# Subject – Math AA(Higher Level)
# Topic - Geometry and Trigonometry
# Year - May 2021 – Nov 2024
# Paper -1
# Question

## Question 1

[Maximum mark: 7]

The plane $\Pi$ has the Cartesian equation $2x + y + 2z = 3$.

The line $L$ has the vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}$, $\mu, p \in \mathbb{R}$. The acute angle between the line $L$ and the plane $\Pi$ is $30°$.

Find the possible values of $p$.

## Question 2

[Maximum mark: 19]

Consider the line $L_1$ defined by the Cartesian equation $\frac{x+1}{2} = y = 3 - z$.

(a) (i) Show that the point $(-1, 0, 3)$ lies on $L_1$.

(ii) Find a vector equation of $L_1$. [4]

Consider a second line $L_2$ defined by the vector equation $\boldsymbol{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$,

where $t \in \mathbb{R}$ and $a \in \mathbb{R}$.

(b) Find the possible values of $a$ when the acute angle between $L_1$ and $L_2$ is $45°$. [8]

It is given that the lines $L_1$ and $L_2$ have a unique point of intersection, A, when $a \neq k$.

(c) Find the value of $k$, and find the coordinates of the point A in terms of $a$. [7]

## Question 3

[Maximum mark: 4]

It is given that $\operatorname{cosec}\theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $\cot\ \theta$.

## Question 4

[Maximum mark: 8]

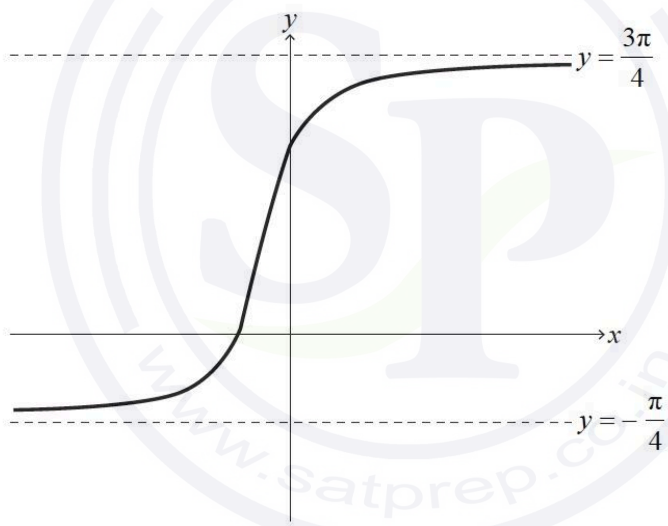
(a) Show that $\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$. [2]

(b) Hence or otherwise, solve $\sin 2x + \cos 2x - 1 + \cos x - \sin x = 0$ for $0 < x < 2\pi$. [6]

## Question 5

[Maximum mark: 19]

The following diagram shows the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, with asymptotes at $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$.




(a) Describe a sequence of transformations that transforms the graph of $y = \arctan x$ to the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$. [3]

(b) Show that $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where $p$, $q > 0$ and $pq < 1$. [4]

(c) Verify that $\arctan(2x+1) = \arctan\left(\frac{x}{x+1}\right) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, $x > 0$. [3]

(d) Using mathematical induction and the result from part (b), prove that $\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$. [9]

## Question 6

[Maximum mark: 8]

The lines $l_1$ and $l_2$ have the following vector equations where $\lambda, \mu \in \mathbb{R}$.

$$l_1 : \boldsymbol{r}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$l_2 : \boldsymbol{r}_2 = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(a) Show that $l_1$ and $l_2$ do not intersect. [3]

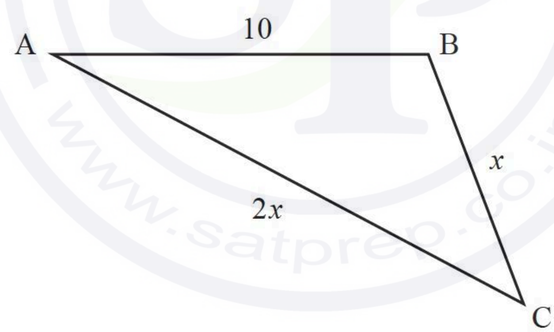
(b) Find the minimum distance between $l_1$ and $l_2$. [5]

## Question 7

[Maximum mark: 7]

The following diagram shows triangle ABC, with $\mathrm{AB} = 10$, $\mathrm{BC} = x$ and $\mathrm{AC} = 2x$.

