{\left(1+x-\frac{1}{3}x^3-\frac{1}{6}x^4+\ldots\right)-1-x}{x^3}$ **M1**

$=\lim\limits_{x\to 0}\left(-\frac{1}{3}-\frac{1}{6}x+\ldots\right)$ **(A1)**

$=-\frac{1}{3}$ **A1**

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METHOD 2

$\lim\limits_{x\to0}\dfrac{e^x\cos x-1-x}{x^3}=\dfrac{0}{0}$ indeterminate form, attempt to apply l'Hôpital's rule ***M1***

$=\lim\limits_{x\to0}\dfrac{e^x\cos x-e^x\sin x-1}{3x^2}\left(=\lim\limits_{x\to0}\dfrac{g'(x)-1}{3x^2}\right)$

$=\dfrac{0}{0}$, using l'Hôpital's rule again

$=\lim\limits_{x\to0}\dfrac{-2e^x\sin x}{6x}\left(=\lim\limits_{x\to0}\dfrac{g''(x)}{6x}\right)$

$=\dfrac{0}{0}$, using l'Hôpital's rule again

$=\lim\limits_{x\to0}\dfrac{-2e^x\sin x-2e^x\cos x}{6}\left(=\lim\limits_{x\to0}\dfrac{g'''(x)}{6}\right)$ ***A1***

$=-\dfrac{1}{3}$ ***A1***

[3 marks]

Total [21 marks]

Question 18

evidence of using product rule ***(M1)***

$\dfrac{dy}{dx}=(2x-1)\times\left(ke^{kx}\right)+2\times e^{kx}\quad\left(=e^{kx}(2kx-k+2)\right)$ ***A1***

correct working for one of (seen anywhere) ***A1***

$\dfrac{dy}{dx}$ at $x=1\Rightarrow ke^k+2e^k$

OR

slope of tangent is $5e^k$

their $\dfrac{dy}{dx}$ at $x=1$ equals the slope of $y=5e^kx\ \left(=5e^k\right)$ (seen anywhere) ***(M1)***

$ke^k+2e^k=5e^k$

$k=3$ ***A1***

[5 marks]

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Question 19

$$\int \frac{3\sqrt{x}-5}{\sqrt{x}}\mathrm{d}x = \int\left(3-5x^{-\frac{1}{2}}\right)\mathrm{d}x$$ **(A1)**

$$\int \frac{3\sqrt{x}-5}{\sqrt{x}}\mathrm{d}x = 3x-10x^{\frac{1}{2}}\ (+c)$$ **A1A1**

substituting limits into their integrated function and subtracting **(M1)**

$3(9)-10(9)^{\frac{1}{2}}-\left(3(1)-10(1)^{\frac{1}{2}}\right)$ OR $27-10\times3-(3-10)$

$=4$ **A1**

[5 marks]

Question 20

$u=\sec x \Rightarrow \mathrm{d}u=\sec x\tan x\ \mathrm{d}x$ **(A1)**

attempts to express the integral in terms of u **M1**

$\int_1^2 u^{n-1}\mathrm{d}u$ **A1**

$=\frac{1}{n}\left[u^n\right]_1^2\ \ \left(=\frac{1}{n}\left[\sec^n x\right]_0^{\frac{\pi}{3}}\right)$ **A1**

Note: Condone the absence of or incorrect limits up to this point.

$=\frac{2^n-1^n}{n}$ **M1**

$=\frac{2^n-1}{n}$ **A1**

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Question 21

(a) attempts to replace x with $-x$ ***M1***

$$f(-x)=-x\sqrt{1-(-x)^2}$$

$$=-x\sqrt{1-(-x)^2}\ (=-f(x))$$ ***A1***

> **Note:** Award ***M1A1*** for an attempt to calculate both $f(-x)$ and $-f(-x)$ independently, showing that they are equal.
>
> **Note:** Award ***M1A0*** for a graphical approach including evidence that **either** the graph is invariant after rotation by 180° about the origin **or** the graph is invariant after a reflection in the y- axis and then in the x- axis (or vice versa).

so f is an odd function ***AG***

[2 marks]

(b) attempts both product rule and chain rule differentiation to find $f'(x)$ ***M1***

$$f'(x)=x\times\frac{1}{2}\times(-2x)\times\left(1-x^2\right)^{-\frac{1}{2}}+\left(1-x^2\right)^{\frac{1}{2}}\times 1\ \left(=\sqrt{1-x^2}-\frac{x^2}{\sqrt{1-x^2}}\right)$$ ***A1***

$$=\frac{1-2x^2}{\sqrt{1-x^2}}$$

sets their $f'(x)=0$ ***M1***

$$\Rightarrow x=\pm\frac{1}{\sqrt{2}}$$ ***A1***

attempts to find at least one of $f\left(\pm\frac{1}{\sqrt{2}}\right)$ ***(M1)***

> **Note:** Award ***M1*** for an attempt to evaluate $f(x)$ at least at one of their $f'(x)=0$ roots.

$a=-\frac{1}{2}$ and $b=\frac{1}{2}$ ***A1***

> **Note:** Award ***A1*** for $-\frac{1}{2}\le y\le\frac{1}{2}$.

[6 marks]

Total [8 marks]

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Question 22

(a) $\cos^2 x - 3\sin^2 x = 0$

valid attempt to reduce equation to one involving one trigonometric function ***(M1)***

$\frac{\sin^2 x}{\cos^2 x} = \frac{1}{3}$ OR $1 - \sin^2 x - 3\sin^2 x = 0$ OR $\cos^2 x - 3\left(1 - \cos^2 x\right) = 0$

OR $\cos 2x - 1 + \cos 2x = 0$

correct equation ***(A1)***

$\tan^2 x = \frac{1}{3}$ OR $\cos^2 x = \frac{3}{4}$ OR $\sin^2 x = \frac{1}{4}$ OR $\cos 2x = \frac{1}{2}$

$\tan x = \pm\frac{1}{\sqrt{3}}$ OR $\cos x = \pm\frac{\sqrt{3}}{2}$ OR $\sin x = (\pm)\frac{1}{2}$ OR $2x = \frac{\pi}{3}\left(, \frac{5\pi}{3}\right)$ ***(A1)***

$x = \frac{\pi}{6},\ x = \frac{5\pi}{6}$ ***A1A1***

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(b) (i) attempt to use the chain rule (may be evidenced by at least one $\cos x \sin x$ term) ***(M1)***

$f'(x) = -2\cos x \sin x - 6\sin x \cos x \left(= -8\sin x \cos x = -4\sin 2x\right)$ ***A1***

(ii) valid attempt to solve their $f'(x) = 0$ ***(M1)***

at least 2 correct x-coordinates (may be seen in coordinates) ***(A1)***

$x = 0$, $x = \dfrac{\pi}{2}$, $x = \pi$

Note: Accept additional correct solutions outside the domain.
Award ***A0*** if any additional incorrect solutions are given.

correct coordinates (may be seen in graph for part (c)) ***A1A1A1***

$(0,1)$, $(\pi,1)$, $\left(\dfrac{\pi}{2},-3\right)$

Note: Award a maximum of ***M1A1A1A1A0*** if any additional solutions are given.

Note: If candidates do not find at least two correct x-coordinates, it is possible to award the appropriate final marks for their correct coordinates, such as ***M1A0A0A1A0.***

[7 marks]

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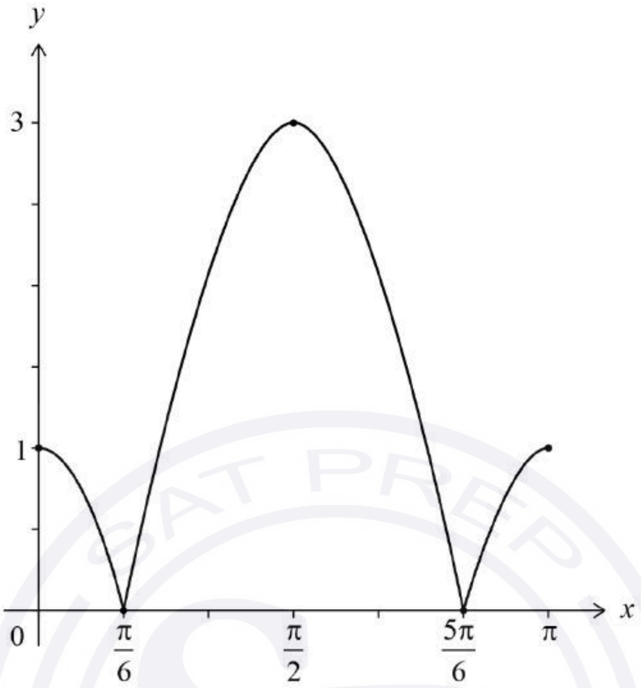
(c)



attempt to reflect the negative part of the graph of f in the x-axis **M1**

endpoints have coordinates $(0,1)$, $(\pi,1)$ **A1**

smooth maximum at $\left(\frac{\pi}{2},3\right)$ **A1**

sharp points (cusps) at x-intercepts $\frac{\pi}{6}$, $\frac{5\pi}{6}$ **A1**

[4 marks]

