Subject – Math AA(Higher Level)
Topic - Calculus
Year - May 2021 – Nov 2022
Paper -1
Questions

Question 1

[Maximum mark: 21]

The function f is defined by $f(x) = e^{\sin x}$.

(a) Find the first two derivatives of $f(x)$ and hence find the Maclaurin series for $f(x)$ up to and including the x^2 term. [8]

(b) Show that the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero. [4]

(c) Using the Maclaurin series for $\arctan x$ and $e^{3x} - 1$, find the Maclaurin series for $\arctan(e^{3x} - 1)$ up to and including the x^3 term. [6]

(d) Hence, or otherwise, find $\lim_{x \to 0} \frac{f(x) - 1}{\arctan\left(e^{3x} - 1\right)}$. [3]

Question 2

[Maximum mark: 16]

Let $f(x) = \frac{\ln 5x}{kx}$ where $x > 0$, $k \in \mathbb{R}^+$.

(a) Show that $f'(x) = \frac{1 - \ln 5x}{kx^2}$. [3]

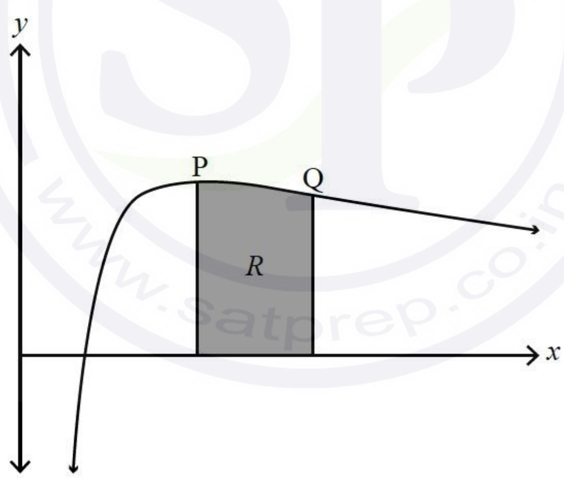
The graph of f has exactly one maximum point P.

(b) Find the x-coordinate of P. [3]

The second derivative of f is given by $f''(x) = \frac{2\ln 5x - 3}{kx^3}$. The graph of f has exactly one point of inflexion Q.

(c) Show that the x-coordinate of Q is $\frac{1}{5}e^{\frac{3}{2}}$. [3]

The region R is enclosed by the graph of f, the x-axis, and the vertical lines through the maximum point P and the point of inflexion Q.

