Subject – Math AA(Higher Level)
Topic - Geometry and Trigonometry
Year - May 2021 – Nov 2022
Paper -1
Question

Question 1

[Maximum mark: 7]

The plane Π has the Cartesian equation $2x + y + 2z = 3$.

The line L has the vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}$, $\mu, p \in \mathbb{R}$. The acute angle between the line L and the plane Π is 30°.

Find the possible values of p.

Question 2

[Maximum mark: 19]

Consider the line L_1 defined by the Cartesian equation $\frac{x+1}{2} = y = 3 - z$.

(a) (i) Show that the point $(-1, 0, 3)$ lies on L_1.

(ii) Find a vector equation of L_1. [4]

Consider a second line L_2 defined by the vector equation $\boldsymbol{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$,

where $t \in \mathbb{R}$ and $a \in \mathbb{R}$.

(b) Find the possible values of a when the acute angle between L_1 and L_2 is 45°. [8]

It is given that the lines L_1 and L_2 have a unique point of intersection, A, when $a \neq k$.

(c) Find the value of k, and find the coordinates of the point A in terms of a. [7]

Question 3

[Maximum mark: 4]

It is given that $\operatorname{cosec}\theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of cot θ.

Question 4

[Maximum mark: 8]

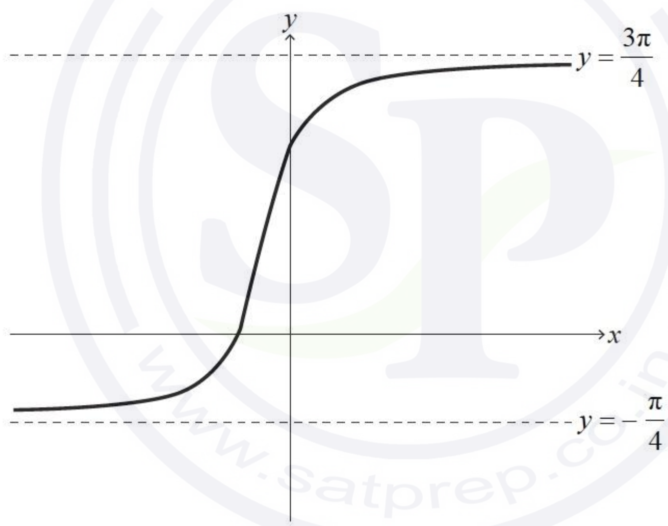
(a) Show that $\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$. [2]

(b) Hence or otherwise, solve $\sin 2x + \cos 2x - 1 + \cos x - \sin x = 0$ for $0 < x < 2\pi$. [6]

Question 5

[Maximum mark: 19]

The following diagram shows the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, with asymptotes at $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$.



(a) Describe a sequence of transformations that transforms the graph of $y = \arctan x$ to the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$. [3]

(b) Show that $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where p, $q > 0$ and $pq < 1$. [4]

(c) Verify that $\arctan(2x+1) = \arctan\left(\frac{x}{x+1}\right) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, $x > 0$. [3]

(d) Using mathematical induction and the result from part (b), prove that $\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$. [9]

Question 6

[Maximum mark: 8]

The lines l_1 and l_2 have the following vector equations where $\lambda, \mu \in \mathbb{R}$.

$$l_1 : \boldsymbol{r}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$l_2 : \boldsymbol{r}_2 = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(a) Show that l_1 and l_2 do not intersect. [3]

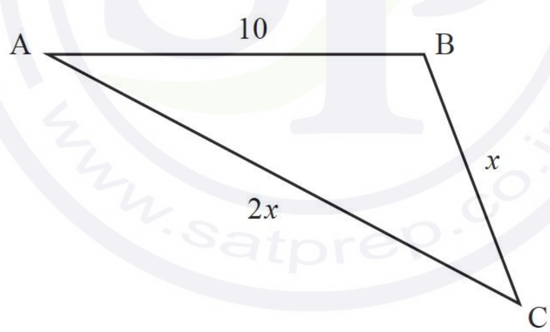
(b) Find the minimum distance between l_1 and l_2. [5]

Question 7

[Maximum mark: 7]

The following diagram shows triangle ABC, with $\text{AB} = 10$, $\text{BC} = x$ and $\text{AC} = 2x$.

diagram not to scale



Given that $\cos \hat{\text{C}} = \frac{3}{4}$, find the area of the triangle.