**diagram not to scale**




Given that $\cos \hat{\mathrm{C}} = \dfrac{3}{4}$, find the area of the triangle.

Give your answer in the form $\dfrac{p\sqrt{q}}{2}$ where $p, q \in \mathbb{Z}^+$.

## Question 8

[Maximum mark: 4]

Given any two non-zero vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, show that $|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 \, |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$.

## Question 9

[Maximum mark: 7]

Solve the equation $2\cos^2 x + 5\sin x = 4$, $0 \le x \le 2\pi$.

## Question 10

[Maximum mark: 7]

(a) Show that $2x - 3 - \frac{6}{x-1} = \frac{2x^2 - 5x - 3}{x-1}$, $x \in \mathbb{R}$, $x \ne 1$. [2]

(b) Hence or otherwise, solve the equation $2\sin 2\theta - 3 - \frac{6}{\sin 2\theta - 1} = 0$ for $0 \le \theta \le \pi$, $\theta \ne \frac{\pi}{4}$. [5]

## Question 11

[Maximum mark: 15]

Consider the three planes

$$\Pi_1 : 2x - y + z = 4$$
$$\Pi_2 : x - 2y + 3z = 5$$
$$\Pi_3 : -9x + 3y - 2z = 32$$

(a) Show that the three planes do not intersect. [4]

(b) (i) Verify that the point $\mathrm{P}(1, -2, 0)$ lies on both $\Pi_1$ and $\Pi_2$.

(ii) Find a vector equation of $L$, the line of intersection of $\Pi_1$ and $\Pi_2$. [5]

(c) Find the distance between $L$ and $\Pi_3$. [6]

## Question 12

[Maximum mark: 7]

Consider the functions $f(x) = \sqrt{3}\sin x + \cos x$ where $0 \le x \le \pi$ and $g(x) = 2x$ where $x \in \mathbb{R}$.

(a) Find $(f \circ g)(x)$. [2]

(b) Solve the equation $(f \circ g)(x) = 2\cos 2x$ where $0 \le x \le \pi$. [5]

## Question 13

[Maximum mark: 5]

Find the least positive value of $x$ for which $\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$.

## Question 14

[Maximum mark: 20]

The function $f$ is defined by $f(x)=\cos^2 x-3\sin^2 x,\ 0\le x\le \pi$.

(a) Find the roots of the equation $f(x)=0$. [5]

(b) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the points on the graph of $y=f(x)$ where $f'(x)=0$. [7]

(c) Sketch the graph of $y=|f(x)|$, clearly showing the coordinates of any points where $f'(x)=0$ and any points where the graph meets the coordinate axes. [4]

(d) Hence or otherwise, solve the inequality $|f(x)|>1$. [4]

## Question 15

[Maximum mark: 5]

Let $a$ be a constant, where $a>1$.

(a) Show that $a^2+\left(\frac{a^2-1}{2}\right)^2=\left(\frac{a^2+1}{2}\right)^2$. [3]

Consider a right-angled triangle with sides of length $a$, $\left(\frac{a^2-1}{2}\right)$ and $\left(\frac{a^2+1}{2}\right)$.

(b) Find an expression for the area of the triangle in terms of $a$. [2]

**Question 16**

[Maximum mark: 7]

Consider a circle with a diameter AB, where A has coordinates $(1, 4, 0)$ and B has coordinates $(-3, 2, -4)$.