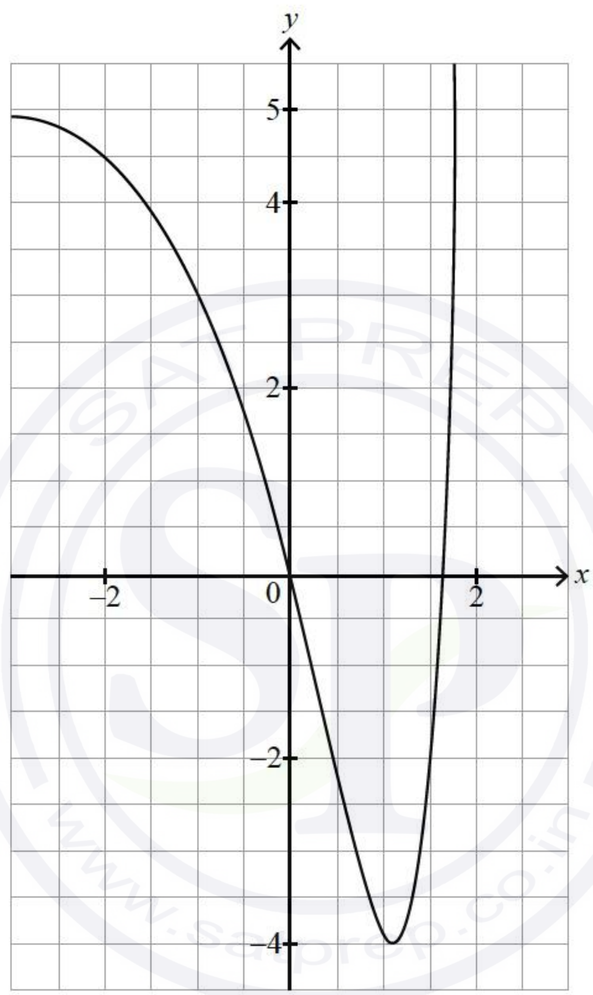


(d) Given that the area of R is 3, find the value of k. [7]

Question 3

[Maximum mark: 8]

The function f is defined by $f(x) = e^{2x} - 6e^{x} + 5$, $x \in \mathbb{R}$, $x \leq a$. The graph of $y = f(x)$ is shown in the following diagram.



(a) Find the largest value of a such that f has an inverse function. [3]

(b) For this value of a, find an expression for $f^{-1}(x)$, stating its domain. [5]

Question 4

[Maximum mark: 5]

Let $f'(x) = \frac{8x}{\sqrt{2x^2 + 1}}$. Given that $f(0) = 5$, find $f(x)$.

Question 5

[Maximum mark: 20]

Let $f(x)=\sqrt{1+x}$ for $x>-1$.

(a) Show that $f''(x)=-\dfrac{1}{4\sqrt{(1+x)^3}}$. [3]

(b) Use mathematical induction to prove that $f^{(n)}(x)=\left(-\dfrac{1}{4}\right)^{n-1}\dfrac{(2n-3)!}{(n-2)!}(1+x)^{\frac{1}{2}-n}$ for $n\in\mathbb{Z},\ n\geq 2$. [9]

Let $g(x)=\mathrm{e}^{mx},\ m\in\mathbb{Q}$.

Consider the function h defined by $h(x)=f(x)\times g(x)$ for $x>-1$.

It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $\dfrac{7}{4}$.

(c) Find the possible values of m. [8]

Question 6

[Maximum mark: 5]

Use l'Hôpital's rule to find $\lim\limits_{x\to 0}\left(\dfrac{\arctan 2x}{\tan 3x}\right)$.

Question 7

[Maximum mark: 7]

Consider the functions $f(x)=-(x-h)^2+2k$ and $g(x)=\mathrm{e}^{x-2}+k$ where $h,\ k\in\mathbb{R}$.

(a) Find $f'(x)$. [1]

The graphs of f and g have a common tangent at $x=3$.

(b) Show that $h=\dfrac{\mathrm{e}+6}{2}$. [3]

(c) Hence, show that $k=\mathrm{e}+\dfrac{\mathrm{e}^2}{4}$. [3]

Question 8

[Maximum mark: 20]

The acceleration, $a\,\text{ms}^{-2}$, of a particle moving in a horizontal line at time t seconds, $t \geq 0$, is given by $a = -(1+v)$ where $v\,\text{ms}^{-1}$ is the particle's velocity and $v > -1$.

At $t = 0$, the particle is at a fixed origin O and has initial velocity $v_0\,\text{ms}^{-1}$.

(a) By solving an appropriate differential equation, show that the particle's velocity at time t is given by $v(t) = (1 + v_0)\text{e}^{-t} - 1$. [6]

(b) Initially at O, the particle moves in the positive direction until it reaches its maximum displacement from O. The particle then returns to O.

Let s metres represent the particle's displacement from O and $s_{\max}$ its maximum displacement from O.

(i) Show that the time T taken for the particle to reach $s_{\max}$ satisfies the equation $\text{e}^T = 1 + v_0$.

(ii) By solving an appropriate differential equation and using the result from part (b) (i), find an expression for $s_{\max}$ in terms of v_0. [7]

Let $v(T-k)$ represent the particle's velocity k seconds before it reaches $s_{\max}$, where

$$v(T-k) = (1 + v_0)\text{e}^{-(T-k)} - 1.$$

(c) By using the result to part (b) (i), show that $v(T-k) = \text{e}^k - 1$. [2]

Similarly, let $v(T+k)$ represent the particle's velocity k seconds after it reaches $s_{\max}$.