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(d) considers points of intersection of $y=\left|f(x)\right|$ and $y=1$ on graph or algebraically ***(M1)***

$-\left(\cos^2 x-3\sin^2 x\right)=1$ or $-\left(1-4\sin^2 x\right)=1$ or $-\left(4\cos^2 x-3\right)=1$ or $-\left(2\cos 2x-1\right)=1$

$\tan^2 x=1$ or $\sin^2 x=\frac{1}{2}$ or $\cos^2 x=\frac{1}{2}$ or $\cos 2x=0$ ***(A1)***

$x=\frac{\pi}{4},\frac{3\pi}{4}$ ***(A1)***

For $\left|f(x)\right|>1$

$\frac{\pi}{4}<x<\frac{3\pi}{4}$ ***A1***

[4 marks]

Total [20 marks]

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Question 23

(a) $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2-2x^2}{xy}$

let $y=vx$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=v+x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

$\Rightarrow v+x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{v^2x^2-2x^2}{vx^2}$ ***(M1)***

$\Rightarrow x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{-2}{v}$ ***(A1)***

$\Rightarrow \int v\mathrm{d}v=-\int\frac{2}{x}\mathrm{d}x$ ***M1***

$\Rightarrow \frac{v^2}{2}=-2\ln|x|(+c)$ ***A1***

Note: Condone the absence of the modulus sign up to this point.

$\Rightarrow \frac{y^2}{2x^2}=-2\ln|x|+c$ ***A1***

attempt to substitute $x=1, y=2$ into their integrated expression to find c ***M1***

$\Rightarrow 2=-2\ln|1|+c\Rightarrow c=2$

$\Rightarrow \frac{y^2}{2x^2}=-2\ln|x|+2$

$\Rightarrow y^2=2x^2(-2\ln|x|+2)(=4x^2(1-\ln|x|))$ ***A1***

[8 marks]

(b) attempt to set $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ in the differential equation ***(M1)***

$y=\sqrt{2}x$ and $y=-\sqrt{2}x$ or $m=\pm\sqrt{2}$ ***A1***

[2 marks]

Total [10 marks]

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Question 24

attempt at implicit differentiation, including use of the product rule ***(M1)***

EITHER

$$\left(2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)y^2+\left(x^2+y^2\right)2y\frac{\mathrm{d}y}{\mathrm{d}x}=8x$$ ***A1A1A1***

Note: Award ***A1*** for each of $\left(2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)y^2$, $\left(x^2+y^2\right)2y\frac{\mathrm{d}y}{\mathrm{d}x}$ and $8x$.

OR

$$x^2y^2+y^4=4x^2$$

$$2xy^2+2x^2y\frac{\mathrm{d}y}{\mathrm{d}x}+4y^3\frac{\mathrm{d}y}{\mathrm{d}x}=8x$$ ***A1A1A1***

Note: Award ***A1*** for each of $2xy^2+2x^2y\frac{\mathrm{d}y}{\mathrm{d}x}$, $4y^3\frac{\mathrm{d}y}{\mathrm{d}x}$ and $8x$.

THEN

at a local maximum or minimum point, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ ***(M1)***

$$2xy^2=8x$$

$$x=0 \text{ or } y^2=4\left(\Rightarrow y=\pm 2\right)$$ ***A1***

Note: Award ***A0*** for $x=0$ or $y=2$

since $x>0$ and $-2<y<2$ there are no solutions ***R1***

hence there are no local maximum or minimum points ***AG***

[7 marks]

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Question 25

recognizing need to integrate **_(M1)_**

$\int \frac{6x}{x^2+1}\mathrm{d}x$ OR $u = x^2+1$ OR $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

$\int \frac{3}{u}\mathrm{d}u$ OR $3\int \frac{2x}{x^2+1}\mathrm{d}x$ **_(A1)_**

$= 3\ln\left(x^2+1\right)(+c)$ or $3\ln u(+c)$ **_A1_**

correct substitution of $x = 1$ and $f(x) = 5$ or $x = 1$ and $u = 2$ into equation

using **their** integrated expression (must involve *c*) **_(M1)_**

$5 = 3\ln 2 + c$

$f(x) = 3\ln\left(x^2+1\right) + 5 - 3\ln 2 \quad \left(= 3\ln\left(x^2+1\right) + 5 - \ln 8 = 3\ln\left(\frac{x^2+1}{2}\right) + 5\right)$ (or equivalent) **_A1_**

Note: Accept the use of the modulus sign in working and the final answer.

[5 marks]

Question 26

$g'(x) = 2x\mathrm{e}^{x^2+1}$ **_(A2)_**

substitute $x = -1$ into **their** derivative **_(M1)_**

$g'(-1) = -2\mathrm{e}^2$ **_A1_**

Note: Award ***A0M0A0*** in cases where candidate's incorrect derivative is $g'(x) = \mathrm{e}^{x^2+1}$.

[4 marks]

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Question 27

(a) $L = \mathrm{AC} + \mathrm{CB}$

$\dfrac{\left(\frac{3}{4}\right)}{\mathrm{AC}} = \cos\alpha \left(\Rightarrow \mathrm{AC} = \dfrac{\frac{3}{4}}{\cos\alpha} \Rightarrow \mathrm{AC} = \dfrac{3}{4}\sec\alpha \right)$ ***A1***

$\dfrac{6}{\mathrm{CB}} = \sin\alpha \left(\Rightarrow \mathrm{CB} = \dfrac{6}{\sin\alpha} \Rightarrow \mathrm{CB} = 6\operatorname{cosec}\alpha \right)$ ***A1***

so $L = \dfrac{3}{4}\sec\alpha + 6\operatorname{cosec}\alpha$ ***AG***

[2 marks]

(b) (i) $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{3}{4}\sec\alpha\tan\alpha - 6\operatorname{cosec}\alpha\cot\alpha$ ***A1***

(ii) attempt to write $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha}$ in terms of $\sin\alpha, \cos\alpha$ or $\tan\alpha$ (may be seen in (i)) ***(M1)***

$\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{\frac{3}{4}\sin\alpha}{\cos^2\alpha} - \dfrac{6\cos\alpha}{\sin^2\alpha}$ OR $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{\frac{3}{4}\tan\alpha}{\cos\alpha} - \dfrac{6}{\sin\alpha\tan\alpha} \left(= \dfrac{\frac{3}{4}\tan^3\alpha - 6}{\cos\alpha\tan^2\alpha} \right)$

$\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = 0 \Rightarrow \dfrac{3}{4}\sin^3\alpha - 6\cos^3\alpha = 0$ OR $\dfrac{3}{4}\tan^3\alpha - 6 = 0$ (or equivalent) ***(A1)***

$\tan^3\alpha = 8$ ***A1***

$\tan\alpha = 2$ ***A1***

$\alpha = \arctan 2$ ***AG***

[5 marks]

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(c) (i) attempt to use product rule (at least once) ***(M1)***

$$\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}=\frac{3}{4}\sec\alpha\tan\alpha\tan\alpha+\frac{3}{4}\sec\alpha\sec^2\alpha$$

$+6\operatorname{cosec}\alpha\cot\alpha\cot\alpha+6\operatorname{cosec}\alpha\operatorname{cosec}^2\alpha$ (or equivalent) ***A1A1***

Note: Award ***A1*** for $\frac{3}{4}\sec\alpha\tan\alpha\tan\alpha+\frac{3}{4}\sec\alpha\sec^2\alpha$ and ***A1*** for $+6\operatorname{cosec}\alpha\cot\alpha\cot\alpha+6\operatorname{cosec}\alpha\operatorname{cosec}^2\alpha$. Allow unsimplified correct answer.

$$\left(\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}=\frac{3}{4}\sec\alpha\tan^2\alpha+\frac{3}{4}\sec^3\alpha+6\operatorname{cosec}\alpha\cot^2\alpha+6\operatorname{cosec}^3\alpha\right)$$

(ii) attempt to find a ratio other than $\tan\alpha$ using an appropriate trigonometric identity OR a right triangle with at least two side lengths seen ***(M1)***

Note: Award ***M0*** for $\alpha=\arctan 2$ substituted into their $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}$ with no further progress.

one correct ratio ***(A1)***

$\sec\alpha=\sqrt{5}$ OR $\operatorname{cosec}\alpha=\frac{\sqrt{5}}{2}$ OR $\cot\alpha=\frac{1}{2}$ OR $\cos\alpha=\frac{1}{\sqrt{5}}$ OR $\sin\alpha=\frac{2}{\sqrt{5}}$

Note: ***M1A1*** may be seen in part (d).