Give your answer in the form $\frac{p\sqrt{q}}{2}$ where $p, q \in \mathbb{Z}^+$.

Question 8

[Maximum mark: 4]

Given any two non-zero vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, show that $|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$.

Question 9

[Maximum mark: 7]

Solve the equation $2\cos^2 x + 5\sin x = 4$, $0 \le x \le 2\pi$.

Question 10

[Maximum mark: 7]

(a) Show that $2x - 3 - \frac{6}{x-1} = \frac{2x^2 - 5x - 3}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$. [2]

(b) Hence or otherwise, solve the equation $2\sin 2\theta - 3 - \frac{6}{\sin 2\theta - 1} = 0$ for $0 \le \theta \le \pi$, $\theta \neq \frac{\pi}{4}$. [5]

Question 11

[Maximum mark: 15]

Consider the three planes

$$\Pi_1 : 2x - y + z = 4$$
$$\Pi_2 : x - 2y + 3z = 5$$
$$\Pi_3 : -9x + 3y - 2z = 32$$

(a) Show that the three planes do not intersect. [4]

(b) (i) Verify that the point $\mathrm{P}(1, -2, 0)$ lies on both Π_1 and Π_2.

(ii) Find a vector equation of L, the line of intersection of Π_1 and Π_2. [5]

(c) Find the distance between L and Π_3. [6]

Question 12

[Maximum mark: 7]

Consider the functions $f(x) = \sqrt{3}\sin x + \cos x$ where $0 \le x \le \pi$ and $g(x) = 2x$ where $x \in \mathbb{R}$.

(a) Find $(f \circ g)(x)$. [2]

(b) Solve the equation $(f \circ g)(x) = 2\cos 2x$ where $0 \le x \le \pi$. [5]

Question 13

[Maximum mark: 5]

Find the least positive value of x for which $\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$.

Question 14

[Maximum mark: 20]

The function f is defined by $f(x)=\cos^2 x-3\sin^2 x,\ 0\le x\le \pi$.

(a) Find the roots of the equation $f(x)=0$. [5]

(b) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the points on the graph of $y=f(x)$ where $f'(x)=0$. [7]

(c) Sketch the graph of $y=|f(x)|$, clearly showing the coordinates of any points where $f'(x)=0$ and any points where the graph meets the coordinate axes. [4]

(d) Hence or otherwise, solve the inequality $|f(x)|>1$. [4]

Question 15

[Maximum mark: 5]

Let a be a constant, where $a>1$.

(a) Show that $a^2+\left(\frac{a^2-1}{2}\right)^2=\left(\frac{a^2+1}{2}\right)^2$. [3]

Consider a right-angled triangle with sides of length a, $\left(\frac{a^2-1}{2}\right)$ and $\left(\frac{a^2+1}{2}\right)$.

(b) Find an expression for the area of the triangle in terms of a. [2]

Question 16

[Maximum mark: 7]

Consider a circle with a diameter AB, where A has coordinates $(1, 4, 0)$ and B has coordinates $(-3, 2, -4)$.

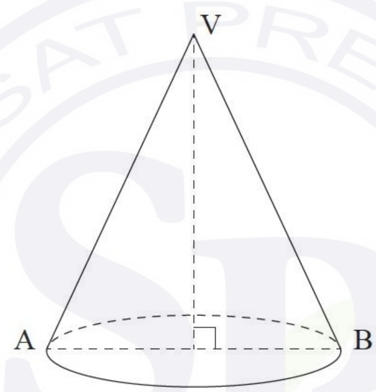
(a) Find

(i) the coordinates of the centre of the circle;

(ii) the radius of the circle. [4]

The circle forms the base of a right cone whose vertex V has coordinates $(-1, -1, 0)$.

diagram not to scale



(b) Find the exact volume of the cone. [3]