(d) Deduce a similar expression for $v(T+k)$ in terms of k. [2]

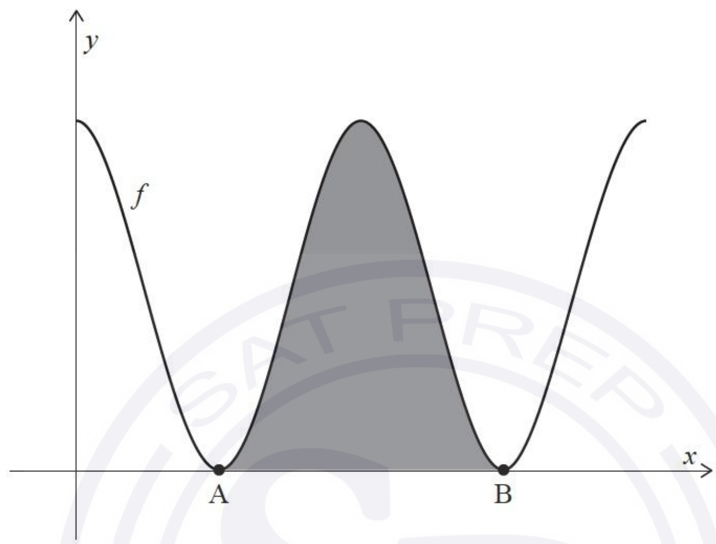
(e) Hence, show that $v(T-k) + v(T+k) \geq 0$. [3]

Question 9

[Maximum mark: 15]

Consider the function f defined by $f(x) = 6 + 6\cos x$, for $0 \leq x \leq 4\pi$.

The following diagram shows the graph of $y = f(x)$.



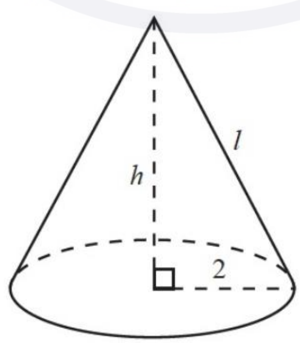
The graph of f touches the x-axis at points A and B, as shown. The shaded region is enclosed by the graph of $y = f(x)$ and the x-axis, between the points A and B.

(a) Find the x-coordinates of A and B. [3]

(b) Show that the area of the shaded region is 12π. [5]

The right cone in the following diagram has a total surface area of 12π, equal to the shaded area in the previous diagram.

The cone has a base radius of 2, height h, and slant height l.



(c) Find the value of l. [3]

(d) Hence, find the volume of the cone. [4]

Question 10

[Maximum mark: 7]

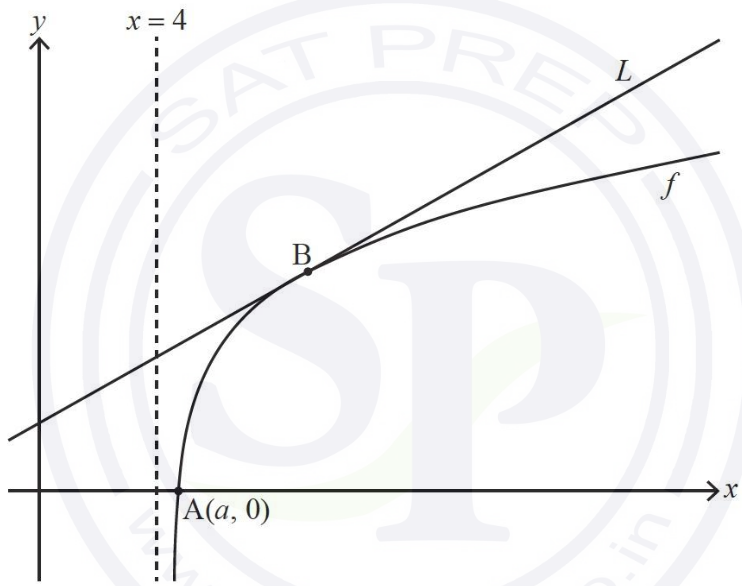
By using the substitution $u = \sin x$, find $\int \frac{\sin x \cos x}{\sin^2 x - \sin x - 2} \mathrm{d}x$.

