Subject – Math AA(Higher Level)
Topic - Functions
Year - May 2021 – Nov 2024
Paper -1
Questions

Question 1

[Maximum mark: 5]

The functions f and g are defined such that $f(x) = \frac{x+3}{4}$ and $g(x) = 8x + 5$.

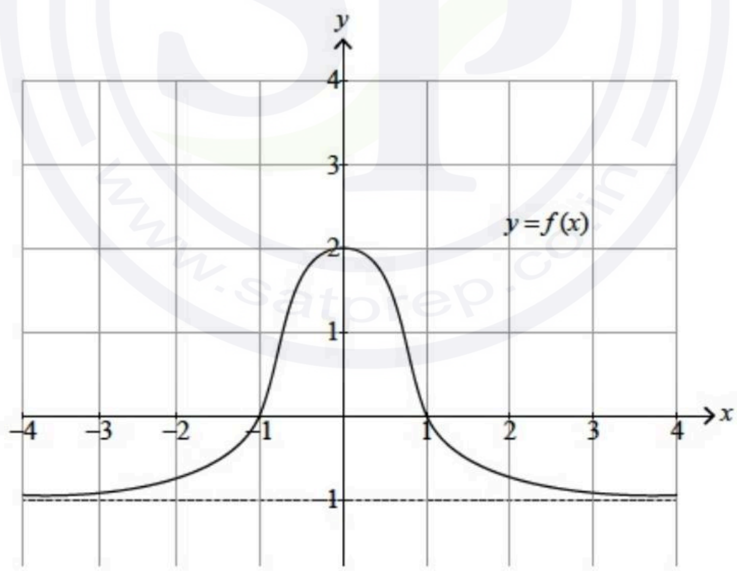
(a) Show that $(g \circ f)(x) = 2x + 11$. [2]

(b) Given that $(g \circ f)^{-1}(a) = 4$, find the value of a. [3]