$\frac{3}{4}\left(\sqrt{5}\right)\left(2^2\right)+\frac{3}{4}\left(\sqrt{5}\right)^3+6\left(\frac{\sqrt{5}}{2}\right)\left(\frac{1}{2}\right)^2+6\left(\frac{\sqrt{5}}{2}\right)^3$ (or equivalent) ***A2***

$$\frac{12\sqrt{5}}{4}+\frac{15\sqrt{5}}{4}+\frac{3\sqrt{5}}{4}+\frac{15\sqrt{5}}{4}$$

Note: Award ***A1*** for only two or three correct terms.
Award a maximum of ***(M1)(A1)A1*** on ***FT*** from c(i).

$\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}=\frac{45}{4}\sqrt{5}$ ***AG***

[7 marks]

(d) (i) $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} > 0$ OR concave up (or equivalent) ***R1***

(and $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0$, when $\alpha = \arctan 2$, hence L is a minimum)

(ii) $\left(L_{\min} =\right)\frac{3}{4}\left(\sqrt{5}\right) + 6\left(\frac{\sqrt{5}}{2}\right)$ ***(A1)***

$= \frac{15\sqrt{5}}{4}$ ***A1***

[3 marks]

(e) $\left(11.25 =\right)\frac{15\sqrt{9}}{4} > \frac{15\sqrt{5}}{4}$ (or equivalent comparative reasoning) ***R1***

the pole cannot be carried (horizontally from the passageway into the room) ***A1***

Note: Do not award ***R0A1***.

[2 marks]

Total [19 marks]

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Question 28

METHOD 1 (subtracting volumes)

radius of cylinder, R is $\sqrt{r^2-\frac{h^2}{4}}$ OR $R^2=r^2-\frac{h^2}{4}$ (seen anywhere) ***(A1)***

correct limits 0 and $\frac{h}{2}$ OR $-\frac{h}{2}$ and $\frac{h}{2}$ (seen anywhere) ***(A1)***

EITHER

volume of part sphere $=\pi\int\left(r^2-y^2\right)\mathrm{d}y$

correct integration ***A1***

$$r^2y-\frac{y^3}{3}$$

attempt to substitute their limits into their integrated expression ***(M1)***

$$\frac{r^2h}{2}-\frac{h^3}{24}$$

recognition that the volume of the ring is $\pi\int\left(r^2-y^2\right)\mathrm{d}y-\pi R^2h$ where $R\neq r$ ***(M1)***

$$\pi\int\left(r^2-y^2\right)\mathrm{d}y-\pi\left(r^2-\frac{h^2}{4}\right)h \text{ (or equivalent)}$$

correct equation ***(A1)***

$$2\pi\left(\frac{r^2h}{2}-\frac{h^3}{24}\right)-\pi r^2h+\frac{\pi h^3}{4}=\pi \quad \text{OR} \quad \frac{h^3}{4}-\frac{h^3}{12}=1 \text{ (or equivalent)}$$

OR

recognition that the volume of the ring is $\pi\int\left(\left(r^2-y^2\right)-\left(r^2-\frac{h^2}{4}\right)\right)\mathrm{d}y$ (or equivalent) ***(M1)***

correct integration ***A1***

$$\frac{h^2}{4}y-\frac{y^3}{3}$$

attempt to substitute their limits into their integrated expression ***(M1)***

$$\frac{h^3}{8}-\frac{h^3}{24}$$

correct equation ***(A1)***

$$2\pi\left(\frac{h^3}{8}-\frac{h^3}{24}\right)=\pi \quad \text{OR} \quad 2\left(\frac{h^3}{8}-\frac{h^3}{24}\right)=1 \text{ (or equivalent)}$$

THEN

$h=\sqrt[3]{6}$ ***A1***

[7 marks]

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METHOD 2 (volume of cylindrical hole)

radius of cylinder, R is $\sqrt{r^2-\frac{h^2}{4}}$ OR $R^2=r^2-\frac{h^2}{4}$ (seen anywhere) ***(A1)***

correct limits $\frac{h}{2}$ and r (seen anywhere) ***(A1)***

volume of part sphere $=\pi\int\left(r^2-y^2\right)\mathrm{d}y$

correct integration ***A1***

$r^2y-\frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{2r^3}{3}-\frac{r^2h}{2}+\frac{h^3}{24}$

recognition that the volume of the cylindrical hole is $\pi\int\left(r^2-y^2\right)\mathrm{d}y+\pi R^2h$ where $R\neq r$ ***(M1)***

$\pi\int\left(r^2-y^2\right)\mathrm{d}y+\pi\left(r^2-\frac{h^2}{4}\right)h\ \left(=\frac{4}{3}\pi r^3-\pi\right)$ (or equivalent)

correct equation ***(A1)***

$2\pi\left(\frac{2r^3}{3}-\frac{r^2h}{2}+\frac{h^3}{24}\right)+\pi r^2h-\frac{\pi h^3}{4}=\frac{4}{3}\pi r^3-\pi$ OR $\frac{h^3}{12}-\frac{h^3}{4}=-1$ (or equivalent)

$h=\sqrt[3]{6}$ ***A1***

[7 marks]

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METHOD 3 (shells)

radius of cylinder, R is $\sqrt{r^2-\frac{h^2}{4}}$ OR $R^2=r^2-\frac{h^2}{4}$ (seen anywhere) ***(A1)***

attempt to use shells method ***(M1)***

$2\pi\int x\sqrt{r^2-x^2}\,\mathrm{d}x$

correct limits *r* and $\sqrt{r^2-\frac{h^2}{4}}$ (seen anywhere) ***(A1)***

correct integration ***A1***

$-\frac{1}{3}\left(r^2-x^2\right)^{\frac{3}{2}}$

attempt to substitute their limits into their integrated expression ***(M1)***

$$-\frac{1}{3}\left(0-\left(r^2-\left(r^2-\frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right)$$

correct equation ***(A1)***

$$2\times\frac{-2\pi}{3}\left(0-\left(r^2-\left(r^2-\frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right)=\pi \quad \text{OR} \quad 2\left(\frac{2\pi}{3}\times\frac{h^3}{8}\right)=\pi$$

$h=\sqrt[3]{6}$ ***A1***

[7 marks]

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Question 29

$A=\frac{1}{2}x^2\sin\frac{\pi}{3}$ OR $A=\frac{1}{2}x^2\sin 60^\circ$ OR triangle height $h=\sqrt{x^2-\left(\frac{x}{2}\right)^2}$ $\left(=\frac{\sqrt{3}}{2}x\right)$ ***(A1)***

$=\frac{1}{2}x^2\left(\frac{\sqrt{3}}{2}\right)$ OR $A=\frac{1}{2}x\left(\frac{\sqrt{3}}{2}x\right)\left(=\frac{\sqrt{3}}{4}x^2\right)$ ***A1***

Note: Award ***A1*** for $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$. This may be seen at a later stage.

attempt to use chain rule or implicit differentiation ***(M1)***

$\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}$

$\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\sqrt{3}}{4}\times 2x\frac{\mathrm{d}x}{\mathrm{d}t}$ OR $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{1}{2}\times\sin\frac{\pi}{3}\times 2x\frac{\mathrm{d}x}{\mathrm{d}t}$ ***(A1)***

$=\frac{2\sqrt{3}}{4}\times 5\sqrt{3}\times 4$

$\frac{\mathrm{d}A}{\mathrm{d}t}=30\,(\mathrm{cm}^2\mathrm{s}^{-1})$ ***A1***

Note: Award a maximum of ***(A1)A1(M1)(A0)A1*** for a correct answer with incorrect derivative notation seen throughout.