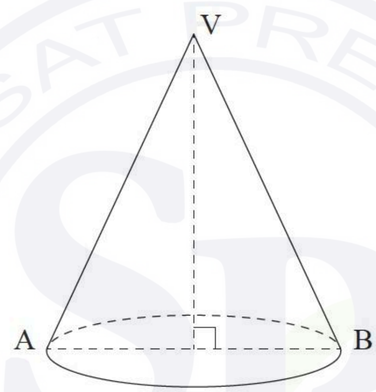
(a) Find

(i) the coordinates of the centre of the circle;

(ii) the radius of the circle. [4]

The circle forms the base of a right cone whose vertex V has coordinates $(-1, -1, 0)$.

**diagram not to scale**



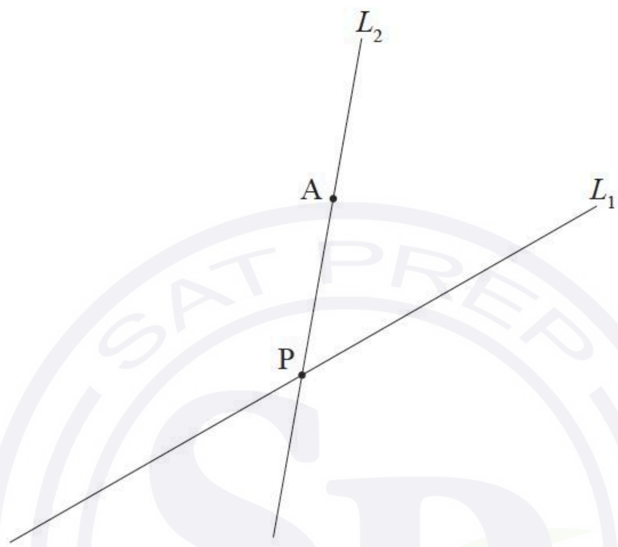

(b) Find the exact volume of the cone. [3]

## Question 17

[Maximum mark: 21]

Two lines, $L_1$ and $L_2$, intersect at point P. Point $A(2t, 8, 3)$, where $t > 0$, lies on $L_2$. This is shown in the following diagram.

**diagram not to scale**




The acute angle between the two lines is $\frac{\pi}{3}$.

The direction vector of $L_1$ is $\begin{pmatrix}1\\1\\0\end{pmatrix}$, and $\vec{PA} = \begin{pmatrix}2t\\0\\3+t\end{pmatrix}$.

(a) Show that $4t = \sqrt{10t^2 + 12t + 18}$. [4]

(b) Find the value of $t$. [4]

(c) Hence or otherwise, find the shortest distance from A to $L_1$. [4]

A plane, $\Pi$, contains $L_1$ and $L_2$.

(d) Find a normal vector to $\Pi$. [2]

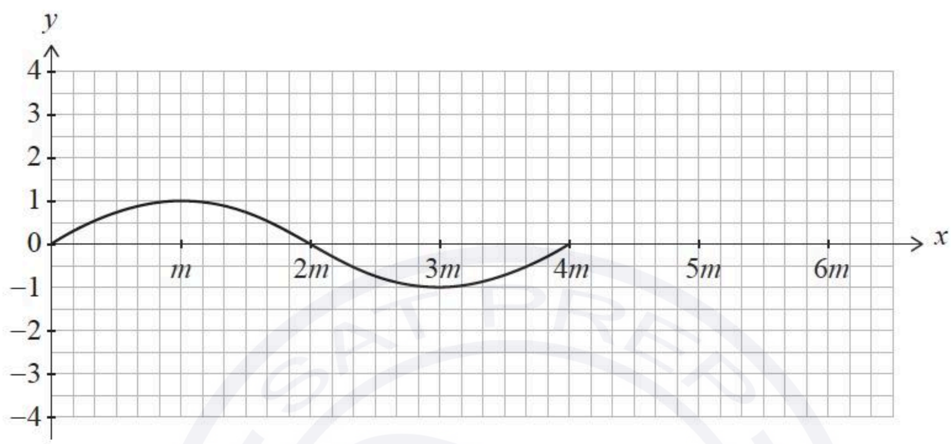
The base of a right cone lies in $\Pi$, centred at A such that $L_1$ is a tangent to its base. The volume of the cone is $90\pi\sqrt{3}$ cubic units.