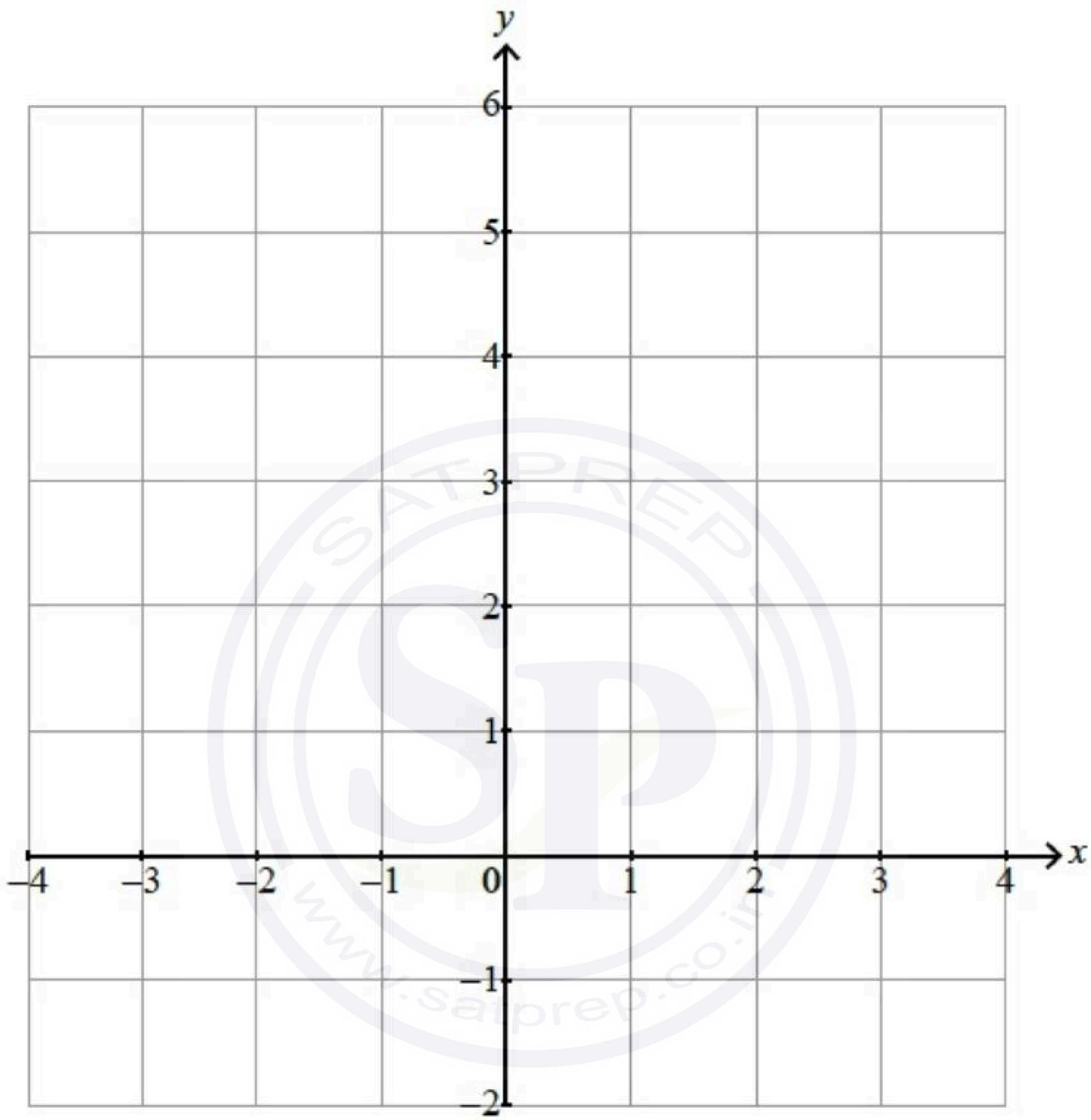
Question 2

[Maximum mark: 5]

The following diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = -1$. The graph crosses the x-axis at $x = -1$ and $x = 1$, and the y-axis at $y = 2$.



On the following set of axes, sketch the graph of $y = [f(x)]^2 + 1$, clearly showing any asymptotes with their equations and the coordinates of any local maxima or minima.


y
x
6
5
4
3
2
1
0
−1
−2
−4
−3
−2
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1
2
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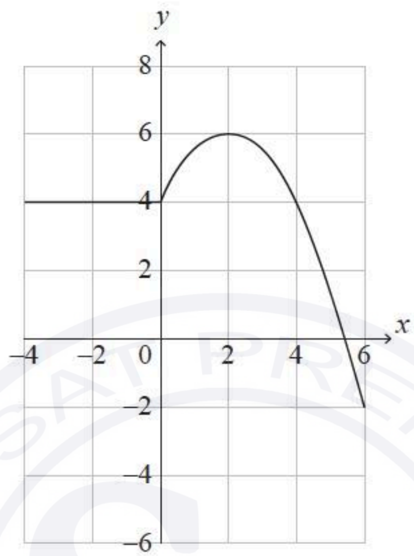
4

Question 3

[Maximum mark: 5]

The graph of $y = f(x)$ for $-4 \le x \le 6$ is shown in the following diagram.



(a) Write down the value of

(i) $f(2)$;

(ii) $(f \circ f)(2)$. [2]

(b) Let $g(x) = \frac{1}{2} f(x) + 1$ for $-4 \le x \le 6$. On the axes above, sketch the graph of g. [3]

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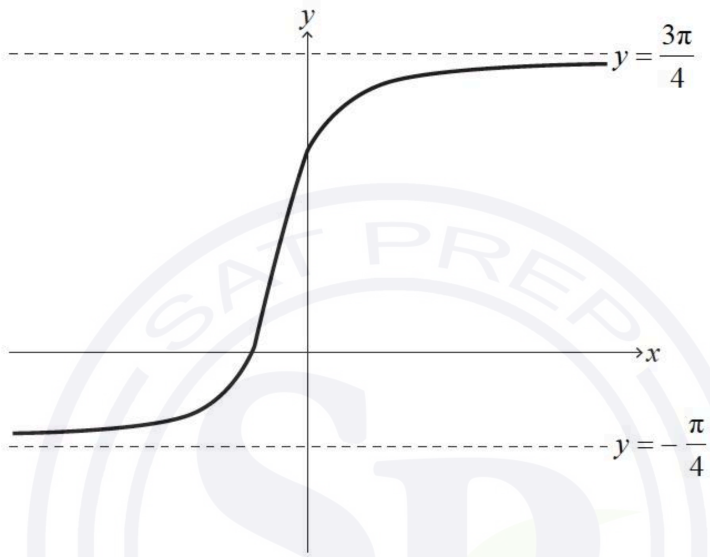
Question 4

[Maximum mark: 19]

The following diagram shows the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, with asymptotes at $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$.



(a) Describe a sequence of transformations that transforms the graph of $y = \arctan x$ to the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$. [3]

(b) Show that $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where p, $q > 0$ and $pq < 1$. [4]

(c) Verify that $\arctan(2x+1) = \arctan\left(\frac{x}{x+1}\right) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, $x > 0$. [3]

(d) Using mathematical induction and the result from part (b), prove that
$\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$. [9]

Question 5

[Maximum mark: 9]

The function f is defined by $f(x) = \dfrac{2x+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$.