[5 marks]

Question 30

(a) let $t = \sqrt{x}$ ***M1***

$t^2 = x \Rightarrow 2t \; \mathrm{d}t = \mathrm{d}x$ (or equivalent) ***A1***

so $\int \cos\sqrt{x} \; \mathrm{d}x = 2\int t\cos t \; \mathrm{d}t$ ***A1***

attempts integration by parts ***(M1)***

$u = 2t$, $\mathrm{d}v = \cos t \; \mathrm{d}t$, $\mathrm{d}u = 2 \; \mathrm{d}t$, $v = \sin t$

$2\int t\cos t \; \mathrm{d}t = 2t\sin t - 2\int \sin t \; \mathrm{d}t$ ***(A1)***

$= 2t\sin t + 2\cos t + C$ ***A1***

substitution of $t = \sqrt{x} \;\; \Rightarrow \int \cos\sqrt{x} \; \mathrm{d}x = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$ ***AG***

[6 marks]

(b) $x_{n+1} = \dfrac{\left(2\left(n+1\right)-1\right)^2 \pi^2}{4} \left(= \dfrac{\left(2n+1\right)^2 \pi^2}{4} \right)$ ***A1***

[1 mark]

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(c) area of R_n is $\left|\int_{x_n}^{x_{n+1}} \cos\sqrt{x}\ \mathrm{d}x\right|$ ***(M1)***

Note: Modulus may be seen at a later stage.

$$=\left|\left[2\sqrt{x}\sin\sqrt{x}+2\cos\sqrt{x}\right]_{\frac{(2n-1)^2\pi^2}{4}}^{\frac{(2n+1)^2\pi^2}{4}}\right|$$ ***A1***

Note: Condone $+C$ at this stage.

attempts to substitute their limits into their integrated expression ***(M1)***

$$=2\left|\frac{(2n+1)\pi}{2}\times\sin\frac{(2n+1)\pi}{2}+\cos\frac{(2n+1)\pi}{2}-\left(\frac{(2n-1)\pi}{2}\times\sin\frac{(2n-1)\pi}{2}+\cos\frac{(2n-1)\pi}{2}\right)\right|$$

A1

$$=2\left|(-1)^n\frac{(2n+1)\pi}{2}-\left((-1)^{n+1}\frac{(2n-1)\pi}{2}\right)\right|$$ (or equivalent) ***A1***

$$=2\left|(-1)^n\frac{(2n+1)\pi}{2}+(-1)^n\frac{(2n-1)\pi}{2}\right|$$ ***A1***

$$=2\left|(-1)^n\frac{4n\pi}{2}\right|$$

$$=4n\pi$$ ***A1***

Note: Award a maximum of ***(M1)A1M1A1A1A0A0*** for only attempting to calculate $\int_{x_n}^{x_{n+1}} \cos\sqrt{x}\ \mathrm{d}x$, and not applying the modulus.

[7 marks]

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(d) **EITHER**

attempts to find $(d =) R_{n+1} - R_n$ ***M1***

$(d =) 4(n+1)\pi - 4n\pi$

$= 4\pi$ ***A1***

Note: Award ***M0*** for consideration of special cases for example R_3 and R_2.
Accept $d = k\pi$.

which is a constant (common difference is 4π) ***R1***

OR

an arithmetic sequence is of the form $u_n = dn + c$ $(u_n = dn + u_1 - d)$ ***M1***

attempts to compare $u_n = dn + c$ $(u_n = dn + u_1 - d)$ and $R_n = 4n\pi$ ***M1***

$d = 4\pi$ and $c = 0$ $(u_1 - d = 0)$ ***A1***

Note: Accept $d = k\pi$.

THEN

so the areas of the regions form an arithmetic sequence ***AG***

[3 marks]

Total [17 marks]

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Question 31

(a) $y^2 = 9 - x^2$ OR $y = \pm\sqrt{9 - x^2}$ ***A1***

(since $y > 0$) $\Rightarrow y = \sqrt{9 - x^2}$ ***AG***

[1 mark]

(b) $b = 2y \left(= 2\sqrt{9 - x^2}\right)$ or $h = x + 3$ ***(A1)***

attempts to substitute their base expression and height expression into $A = \frac{1}{2}bh$ ***(M1)***

$A = \sqrt{9 - x^2}(x+3)$ (or equivalent) $\left(= \frac{2(x+3)\sqrt{9-x^2}}{2} = x\sqrt{9-x^2} + 3\sqrt{9-x^2}\right)$ ***A1***

[3 marks]

(c) **METHOD 1**

attempts to use the product rule to find $\frac{\mathrm{d}A}{\mathrm{d}x}$ ***(M1)***

attempts to use the chain rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{9-x^2}$ ***(M1)***

$\left(\frac{\mathrm{d}A}{\mathrm{d}x} =\right) \sqrt{9-x^2} + (3+x)\left(\frac{1}{2}\right)\left(9-x^2\right)^{-\frac{1}{2}}(-2x) \left(= \sqrt{9-x^2} - \frac{x^2+3x}{\sqrt{9-x^2}}\right)$ ***A1***

$\left(\frac{\mathrm{d}A}{\mathrm{d}x} =\right) \frac{9-x^2}{\sqrt{9-x^2}} - \frac{x^2+3x}{\sqrt{9-x^2}} \left(= \frac{9-x^2-\left(x^2+3x\right)}{\sqrt{9-x^2}}\right)$ ***A1***

$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{9-3x-2x^2}{\sqrt{9-x^2}}$ ***AG***

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METHOD 2

$\frac{dA}{dx} = \frac{dA}{dy} \times \frac{dy}{dx}$

attempts to find $\frac{dA}{dy}$ where $A = y(x+3)$ and $\frac{dy}{dx}$ where $y^2 = 9 - x^2$ ***(M1)***

$\frac{dA}{dy} = y\frac{dx}{dy} + x + 3$ and $\frac{dy}{dx} = -\frac{x}{y}$ (or equivalent) ***A1***

substitutes their $\frac{dA}{dy}$ and their $\frac{dy}{dx}$ into $\frac{dA}{dx} = \frac{dA}{dy} \times \frac{dy}{dx}$ ***(M1)***

$\frac{dA}{dx} = \left(y\left(-\frac{y}{x}\right) + x + 3 \right)\left(-\frac{x}{y}\right)$ (or equivalent)

$= \frac{9 - x^2 - x^2 - 3x}{\sqrt{9 - x^2}}$ (or equivalent) ***A1***

$\frac{dA}{dx} = \frac{9 - 3x - 2x^2}{\sqrt{9 - x^2}}$ ***AG***

[4 marks]

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(d) $\frac{\mathrm{d}A}{\mathrm{d}x}=0\left(\frac{9-3x-2x^2}{\sqrt{9-x^2}}=0\right)$ ***(M1)***

attempts to solve $9-3x-2x^2=0$ (or equivalent) ***(M1)***

$-(2x-3)(x+3)(=0)$ OR $x=\frac{3\pm\sqrt{(-3)^2-4(-2)(9)}}{2(-2)}$ (or equivalent) ***(A1)***

$x=\frac{3}{2}$ ***A1***

Note: Award the above ***A1*** if $x=-3$ is also given.

substitutes their value of x into either $y=\sqrt{9-x^2}$ or $y=-\sqrt{9-x^2}$ ***(M1)***

Note: Do not award the above ***(M1)*** if $x\le 0$.

$y=-\sqrt{9-\left(\frac{3}{2}\right)^2}$

$=-\frac{\sqrt{27}}{2}\left(=-\frac{3\sqrt{3}}{2},=-\sqrt{\frac{27}{4}},=-\sqrt{6.75}\right)$ ***A1***

[6 marks]

Total [14 marks]

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Question 32

$$A=\int_0^c \frac{x}{x^2+2}\,\mathrm{d}x$$

EITHER

attempts to integrate by inspection or substitution using $u=x^2+2$ or $u=x^2$ ***(M1)***

Note: If candidate simply states $u=x^2+2$ or $u=x^2$, but does not attempt to integrate, do not award the ***(M1)***.

Note: If candidate does not explicitly state the u-substitution, award the ***(M1)*** only for expressions of the form $k\ln u$ or $k\ln(u+2)$.

$\left[\frac{1}{2}\ln u\right]_2^{c^2+2}$ OR $\left[\frac{1}{2}\ln(u+2)\right]_0^{c^2}$ OR $\left[\frac{1}{2}\ln(x^2+2)\right]_0^{c}$ ***A1***

Note: Limits may be seen in the substitution step.

OR

attempts to integrate by inspection ***(M1)***

Note: Award the ***(M1)*** only for expressions of the form $k\ln(x^2+2)$.

$\left[\frac{1}{2}\ln(x^2+2)\right]_0^{c}$ ***A1***

Note: Limits may be seen in the substitution step.

THEN

correctly substitutes their limits into their integrated expression ***(M1)***

$\frac{1}{2}\left(\ln(c^2+2)-\ln 2\right)\ (=\ln 3)$ OR $\frac{1}{2}\ln(c^2+2)-\frac{1}{2}\ln 2(=\ln 3)$

correctly applies at least one log law to their expression **(M1)**

$\frac{1}{2}\ln\left(\frac{c^2+2}{2}\right)(=\ln 3)$ OR $\ln\sqrt{c^2+2}-\ln\sqrt{2}(=\ln 3)$ OR $\ln\left(\frac{c^2+2}{2}\right)=\ln 9$

OR $\ln(c^2+2)-\ln 2=\ln 9$ OR $\ln\sqrt{\frac{c^2+2}{2}}(=\ln 3)$ OR $\ln\frac{\sqrt{c^2+2}}{\sqrt{2}}(=\ln 3)$

Note: Condone the absence of $\ln 3$ up to this stage.

$\frac{c^2+2}{2}=9$ OR $\sqrt{\frac{c^2+2}{2}}=3$ **A1**

$c^2=16$

$c=4$ **A1**

Note: Award ***A0*** for $c=\pm4$ as a final answer.