Question 11

[Maximum mark: 9]

Consider the function f defined by $f(x) = \ln(x^2 - 16)$ for $x > 4$.

The following diagram shows part of the graph of f which crosses the x-axis at point A, with coordinates $(a, 0)$. The line L is the tangent to the graph of f at the point B.



(a) Find the exact value of a. [3]

(b) Given that the gradient of L is $\frac{1}{3}$, find the x-coordinate of B. [6]

Question 12

[Maximum mark: 14]

(a) Prove by mathematical induction that $\frac{d^n}{dx^n}\left(x^2e^x\right)=\left[x^2+2nx+n(n-1)\right]e^x$ for $n \in \mathbb{Z}^+$. [7]

(b) Hence or otherwise, determine the Maclaurin series of $f(x)=x^2e^x$ in ascending powers of x, up to and including the term in x^4. [3]

(c) Hence or otherwise, determine the value of $\lim_{x\to 0}\left[\frac{\left(x^2e^x-x^2\right)^3}{x^9}\right]$. [4]

Question 13

[Maximum mark: 16]

A particle P moves along the x-axis. The velocity of P is $v\,\mathrm{m\,s^{-1}}$ at time t seconds, where $v(t)=4+4t-3t^2$ for $0 \le t \le 3$. When $t=0$, P is at the origin O.

(a) (i) Find the value of t when P reaches its maximum velocity.

(ii) Show that the distance of P from O at this time is $\frac{88}{27}$ metres. [7]

(b) Sketch a graph of v against t, clearly showing any points of intersection with the axes. [4]

(c) Find the total distance travelled by P. [5]

Question 14

[Maximum mark: 7]

Solve the differential equation $\frac{dy}{dx}=\frac{\ln 2x}{x^2}-\frac{2y}{x}$, $x>0$, given that $y=4$ at $x=\frac{1}{2}$.

Give your answer in the form $y=f(x)$.

Question 15

[Maximum mark: 7]

The function f is defined for all $x \in \mathbb{R}$. The line with equation $y = 6x - 1$ is the tangent to the graph of f at $x = 4$.

(a) Write down the value of $f'(4)$. [1]

(b) Find $f(4)$. [1]

The function g is defined for all $x \in \mathbb{R}$ where $g(x) = x^2 - 3x$ and $h(x) = f(g(x))$.

(c) Find $h(4)$. [2]

(d) Hence find the equation of the tangent to the graph of h at $x = 4$. [3]

Question 16

[Maximum mark: 4]

Given that $\frac{dy}{dx} = \cos\left(x - \frac{\pi}{4}\right)$ and $y = 2$ when $x = \frac{3\pi}{4}$, find y in terms of x.

Question 17

[Maximum mark: 21]

The function f is defined by $f(x) = e^x \sin x$, where $x \in \mathbb{R}$.

(a) Find the Maclaurin series for $f(x)$ up to and including the x^3 term. [4]

(b) Hence, find an approximate value for $\int_0^1 e^{x^2} \sin(x^2)\,dx$. [4]

The function g is defined by $g(x) = e^x \cos x$, where $x \in \mathbb{R}$.

(c) (i) Show that $g(x)$ satisfies the equation $g''(x) = 2(g'(x) - g(x))$.