(a) Write down the equation of

(i) the vertical asymptote of the graph of f;

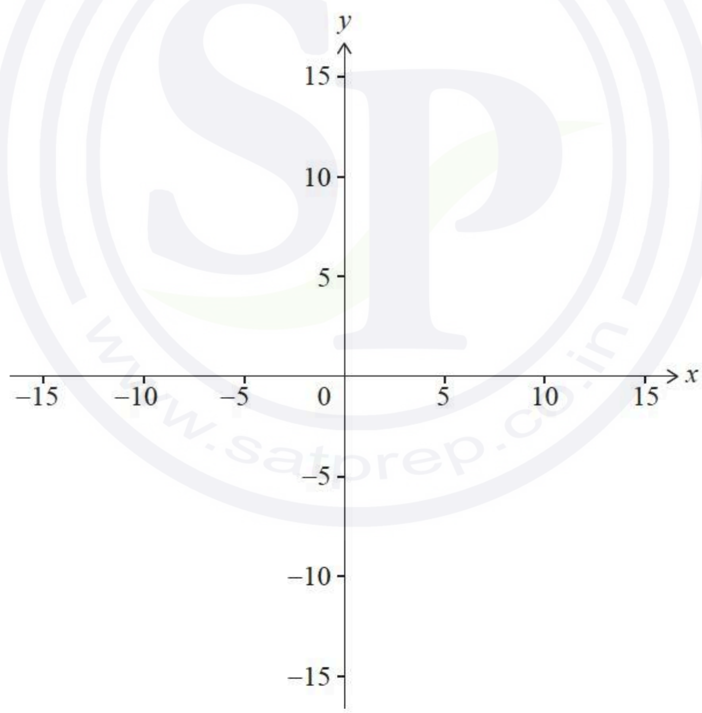
(ii) the horizontal asymptote of the graph of f. [2]

(b) Find the coordinates where the graph of f crosses

(i) the x-axis;

(ii) the y-axis. [2]

(c) Sketch the graph of f on the axes below. [1]



The function g is defined by $g(x) = \dfrac{ax+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$ and $a \in \mathbb{R}$.

(d) Given that $g(x) = g^{-1}(x)$, determine the value of a. [4]

Question 6

[Maximum mark: 7]

Consider $f(x) = 4\sin x + 2.5$ and $g(x) = 4\sin\left(x - \frac{3\pi}{2}\right) + 2.5 + q$, where $x \in \mathbb{R}$ and $q > 0$.

The graph of g is obtained by two transformations of the graph of f.

(a) Describe these two transformations. [2]

The y-intercept of the graph of g is at $(0, r)$.

(b) Given that $g(x) \geq 7$, find the smallest value of r. [5]

Question 7

[Maximum mark: 7]

Consider the functions $f(x) = \sqrt{3}\sin x + \cos x$ where $0 \leq x \leq \pi$ and $g(x) = 2x$ where $x \in \mathbb{R}$.

(a) Find $(f \circ g)(x)$. [2]

(b) Solve the equation $(f \circ g)(x) = 2\cos 2x$ where $0 \leq x \leq \pi$. [5]

Question 8

[Maximum mark: 20]

A function f is defined by $f(x) = \dfrac{1}{x^2 - 2x - 3}$, where $x \in \mathbb{R}, x \neq -1, x \neq 3$.

(a) Sketch the curve $y = f(x)$, clearly indicating any asymptotes with their equations. State the coordinates of any local maximum or minimum points and any points of intersection with the coordinate axes. [6]

A function g is defined by $g(x) = \dfrac{1}{x^2 - 2x - 3}$, where $x \in \mathbb{R}, x > 3$.

(b) The inverse of g is g^{-1}.

(i) Show that $g^{-1}(x) = 1 + \dfrac{\sqrt{4x^2 + x}}{x}$.

(ii) State the domain of g^{-1}. [7]

A function h is defined by $h(x) = \arctan\dfrac{x}{2}$, where $x \in \mathbb{R}$.

(c) Given that $(h \circ g)(a) = \dfrac{\pi}{4}$, find the value of a.

Give your answer in the form $p + \dfrac{q}{2}\sqrt{r}$, where $p, q, r \in \mathbb{Z}^+$. [7]

Question 9

[Maximum mark: 8]

A function f is defined by $f\left(x\right)=\dfrac{2x-1}{x+1}$, where $x\in\mathbb{R}$, $x\neq -1$.

(a) The graph of $y=f(x)$ has a vertical asymptote and a horizontal asymptote.