Total [6 marks]

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Question 33

(a) attempt to use chain rule to find $f'(x)$ ***(M1)***

$f'(x)=(-2\sin 2x)\mathrm{e}^{\cos 2x}(=0)$ ***A1***

$\Rightarrow \sin 2x=0$ ***(M1)***

$2x=0, \pi, 2\pi, \ldots$

$x=0, \frac{\pi}{2}, \pi, \ldots$ ***A1***

Coordinates are $(0, \mathrm{e}), \left(\frac{\pi}{2}, \frac{1}{\mathrm{e}}\right), (\pi, \mathrm{e})$ ***A1***

Note: Special case: For two correct coordinate pairs award ***(M1)A1(M1)A0A1.***

For extra coordinate pairs award ***(M1)A1(M1)A1A0.***

[5 marks]

(b) attempt to differentiate $f'(x)$ using product rule ***(M1)***

$f''(x)=(-2\sin 2x)(-2\sin 2x)\mathrm{e}^{\cos 2x}-(4\cos 2x)\mathrm{e}^{\cos 2x}$ ***A1***

at $x=0$, $f''(x)=-4\mathrm{e}<0$ so maximum **AND**

at $x=\pi$, $f''(x)=-4\mathrm{e}<0$ so maximum ***R1***

at $x=\frac{\pi}{2}$, $f''(x)=\frac{4}{\mathrm{e}}>0$ so minimum ***R1***

Note: The values for the second derivative must be correct in order to award the ***R*** marks.

[4 marks]

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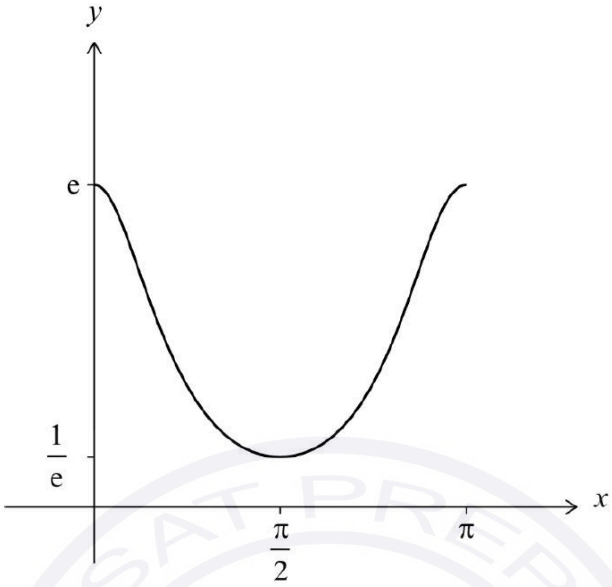
(c)



A1A1A1

Note: Award ***A1*** for general shape, ***A1*** for correct maxima $(0, e), (\pi, e)$ and minimum point $\left(\frac{\pi}{2}, \frac{1}{e}\right)$ and ***A1*** for showing a higher rate of change of gradient at maxima and a lower rate of change of gradient at the minimum point.

[3 marks]

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(d) (i) $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots$ ***(M1)***

$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} \cdots$ ***A1***

(ii) **METHOD 1**

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$

attempt to substitute series for $\cos 2x - 1$ into series for e^x ***(M1)***

Note: Award ***(M0)*** for substituting the Maclaurin series for $\cos 2x$ into the Maclaurin series for e^x.

$e^{\cos 2x - 1} = 1 + \left(-2x^2 + \frac{2x^4}{3}\right) + \frac{\left(-2x^2 + \frac{2x^4}{3}\right)^2}{2} + \ldots$ ***A1***

$\left(= 1 - 2x^2 + \frac{2x^4}{3} + 2x^4 + \ldots\right)$

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

METHOD 2

$e^{\cos 2x - 1} = e^{-2x^2 + \frac{2x^4}{3}} = e^{-2x^2} e^{\frac{2x^4}{3}}$

attempt to find the Maclaurin series for e^{-2x^2} OR $e^{\frac{2x^4}{3}}$ ***(M1)***

$e^{-2x^2} = 1 - 2x^2 + 2x^4 + \ldots$; $e^{\frac{2x^4}{3}} = 1 + \frac{2}{3}x^4 + \ldots$

$e^{-2x^2} e^{\frac{2}{3}x^4} = \left(1 - 2x^2 + 2x^4 + \ldots\right)\left(1 + \frac{2}{3}x^4 + \ldots\right)$ ***A1***

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

(iii) $\left(f(x)\approx\right)\mathrm{e}\left[1-2x^2+\frac{8x^4}{3}+\ldots\right]\left(=\mathrm{e}-2\mathrm{e}x^2+\frac{8\mathrm{e}x^4}{3}+\ldots\right)$ ***A1***

[6 marks]

(e) $\int_0^{\frac{1}{10}}\mathrm{e}^{\cos 2x}\mathrm{d}x\approx\mathrm{e}\int_0^{\frac{1}{10}}\left(1-2x^2\right)\mathrm{d}x$ ***(M1)***

$=\mathrm{e}\left[x-\frac{2x^3}{3}\right]_0^{1/10}$ ***A1***

$=\mathrm{e}\left(\frac{1}{10}-\frac{2}{3000}\right)$ ***A1***

$=\frac{298\mathrm{e}}{3000}$

$=\frac{149\mathrm{e}}{1500}$ ***AG***

Note: If candidate follows through an incorrect expansion from part (d), award a maximum of ***M1A1FTA0***

[3 marks]

Total [21 marks]

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Question 34

(a) **METHOD 1**

attempt to integrate by parts *(M1)*

$u = (\ln x)^2$, $\mathrm{d}v = x\mathrm{d}x$ *(M1)*

$\int x(\ln x)^2 \ \mathrm{d}x = \frac{x^2(\ln x)^2}{2} - \int x \ln x \ \mathrm{d}x$ ***A1***

attempt to integrate $x\ln x$ by parts, with $u = \ln x$ *(M1)*

$\int x \ln x \ \mathrm{d}x = \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ \mathrm{d}x\right]$ ***A1***

$\int x(\ln x)^2 \ \mathrm{d}x = \frac{x^2(\ln x)^2}{2} - \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ \mathrm{d}x\right]$

$= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

METHOD 2 (knowing $\int \ln x \ \mathrm{d}x = x\ln x - x$ **)**

attempt to integrate by parts *(M1)*

$u = x\ln x$, $\mathrm{d}v = \ln x \mathrm{d}x$ *(M1)*

$\int x \ln x(\ln x) \ \mathrm{d}x = x\ln x(x\ln x - x) - \int (\ln x + 1)(x\ln x - x)\mathrm{d}x$ ***A1***

$= x\ln x(x\ln x - x) - \int x(\ln x)^2 \mathrm{d}x + \int x\mathrm{d}x$ ***A1***

$2I = x\ln x(x\ln x - x) + \frac{x^2}{2} + c$ ***M1***

$I = \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

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METHOD 3 (knowing $\int x \ln x \, \mathrm{d}x$ **)**

$$\int x \ln x \, \mathrm{d}x = \frac{x^2(\ln x)}{2} - \frac{x^2}{4}$$

attempt to integrate by parts *(M1)*

$u = \ln x$, $\mathrm{d}v = x \ln x \mathrm{d}x$ *(M1)*

$$\int (x \ln x) \ln x \, \mathrm{d}x = \ln x \left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4} \right) - \int \frac{1}{x} \left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4} \right) \mathrm{d}x$$ ***A1***

$$= \ln x \left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4} \right) - \int \left(\frac{x(\ln x)}{2} - \frac{x}{4} \right) \mathrm{d}x$$ ***A1***

$$= \ln x \left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4} \right) - \frac{x^2 \ln x}{4} + \frac{x^2}{8} + \frac{x^2}{8}$$ ***A1***

$$= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$$ ***A1***

[6 marks]

(b) attempt to substitute limits into their integrated expression *(M1)*

$$\left[\frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} \right]_1^4 = \left(8(\ln 4)^2 - 8 \ln 4 + 4 \right) - \left(\frac{1}{4} \right)$$

attempt to replace any $\ln 4$ term with $2 \ln 2$ *(M1)*

$$= 8(2 \ln 2)^2 - 8(2 \ln 2) + 4 - \frac{1}{4}$$ ***A1***

$$= 32(\ln 2)^2 - 16 \ln 2 + \frac{15}{4}$$ ***AG***

[3 marks]

Total [9 marks]

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Question 35

(a) $n=1: LHS = f^{(1)}(x) = -\frac{1}{2}\times -a(1-ax)^{-\frac{3}{2}}\left(=\frac{a}{2}(1-ax)^{-\frac{3}{2}}\right)$ ***A1***