(ii) Hence, deduce that $g^{(4)}(x) = 2(g'''(x) - g''(x))$. [5]

(d) Using the result from part (c), find the Maclaurin series for $g(x)$ up to and including the x^4 term. [5]

(e) Hence, or otherwise, determine the value of $\lim_{x \to 0} \frac{e^x \cos x - 1 - x}{x^3}$. [3]

Question 18

[Maximum mark: 5]

Consider the curve with equation $y = (2x - 1)e^{kx}$, where $x \in \mathbb{R}$ and $k \in \mathbb{Q}$.

The tangent to the curve at the point where $x = 1$ is parallel to the line $y = 5e^{k}x$.

Find the value of k.

Question 19

[Maximum mark: 5]

Find the value of $\int_{1}^{9} \left(\frac{3\sqrt{x} - 5}{\sqrt{x}} \right) dx$.

Question 20

[Maximum mark: 6]

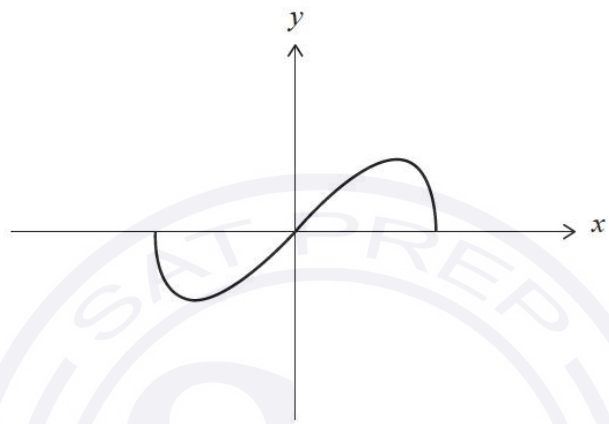
By using the substitution $u = \sec x$ or otherwise, find an expression for $\int_{0}^{\frac{\pi}{3}} \sec^{n} x \tan x \, dx$ in terms of n, where n is a non-zero real number.

Question 21

[Maximum mark: 8]

A function f is defined by $f(x)=x\sqrt{1-x^2}$ where $-1 \le x \le 1$.

The graph of $y=f(x)$ is shown below.



(a) Show that f is an odd function. [2]

The range of f is $a \le y \le b$, where $a, b \in \mathbb{R}$.

(b) Find the value of a and the value of b. [6]

Question 22

[Maximum mark: 20]

The function f is defined by $f(x)=\cos^2 x - 3\sin^2 x,\ 0 \le x \le \pi$.

(a) Find the roots of the equation $f(x)=0$. [5]

(b) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the points on the graph of $y=f(x)$ where $f'(x)=0$. [7]

(c) Sketch the graph of $y=|f(x)|$, clearly showing the coordinates of any points where $f'(x)=0$ and any points where the graph meets the coordinate axes. [4]

(d) Hence or otherwise, solve the inequality $|f(x)|>1$. [4]

Question 23

[Maximum mark: 10]

Consider the homogeneous differential equation $\frac{dy}{dx} = \frac{y^2 - 2x^2}{xy}$, where $x, y \neq 0$.

It is given that $y = 2$ when $x = 1$.

(a) By using the substitution $y = vx$, solve the differential equation. Give your answer in the form $y^2 = f(x)$. [8]

The points of zero gradient on the curve $y^2 = f(x)$ lie on two straight lines of the form $y = mx$ where $m \in \mathbb{R}$.

(b) Find the values of m. [2]

Question 24

[Maximum mark: 7]

Consider the curve with equation $(x^2 + y^2)y^2 = 4x^2$ where $x \geq 0$ and $-2 < y < 2$.

Show that the curve has no local maximum or local minimum points for $x > 0$.

Question 25

[Maximum mark: 5]

The derivative of the function f is given by $f'(x) = \frac{6x}{x^2 + 1}$.

The graph of $y = f(x)$ passes through the point $(1, 5)$. Find an expression for $f(x)$.

Question 26

[Maximum mark: 4]

The function g is defined by $g(x) = e^{x^2 + 1}$, where $x \in \mathbb{R}$.

Find $g'(-1)$.