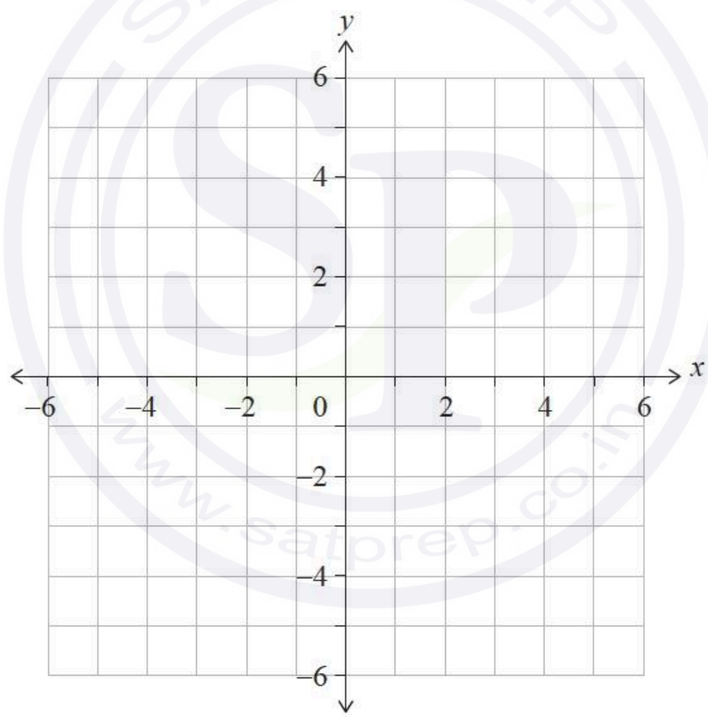
Write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(b) On the set of axes below, sketch the graph of $y=f(x)$.

On your sketch, clearly indicate the asymptotes and the position of any points of intersection with the axes. [3]



(c) Hence, solve the inequality $0<\dfrac{2x-1}{x+1}<2$. [1]

(d) Solve the inequality $0<\dfrac{2|x|-1}{|x|+1}<2$. [2]

Question 10

[Maximum mark: 5]

Let $f(x) = \cos(x - k)$, where $0 \leq x \leq a$ and $a, k \in \mathbb{R}^{+}$.

(a) Consider the case where $k = \frac{\pi}{2}$.

By sketching a suitable graph, or otherwise, find the largest value of a for which the inverse function f^{-1} exists. [2]

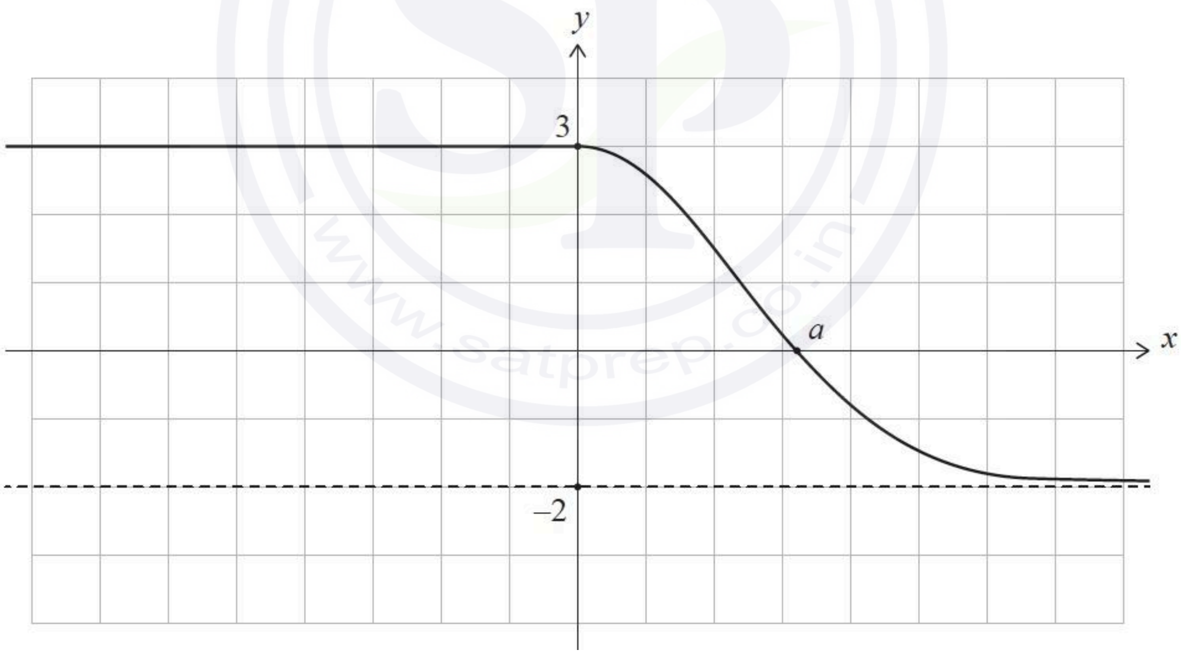
(b) Find the largest value of a for which the inverse function f^{-1} exists in the case where $k = \pi$. [1]