$RHS = \frac{a(1)!(1-ax)^{-\frac{3}{2}}}{2^1(0)!}$ therefore true for $n=1$ ***R1***

assume (that the result is) true for $n=k$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n=k$" or "assume that $n=k$ is true". The assumption of truth must be clear.

$$f^{(k)}(x) = \frac{a^k(2k-1)!(1-ax)^{-\frac{(2k+1)}{2}}}{2^{2k-1}(k-1)!}$$

attempt to differentiate the right-hand side with respect to x: ***M1***

$$f^{(k+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(f^{(k)}(x)\right)$$

$$= \frac{-(2k+1)\times -a}{2}\times\frac{a^k(2k-1)!(1-ax)^{-\frac{(2k+1)}{2}-1}}{2^{2k-1}(k-1)!}$$ (or equivalent) ***A1***

attempt to multiply top and bottom by $2k$ ***M1***

$$= \frac{(2k+1)}{2}\times\frac{a^{k+1}(2k-1)!(1-ax)^{-\frac{2k+3}{2}}}{2^{2k-1}(k-1)!}\times\frac{2k}{2k}$$

$$= \frac{a^{k+1}(2k+1)!(1-ax)^{-\frac{2k+3}{2}}}{2^{2k+1}(k)!}$$ ***A1***

hence if the result is true for $n=k$ then it is true for $n=k+1$ and as it is true for $n=1$ it is true for all $n\in\mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, at least five of the previous marks must have been awarded.

[8 marks]

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(b) $f(x)=f(0)+xf'(0)+\frac{x^2}{2}f''(0)+\dots$

$f''(x)=\frac{a^2(3)!(1-ax)^{-\frac{5}{2}}}{2^3(1)!}$ OR $\frac{3}{4}a^2(1-ax)^{-\frac{5}{2}}$ OR $f''(0)=\frac{a^2(3)!}{2^3}$ ***A1***

$f(0)=1,\ f'(0)=\frac{a}{2},\ f''(0)=\frac{6a^2}{8}$ ***A1***

$f(x)=1+\frac{1}{2}ax+\frac{3}{8}a^2x^2$ + …. ***AG***

[2 marks]

(c) attempt to use $a=2$ or $a=4$ in the expansion ***M1***

$(1-2x)^{-\frac{1}{2}}=1+\frac{2x}{2}+\frac{3\times 4x^2}{8}\left(=1+x+\frac{3}{2}x^2+\dots\right)$

$(1-4x)^{-\frac{1}{2}}=1+\frac{4x}{2}+\frac{3\times 16x^2}{8}\left(=1+2x+6x^2+\dots\right)$ ***A1***

Note: Award ***A1*** for at least one correct.

attempt to multiply their two expansions together ***M1***

$\left(1+x+\frac{3}{2}x^2+\dots\right)\left(1+2x+6x^2+\dots\right)=1+2x+6x^2+x+2x^2+\frac{3}{2}x^2+\dots$

$=1+3x+\frac{19}{2}x^2+\dots$ OR $\frac{2+4x+12x^2+2x+4x^2+3x^2+\dots}{2}$ ***A1***

$(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}}\approx\frac{2+6x+19x^2}{2}$ ***AG***

[4 marks]

(d) $|x|<\frac{1}{4}$ ***A1***

[1 mark]

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(e)

$\left(\frac{2+6x+19x^2}{2}\right)=\frac{2+0.6+0.19}{2}\left(=\frac{279}{200}\right)$ (or equivalent) ***A1***

attempt to substitute $x=\frac{1}{10}$ into $(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}}$ ***(M1)***

$\left((1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}}=\right)\frac{10}{\sqrt{48}}$ (or equivalent) ***A1***

$\frac{10}{4\sqrt{3}}\approx\frac{279}{200}$ (or equivalent in terms of $\sqrt{3}$) ***A1***

$\frac{1}{\sqrt{3}}\approx\frac{279}{500}$

$\sqrt{3}\approx\frac{500}{279}$ ***A1***

[5 marks]

Total [20 marks]

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Question 36

Note: To award full marks limit notation $\lim\limits_{x\to 0}$ must be seen at least once in their working. If no limit notation is seen but otherwise all correct, do not award the final ***A1.***

$$\lim_{x\to 0}\frac{4\sec^4 x\tan x + 2\sin x\cos x}{4x^3 - 2x}$$ ***A1A1***

Note: Award ***A1*** for numerator and ***A1*** for denominator.

$$=\lim_{x\to 0}\frac{16\sec^4 x\tan^2 x + 4\sec^6 x - 2\sin^2 x + 2\cos^2 x}{12x^2 - 2}$$ ***M1A1A1***

Note: Award ***M1 for second use of*** l'Hôpital's rule providing their expression is in indeterminate form as $x\to 0$ and providing there is **no third attempt** at using l'Hôpital's Rule.

$= -3$ ***A1***

[6 marks]

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Question 37

(a) attempt to substitute -1 into $P(x)$ OR use of synthetic division OR long division **M1**

$3(-1)^3+5(-1)^2+(-1)-1=0$ OR

	3	5	1	-1
-1		-3	-2	1
	3	2	-1	0

OR

$$\begin{array}{r}
3x^2+2x-1 \\
x+1\overline{)3x^3+5x^2+x-1} \\
3x^3+3x^2 \quad\quad\quad \\
\hline
2x^2+x \quad\; \\
2x^2+2x \quad\; \\
\hline
-x-1 \\
-x-1 \\
\hline
0
\end{array}$$

A1

[2 marks]

(b) attempt to divide $P(x)$ by $(x+1)$ e.g. using long division or synthetic division **(M1)**

$P(x)=(x+1)(3x^2+2x-1)$ **(A1)**

$=(x+1)(x+1)(3x-1)\left(=(x+1)^2(3x-1)\right)$ **A1**

[3 marks]

(c) $\dfrac{1}{(x+1)(2x+1)}\equiv\dfrac{A}{x+1}+\dfrac{B}{2x+1}\Rightarrow 1\equiv A(2x+1)+B(x+1)$

attempt to equate both coefficients OR substitute two values eg -1 and $-\dfrac{1}{2}$ **(M1)**

$2A+B=0$ and $A+B=1$ OR $1=-A$ and $1=\dfrac{1}{2}B$

$A=-1$ and $B=2$ **A1A1**

Note: Award ***A1*** for each value.

$\dfrac{1}{(x+1)(2x+1)}=-\dfrac{1}{x+1}+\dfrac{2}{2x+1}$

[3 marks]

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(d) $\dfrac{1}{(x+1)(x+1)(2x+1)}$

$= \dfrac{1}{(x+1)}\left(-\dfrac{1}{x+1}+\dfrac{2}{2x+1}\right)$ ***(A1)***

$= -\dfrac{1}{(x+1)^2}+\dfrac{2}{(2x+1)(x+1)}\left(=-\dfrac{1}{(x+1)^2}+2\left(-\dfrac{1}{x+1}+\dfrac{2}{2x+1}\right)\right)$ ***A1***

$= \dfrac{4}{2x+1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}$ ***AG***

Note: Award ***A1A0*** for follow through from incorrect values in part (c).

[2 marks]

(e) attempt to integrate at least one term in $\left(\dfrac{4}{2x+1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}\right)$ ***(M1)***

$\int\left(\dfrac{4}{2x+1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}\right)\mathrm{d}x$

$= 2\ln|2x+1|-2\ln|x+1|+\dfrac{1}{x+1}(+c)$ ***A1A1A1***

Note: Award ***A1*** for each correct term.
Award a maximum of ***M1A1A0A1*** if modulus signs are omitted.
Condone the absence of $+c$.