(c) Find the largest value of a for which the inverse function f^{-1} exists in the case where $\pi < k < 2\pi$. Give your answer in terms of k. [2]

Question 11

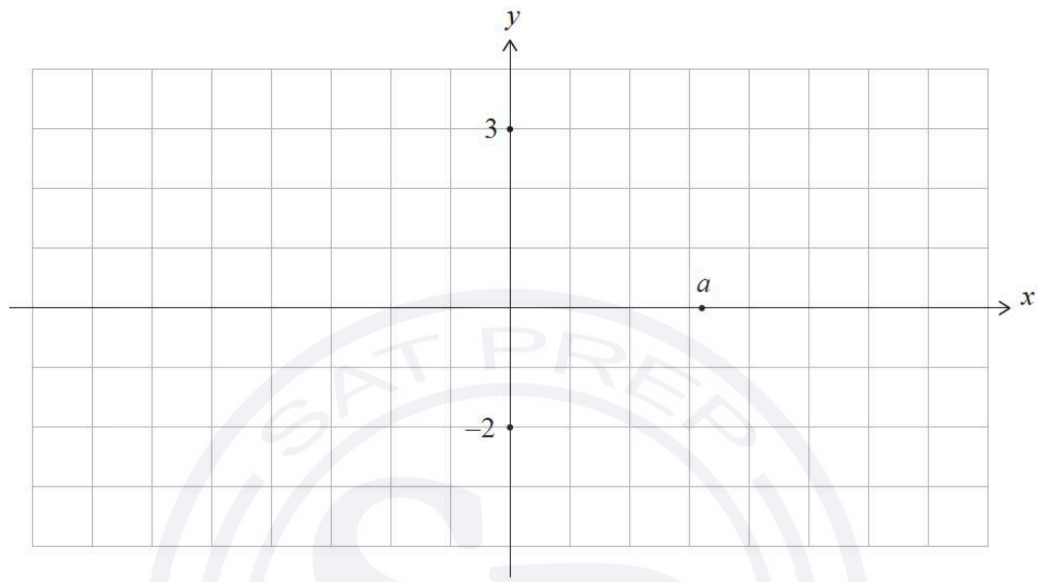
[Maximum mark: 7]

Part of the graph of a function, f, is shown in the following diagram. The graph of $y = f(x)$ has a y-intercept at $(0, 3)$, an x-intercept at $(a, 0)$ and a horizontal asymptote $y = -2$.



Consider the function $g(x) = |f(|x|)|$.

(a) On the following grid, sketch the graph of $y = g(x)$, labelling any axis intercepts and giving the equation of the asymptote. [4]



(b) Find the possible values of k such that $(g(x))^2 = k$ has exactly two solutions. [3]

Question 12

[Maximum mark: 7]

The function f is defined by $f(x) = \dfrac{7x+7}{2x-4}$ for $x \in \mathbb{R}$, $x \neq 2$.

(a) Find the zero of $f(x)$. [2]

(b) For the graph of $y = f(x)$, write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(c) Find $f^{-1}(x)$, the inverse function of $f(x)$. [3]

Question 13

[Maximum mark: 7]

The functions f and g are defined by

$$f(x) = \cos x,\ 0 \le x \le \frac{\pi}{2}$$

$$g(x) = \tan x,\ 0 \le x < \frac{\pi}{2}.$$

The curves $y = f(x)$ and $y = g(x)$ intersect at a point P whose x-coordinate is k, where $0 < k < \dfrac{\pi}{2}$.

(a) Show that $\cos^2 k = \sin k$. [1]

(b) Hence, show that the tangent to the curve $y = f(x)$ at P and the tangent to the curve $y = g(x)$ at P intersect at right angles. [3]

(c) Find the value of $\sin k$. Give your answer in the form $\dfrac{a+\sqrt{b}}{c}$, where a, $c \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. [3]

Question 14

[Maximum mark: 7]

The functions f and g are defined for $x \in \mathbb{R}$ by

$$f(x) = ax + b, \text{ where } a,\ b \in \mathbb{Z}$$

$$g(x) = x^2 + x + 3.$$

Find the two possible functions f such that $(g \circ f)(x) = 4x^2 - 14x + 15$.

Question 15

[Maximum mark: 5]

A function f is defined by $f(x) = 1 - \frac{1}{x-2}$, where $x \in \mathbb{R}$, $x \neq 2$.

(a) The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

Write down the equation of

(i) the vertical asymptote;

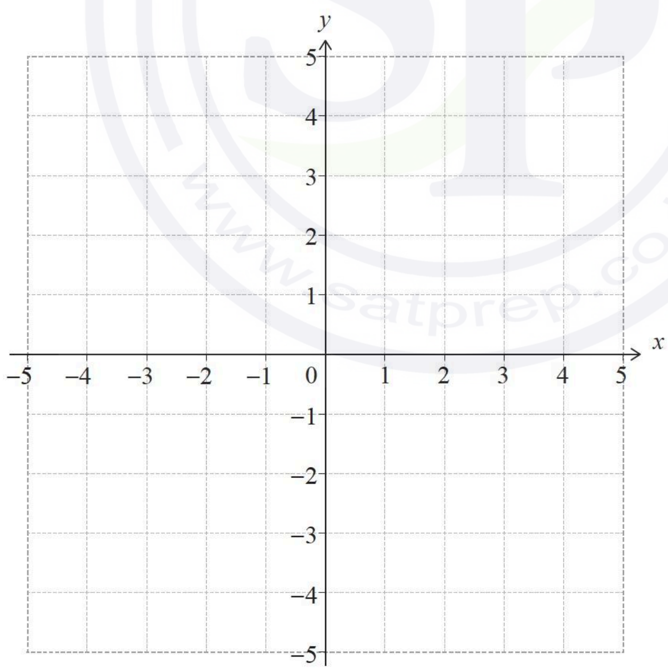
(ii) the horizontal asymptote. [2]

(b) Find the coordinates of the point where the graph of $y = f(x)$ intersects

(i) the y-axis;

(ii) the x-axis. [2]

(c) On the following set of axes, sketch the graph of $y = f(x)$, showing all the features found in parts (a) and (b). [1]



Question 16

[Maximum mark: 15]

The functions f and g are defined by

$f(x) = \ln(2x - 9)$, where $x > \dfrac{9}{2}$

$g(x) = 2\ln x - \ln d$, where $x > 0$, $d \in \mathbb{R}^{+}$.

(a) State the equation of the vertical asymptote to the graph of $y = g(x)$. [1]

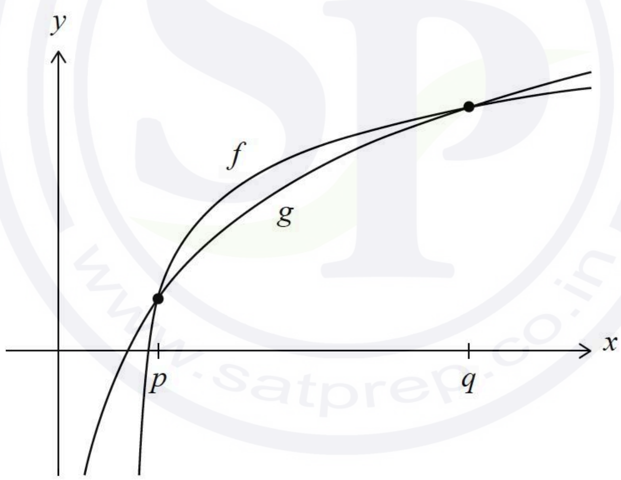
The graphs of $y = f(x)$ and $y = g(x)$ intersect at two distinct points.

(b) (i) Show that, at the points of intersection, $x^2 - 2dx + 9d = 0$.

(ii) Hence show that $d^2 - 9d > 0$.

(iii) Find the range of possible values of d. [9]

The following diagram shows part of the graphs of $y = f(x)$ and $y = g(x)$.



The graphs intersect at $x = p$ and $x = q$, where $p < q$.

(c) In the case where $d = 10$, find the value of $q - p$. Express your answer in the form $a\sqrt{b}$, where a, $b \in \mathbb{Z}^{+}$. [5]

Question 17

[Maximum mark: 8]

Consider the function $f(x)=\dfrac{\sin^2(kx)}{x^2}$, where $x \neq 0$ and $k \in \mathbb{R}^+$.

(a) Show that f is an even function. [2]

(b) Given that $\lim_{x \to 0} f(x) = 16$, find the value of k. [6]

Question 18

[Maximum mark: 5]

Consider the functions $f(x) = x - 3$ and $g(x) = x^2 + k^2$, where k is a real constant.

(a) Write down an expression for $(g \circ f)(x)$. [2]