[4 marks]

(f) (i) **METHOD 1**

attempt to cancel factors and substitute $x=-1$ ***(M1)***

$\lim\limits_{x\to-1} f(x)=\lim\limits_{x\to-1}\left(\dfrac{(x+1)^2(3x-1)}{(x+1)^2(2x+1)}\right)=\lim\limits_{x\to-1}\left(\dfrac{3x-1}{2x+1}\right)=\dfrac{3(-1)-1}{2(-1)+1}$

$=4$ ***A1***

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METHOD 2

attempt to expand denominator, differentiate numerator and denominator twice and substitute $x=-1$ ***(M1)***

$$\lim_{x\to-1} f(x)=\lim_{x\to-1}\left(\frac{3x^3+5x^2+x-1}{2x^3+5x^2+4x+1}\right)=\lim_{x\to-1}\left(\frac{9x^2+10x+1}{6x^2+10x+4}\right)=\lim_{x\to-1}\left(\frac{18x+10}{12x+10}\right)=\frac{18(-1)+10}{12(-1)+10}$$

$=4$ ***A1***

(ii) **METHOD 1**

attempt to consider coefficients of x^3 or divide all terms by x^3

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left(\frac{3x^3+\ldots}{2x^3+\ldots}\right) \text{ or } \lim_{x\to\infty}\left(\frac{3+\text{terms which tend to } 0}{2+\text{terms which tend to } 0}\right)$$

$=\frac{3}{2}$ ***A1***

METHOD 2

attempt to cancel factors and consider coefficients of x or divide all terms by x

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left(\frac{(x+1)^2(3x-1)}{(x+1)^2(2x+1)}\right)=\lim_{x\to\infty}\left(\frac{3x-1}{2x+1}\right) \text{ or } \lim_{x\to\infty}\left(\frac{3-\frac{1}{x}}{2+\frac{1}{x}}\right)$$

$=\frac{3}{2}$ ***A1***

METHOD 3

attempt to expand denominator, differentiate numerator and denominator three times

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left(\frac{3x^3+5x^2+x-1}{2x^3+5x^2+4x+1}\right)=\lim_{x\to\infty}\left(\frac{9x^2+10x+1}{6x^2+10x+4}\right)=\lim_{x\to\infty}\left(\frac{18x+10}{12x+10}\right)=\lim_{x\to\infty}\left(\frac{18}{12}\right)$$

$=\frac{3}{2}$ ***A1***

Note: If the ***M1*** has not been awarded in part (i) it can be awarded in part (ii).

[3 marks]

Total [17 marks]

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Question 38

(a) (i) $\sin\left(x^2\right) = x^2 - \frac{\left(x^2\right)^3}{3!} + \ldots$

$\sin\left(x^2\right) = x^2 - \frac{x^6}{3!} + \ldots \left(= x^2 - \frac{x^6}{6} + \ldots \right)$ ***A1A1***

Note: Award ***A1*** for each term.

(ii) **METHOD 1**

attempt to square their series for $\sin\left(x^2\right)$ ***(M1)***

$\left(\sin\left(x^2\right)\right)^2 = \left(x^2 - \frac{x^6}{3!} + \ldots\right)^2$

Note: Award ***M0*** for $\left(x^2\right)^2 - \left(\frac{x^6}{3!}\right)^2 + \ldots.$

$= x^4 - \frac{2x^8}{3!} + \ldots \left(= x^4 - \frac{x^8}{3} + \ldots \right)$ ***A1A1***

Note: Award ***A1*** for each term.

METHOD 2

attempt to use the identity $\sin^2\left(x^2\right) = \frac{1-\cos\left(2x^2\right)}{2}$ ***(M1)***

$\sin^2\left(x^2\right) = \frac{1}{2}\left(1 - \left(1 - \frac{\left(2x^2\right)^2}{2!} + \frac{\left(2x^2\right)^4}{4!}\right)\right)$

$= x^4 - \frac{8x^8}{4!} + \ldots \left(= x^4 - \frac{x^8}{3} + \ldots \right)$ ***A1A1***

Note: Award ***A1*** for each term.

[5 marks]

(b) **METHOD 1**

recognition that $4x\sin\left(x^2\right)\cos\left(x^2\right)=\frac{\mathrm{d}\left(\left(\sin\left(x^2\right)\right)^2\right)}{\mathrm{d}x}$ ***(M1)***

$=4x^3-\frac{8x^7}{3}+\ldots$ ***A1***

METHOD 2

recognition that $4x\sin\left(x^2\right)\cos\left(x^2\right)=2x\sin\left(2x^2\right)$ ***(M1)***

$$=2x\left(2x^2-\frac{\left(2x^2\right)^3}{3!}+\ldots\right)$$

$=4x^3-\frac{8x^7}{3}+\ldots$ ***A1***

METHOD 3

$$4x\sin\left(x^2\right)\cos\left(x^2\right)$$

$=4x\left(x^2-\frac{x^6}{3!}+\ldots\right)\left(1-\frac{x^4}{2!}+\ldots\right)$ ***(A1)***

$=4x^3-\frac{8x^7}{3}+\ldots$ ***A1***

continued…

Question 8 continued

METHOD 4

recognition that $2x\cos\left(x^2\right)=\frac{\mathrm{d}\left(\sin\left(x^2\right)\right)}{\mathrm{d}x}$ ***(M1)***

$$4x\sin\left(x^2\right)\cos\left(x^2\right)$$

$$=2\left(x^2-\frac{x^6}{3!}+\ldots\right)\left(2x-\frac{6x^5}{2!}+\ldots\right)$$

$=4x^3-\frac{8x^7}{3}+\ldots$ ***A1***

[2 marks]

Total [7 marks]

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Question 39

METHOD 1

attempt to find an integral involving π and the square of $f(x)$ ***M1***

lote: Condone incorrect or absent limits for this ***M1.***

$$\pi\int_0^{\frac{\sqrt{\pi}}{2}}\left(f(x)\right)^2\mathrm{d}x$$

$$\pi\int_0^{\frac{\sqrt{\pi}}{2}}x\sin\left(x^2\right)\mathrm{d}x$$ ***A1***

EITHER

attempt to use integration by substitution ***M1***

$$\frac{\pi}{2}\int_0^{\frac{\pi}{4}}\sin(u)\mathrm{d}u$$

lote: Award ***M1*** for $u=x^2\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=2x$

$$=\left[-\frac{\pi}{2}\cos(u)\right]_0^{\frac{\pi}{4}}$$ ***A1***

OR

attempt to integrate by inspection ***(M1)***

$\frac{\pi}{2}\int_0^{\frac{\sqrt{\pi}}{2}}2x\sin\left(x^2\right)\mathrm{d}x$ OR $\frac{\pi}{2}\int_0^{\frac{\sqrt{\pi}}{2}}\sin\left(x^2\right)\mathrm{d}\left(x^2\right)$

$$=\left[-\frac{\pi}{2}\cos\left(x^2\right)\right]_0^{\frac{\sqrt{\pi}}{2}}$$ ***A1***

lote: Condone incorrect or absent limits for ***M1***.

The correct limits may be seen or implied by later work for the ***A1***.

THEN

$=\left(-\frac{\pi}{2}\cos\left(\frac{\pi}{4}\right)\right)-\left(-\frac{\pi}{2}\cos(0)\right)$ (or equivalent) ***(A1)***

$=-\frac{\pi}{2\sqrt{2}}+\frac{\pi}{2}$ OR $-\frac{\pi\sqrt{2}}{4}+\frac{\pi}{2}$ OR $\frac{\pi}{2}\left(-\frac{1}{\sqrt{2}}+1\right)$ OR $\frac{\pi}{2}\left(-\frac{\sqrt{2}}{2}+1\right)$ ***A1***

$$=\frac{\pi\left(2-\sqrt{2}\right)}{4}$$ ***AG***

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attempt to find an integral involving π and the square of $f\left(x\right)$ **_M1_**

Note: Condone incorrect or absent limits for this **_M1_**.

$$\pi\int_{0}^{\frac{\sqrt{\pi}}{2}}\left(f\left(x\right)\right)^{2}\mathrm{d}x$$

$$\pi\int_{0}^{\frac{\sqrt{\pi}}{2}}x\sin\left(x^{2}\right)\mathrm{d}x$$ **_A1_**

attempt to use integration by substitution **_M1_**

$$u=\cos\left(x^{2}\right)\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=-2x\sin\left(x^{2}\right)$$

Note: Award **_M1_** for $u=\cos\left(x^{2}\right)$

$$=-\frac{\pi}{2}\int_{-\frac{1}{\sqrt{2}}}^{-1}\mathrm{d}u$$

$$=\left[-\frac{\pi}{2}u\right]_{-\frac{1}{\sqrt{2}}}^{-1}$$ (or equivalent) **_A1A1_**

Note: Condone incorrect or absent limits for **_M1_**.

_A1_ for $-\frac{\pi}{2}u$ and **_A1_** for both correct limits.

$=\frac{\pi}{2}-\frac{\pi}{2\sqrt{2}}$ OR $\frac{\pi}{2}-\frac{\pi\sqrt{2}}{4}$ OR $\frac{\pi}{2}\left(1-\frac{1}{\sqrt{2}}\right)$ OR $\frac{\pi}{2}\left(1-\frac{\sqrt{2}}{2}\right)$ **_A1_**

$$=\frac{\pi\left(2-\sqrt{2}\right)}{4}$$ **_AG_**

Total [6 marks]

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Question 40

(a) recognising $\cos x = 2\sin x\cos x$ ***(M1)***

$(\cos x \neq 0)$ so $\sin x = \frac{1}{2}$ OR one correct value (accept degrees) ***(A1)***

x - coordinates $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ ***A1***

Note: Award ***(M1)(A1)A0*** for solutions of 30° and 150°.

[3 marks]

(b) **METHOD 1**

attempt to integrate $\pm(\cos x - \sin 2x)$ ***(M1)***

$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (\cos x - \sin 2x)\,dx$ OR $\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (\cos x - 2\sin x\cos x)\,dx$

$= \left[\sin x + \frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ OR $= \left[\sin x - \sin^2 x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ ***A1***

Note: Award ***A1*** for ±correct integration. Condone incorrect or absent limits up to this point.