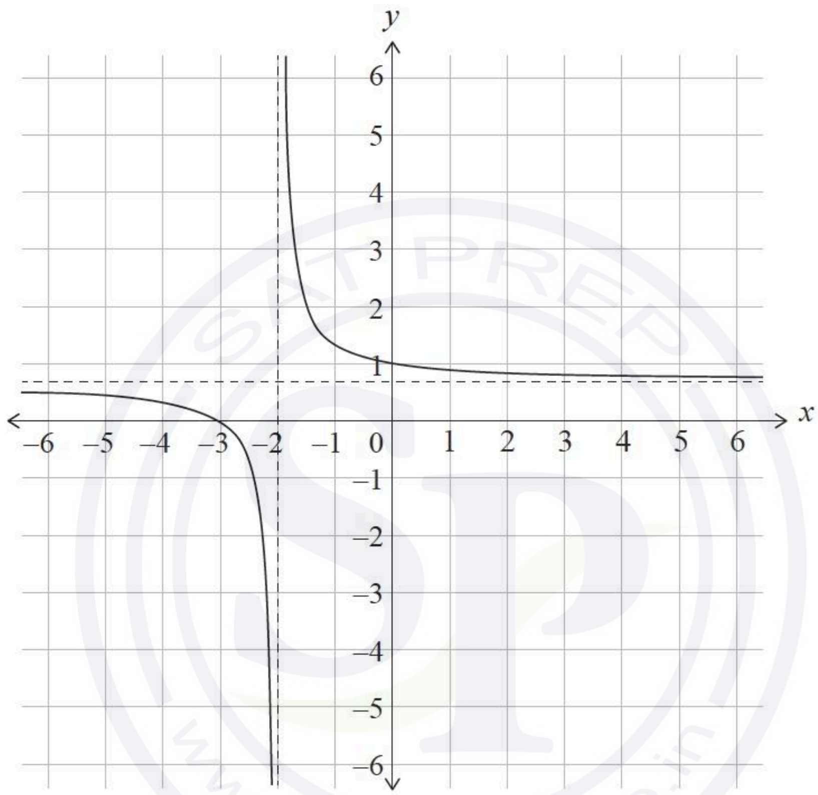
(b) Given that $(g \circ f)(2) = 10$, find the possible values of k. [3]

Question 19

[Maximum mark: 8]

A function f is defined by $f(x) = \frac{2(x+3)}{3(x+2)}$, where $x \in \mathbb{R}$, $x \neq -2$.

The graph $y = f(x)$ is shown below.



(a) Write down the equation of the horizontal asymptote. [1]

Consider $g(x) = mx + 1$, where $m \in \mathbb{R}$, $m \neq 0$.

(b) (i) Write down the number of solutions to $f(x) = g(x)$ for $m > 0$.

(ii) Determine the value of m such that $f(x) = g(x)$ has only one solution for x.

(iii) Determine the range of values for m, where $f(x) = g(x)$ has two solutions for $x \geq 0$. [7]

Question 20

[Maximum mark: 16]

Consider the function $f(x) = \frac{4x+2}{x-2}$, $x \neq 2$.

(a) Sketch the graph of $y = f(x)$. On your sketch, indicate the values of any axis intercepts and label any asymptotes with their equations. [5]

(b) Write down the range of f. [1]

Consider the function $g(x) = x^2 + bx + c$. The graph of g has an axis of symmetry at $x = 2$.

The two roots of $g(x) = 0$ are $-\frac{1}{2}$ and p, where $p \in \mathbb{Q}$.

(c) Show that $p = \frac{9}{2}$. [1]

(d) Find the value of b and the value of c. [3]

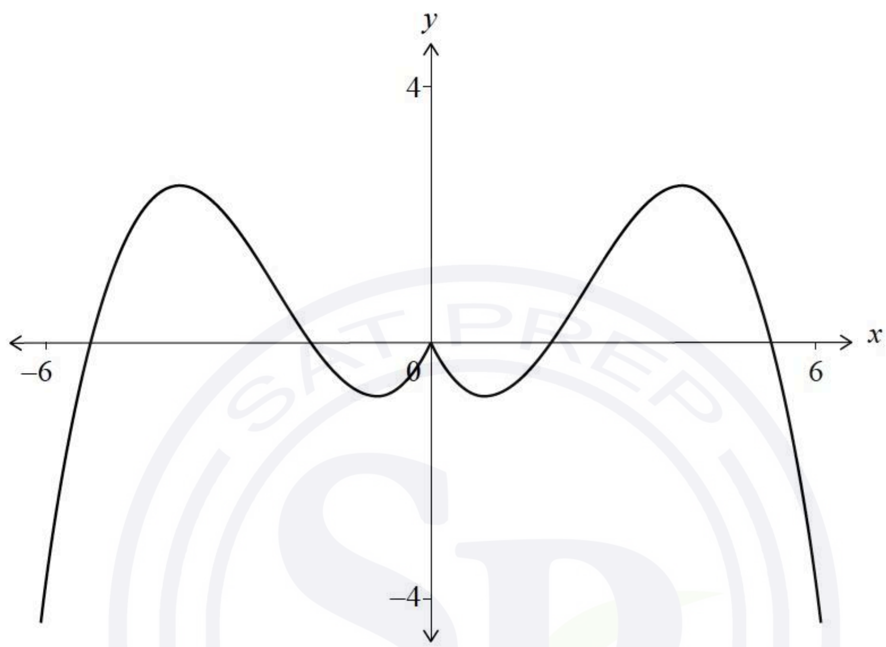
(e) Find the y-coordinate of the vertex of the graph of $y = g(x)$. [2]