---

attempt to substitute their limits into their integral and subtract ***M1***

$= \left(\sin\left(\frac{5\pi}{6}\right) + \frac{1}{2}\cos\left(\frac{5\pi}{3}\right)\right) - \left(\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\cos(\pi)\right)$ OR

$\left(\sin\left(\frac{5\pi}{6}\right) - \sin^2\left(\frac{5\pi}{6}\right)\right) - \left(\sin\left(\frac{\pi}{2}\right) - \sin^2\left(\frac{\pi}{2}\right)\right)$

$= \left(\frac{1}{2} + \frac{1}{4}\right) - \left(1 - \frac{1}{2}\right)$ OR $= \left(\frac{1}{2} - \frac{1}{4}\right) - (1 - 1)$

area $= \frac{1}{4}$ ***A1***

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METHOD 2

$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos x \mathrm{d}x = [\sin x]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ and $\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin 2x \mathrm{d}x = \left[-\frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ ***A1***

Note: Award ***A1*** for correct integration. Condone incorrect or absent limits up to this point.

attempt to substitute their limits into their integral and subtract (for both integrals) ***M1***

$\sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{2}\right)$ and $-\frac{1}{2}\cos\left(\frac{5\pi}{3}\right) + \frac{1}{2}\cos(\pi)$

attempt to subtract the two integrals in either order (seen anywhere) ***(M1)***

$\left(\sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{2}\right)\right) - \left(-\frac{1}{2}\cos\left(\frac{5\pi}{3}\right) + \frac{1}{2}\cos(\pi)\right)$ OR $\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos x \mathrm{d}x - \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin 2x \mathrm{d}x$

$= \left(\frac{1}{2} - 1\right) - \left(\frac{1}{4} - \frac{1}{2}\right) \quad \left(= -\frac{1}{4}\right)$

area $= \frac{1}{4}$ ***A1***

Note: Award all corresponding marks as appropriate for finding the area between A and B.

Accept work done in degrees.

[4 marks]

Total [7 marks]

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Question 41

(a) **EITHER**

attempt to use l'Hôpital's *(M1)*

$$\lim_{x\to\infty}\left(\frac{e^{2x}-1}{e^{2x}+1}\right)=\lim_{x\to\infty}\frac{2e^{2x}}{2e^{2x}} \qquad \textbf{\textit{A1}}$$

OR

attempt to divide each term by e^{2x} *(M1)*

$$\lim_{x\to\infty}\left(\frac{e^{2x}-1}{e^{2x}+1}\right)=\lim_{x\to\infty}\left(\frac{1-e^{-2x}}{1+e^{-2x}}\right) \qquad \textbf{\textit{A1}}$$

THEN

$$=1 \qquad \textbf{\textit{AG}}$$

[2 marks]

(b) (i) attempt to use quotient rule or product rule ***M1***

$$\frac{dy}{dx}=\frac{\left(e^{2x}+1\right)2e^{2x}-\left(e^{2x}-1\right)2e^{2x}}{\left(e^{2x}+1\right)^2} \qquad \textbf{\textit{A1A1}}$$

(ii) attempt to substitute for y and express as a single fraction ***M1***

$$1-y^2=1-\frac{\left(e^{2x}-1\right)^2}{\left(e^{2x}+1\right)^2}$$

$$=\frac{\left(e^{2x}+1\right)^2-\left(e^{2x}-1\right)^2}{\left(e^{2x}+1\right)^2} \text{ or equivalent} \qquad \textit{(A1)}$$

$$=\frac{e^{4x}+2e^{2x}+1-\left(e^{4x}-2e^{2x}+1\right)}{\left(e^{2x}+1\right)^2} \text{ OR } \frac{2e^{2x}\times 2}{\left(e^{2x}+1\right)^2} \qquad \textbf{\textit{A1}}$$

$$=\frac{4e^{2x}}{\left(e^{2x}+1\right)^2} \qquad \textbf{\textit{AG}}$$

$$=\frac{4e^{2x}}{\left(e^{2x}+1\right)^2} \qquad \textbf{\textit{AG}}$$

[6 marks]

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(c) (i) attempt to use implicit differentiation **M1**

$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\right)-2y\frac{\mathrm{d}y}{\mathrm{d}x}$ **A1**

$=-2y\left(1-y^2\right)$ **A1**

$=2y^3-2y$ **AG**

(ii) $\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\right)\left(6y^2-2\right)\frac{\mathrm{d}y}{\mathrm{d}x}$ **A1**

$=\left(6y^2-2\right)\left(1-y^2\right)\left(=8y^2-6y^4-2\right)$ **A1**

[5 marks]

(d) attempt to evaluate at least three of y, y', y'', y''' at $x=0$ ***(M1)***

$y(0)=0$, $y'(0)=1$, $y''(0)=0$, $y'''(0)=-2$ **A1**

attempt to use Maclaurin series $y \approx y(0)+y'(0)x+\frac{y''(0)}{2!}x^2+\frac{y'''(0)}{3!}x^3+\ldots$

M1

$y \approx 0+1x+0x^2+\frac{(-2)}{3!}x^3+\ldots$

$y \approx x-\frac{x^3}{3}+\ldots$ **A1**

[4 marks]

Total [17 marks]

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Question 42

(a) outer curved surface area is $2\pi(4r)h$ AND inner curved surface area is $2\pi rh$

(A1)

area of each base (top and bottom) is $\pi(4r)^2 - \pi r^2$ ***(A1)***

$S = 2\left[\pi(4r)^2 - \pi r^2\right] + 2\pi(4r)h + 2\pi rh$ ***A1***

$= 30\pi r^2 + 10\pi rh$ ***AG***

[3 marks]

(b) $30\pi r^2 + 10\pi rh = 240\pi$

attempt to solve their equation for h or rh in terms of r (must isolate h or rh)

(M1)

$h = \dfrac{240 - 30r^2}{10r}\left(= \dfrac{24 - 3r^2}{r}\right)$ OR $rh = \dfrac{240 - 30r^2}{10}\left(= 24 - 3r^2\right)$ (or equivalent)

A1

uses volume = large cylinder – small cylinder ***(M1)***

$V = \pi(4r)^2 h - \pi r^2 h \quad \left(= 16\pi r^2 h - \pi r^2 h = 15\pi r^2 h\right)$ ***A1***

attempt to substitute in for h or rh ***(M1)***

$V = 15\pi r^2\left(\dfrac{24 - 3r^2}{r}\right)$ OR $V = 15\pi r\left(\dfrac{240 - 30r^2}{10}\right)\left(= 15\pi r\left(24 - 3r^2\right)\right)$ OR

$384\pi r - 48\pi r^3 - 24\pi r + 3\pi r^3$ ***A1***

$= 360\pi r - 45\pi r^3$ ***AG***

[6 marks]

(c) $\dfrac{\mathrm{d}V}{\mathrm{d}r} = 360\pi - 135\pi r^2$ ***A1A1***

[2 marks]

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(d) **METHOD 1** (working with r)

recognition that (for a maximum) $\frac{\mathrm{d}V}{\mathrm{d}r}=0$ ***M1***

$360\pi-135\pi r^2=0$

$r^2=\frac{360}{135}\left(=\frac{8}{3}\right)$

$r=\sqrt{\frac{360}{135}}\left(=\sqrt{\frac{8}{3}}\right)$ ***A1***

$p=2$ OR $r=2\sqrt{\frac{2}{3}}$ ***A1***

METHOD 2 (working with $p\sqrt{\frac{2}{3}}$)

recognition that (for a maximum) $\frac{\mathrm{d}V}{\mathrm{d}r}=0$ ***M1***

$360\pi-135\pi\left(p\sqrt{\frac{2}{3}}\right)^2=0$

$360-90p^2=0$

$p^2=4$ ***A1***

$p=2$ OR $r=2\sqrt{\frac{2}{3}}$ ***A1***

[3 marks]

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(e) attempt to substitute their value of r into $V = 360\pi r - 45\pi r^3$ **_M1_**

$$V = 360\pi \times 2\sqrt{\frac{2}{3}} - 45\pi \times \left(2\sqrt{\frac{2}{3}}\right)^3$$

$$= 360\pi \times 2\sqrt{\frac{2}{3}} - 45\pi \times 8 \times \frac{2}{3} \times \sqrt{\frac{2}{3}}$$

$$\left(= 720\pi\sqrt{\frac{2}{3}} - 240\pi\sqrt{\frac{2}{3}}\right)$$ **_(A1)_**

$$= 480\pi\sqrt{\frac{2}{3}}$$ **_A1_**

[3 marks]

Total [17 marks]

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