(f) Find the product of the solutions of the equation $f(x) = g(x)$. [4]

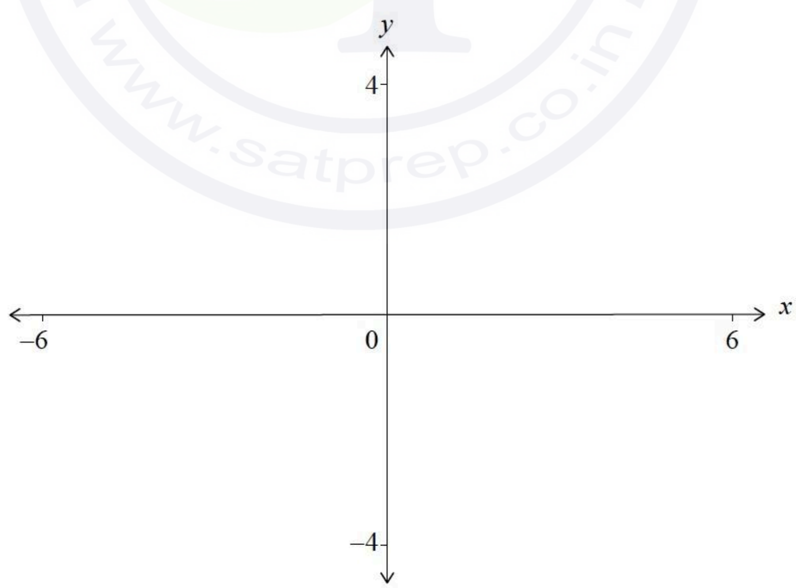
Question 21

[Maximum mark: 6]

The graph of $y = f(|x|)$ for $-6 \le x \le 6$ is shown in the following diagram.

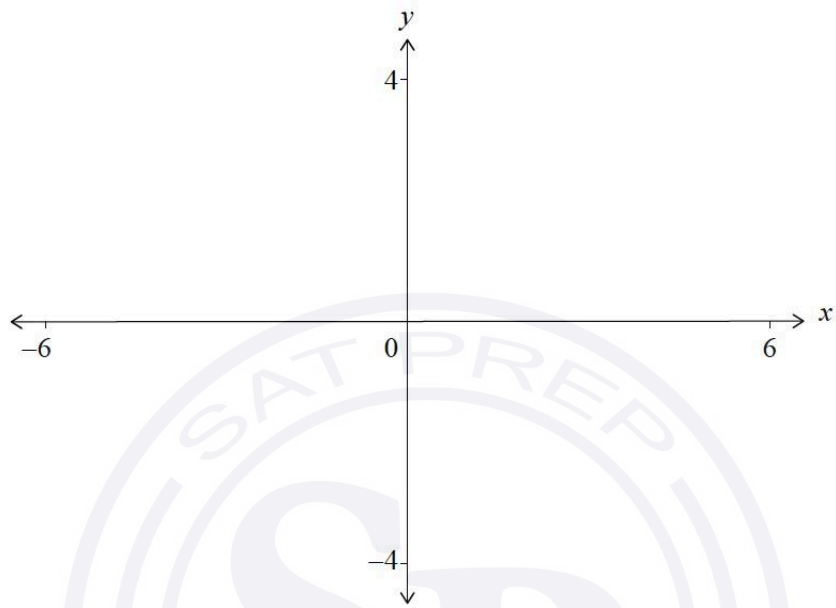


(a) On the following axes, sketch the graph of $y = |f(|x|)|$ for $-6 \le x \le 6$. [2]



It is given that f is an odd function.

(b) On the following axes, sketch the graph of $y = f(x)$ for $-6 \le x \le 6$. [2]



It is also given that $\int_0^4 f(|x|)\,dx = 1.6$.

(c) Write down the value of

(i) $\int_{-4}^{0} f(x)\,dx$;

(ii) $\int_{-4}^{4} \left(f(|x|) + f(x) \right) dx$. [2]

Question 22

[Maximum mark: 7]

Consider the function $f(x) = \sec\left(x - \frac{\pi}{4}\right)$, for $0 \le x \le \frac{\pi}{2}$.

(a) Determine the range of f. [3]

The region bounded by the graph of $y = f(x)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{2}$ is rotated 2π radians about the x-axis.

(b) Find the volume of revolution generated. [4]

Question 23

[Maximum mark: 5]

(a) Solve $2x^2 - 15x + 18 < 0$. [3]

(b) The function f is defined by $f(x) = \sqrt{2x^2 - 15x + 18}$, where $x \in \mathbb{R}$, $x \leq k$.

Find the greatest value of k for which f^{-1} exists, justifying your answer. [2]