(e) Find the two possible positions of the vertex of the cone. [7]

## Question 18

[Maximum mark: 6]

The function $f$ is defined by $f(x)=\sin qx$, where $q>0$. The following diagram shows part of the graph of $f$ for $0\le x\le 4m$, where $x$ is in radians. There are $x$-intercepts at $x=0$, $2m$ and $4m$.




(a) Find an expression for $m$ in terms of $q$. [2]

The function $g$ is defined by $g(x)=3\sin\frac{2qx}{3}$, for $0\le x\le 6m$.

(b) On the axes above, sketch the graph of $g$. [4]

## Question 19

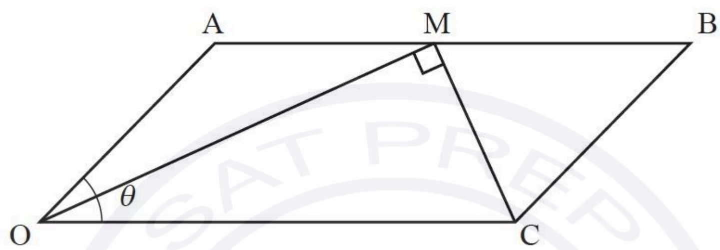
[Maximum mark: 6]

Solve $\cos 2x = \sin x$, where $-\pi \le x \le \pi$.

## Question 20

[Maximum mark: 9]

The following diagram shows parallelogram OABC with $\vec{OA} = \boldsymbol{a}$, $\vec{OC} = \boldsymbol{c}$ and $|\boldsymbol{c}| = 2|\boldsymbol{a}|$, where $|\boldsymbol{a}| \ne 0$.




The angle between $\vec{OA}$ and $\vec{OC}$ is $\theta$, where $0 < \theta < \pi$.

Point M is on [AB] such that $\vec{AM} = k\vec{AB}$, where $0 \le k \le 1$ and $\vec{OM} \bullet \vec{MC} = 0$.

(a) Express $\vec{OM}$ and $\vec{MC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$. [2]

(b) Hence, use a vector method to show that $|\boldsymbol{a}|^2 (1-2k)\left(2\cos\theta - (1-2k)\right) = 0$. [3]

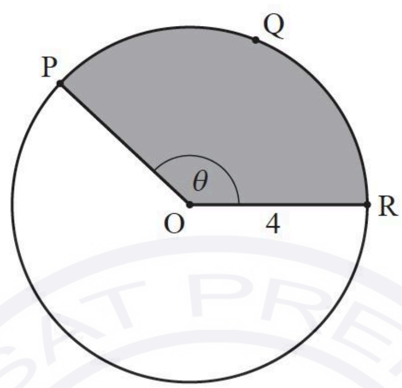
(c) Find the range of values for $\theta$ such that there are two possible positions for M. [4]

## Question 21

[Maximum mark: 6]

The following diagram shows a circle with centre O and radius $4\,\text{cm}$.

**diagram not to scale**




The points P, Q and R lie on the circumference of the circle and $\text{P}\hat{\text{O}}\text{R} = \theta$, where $\theta$ is measured in radians.

The length of arc PQR is $10\,\text{cm}$.

(a) Find the perimeter of the shaded sector. [2]

(b) Find $\theta$. [2]

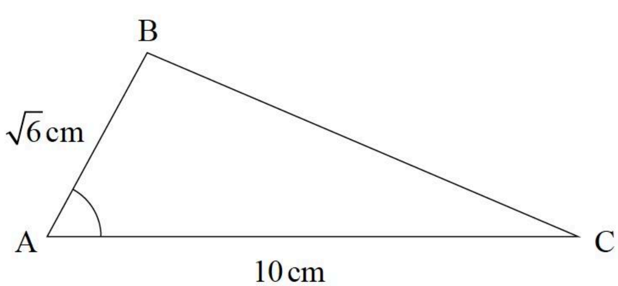
(c) Find the area of the shaded sector. [2]

## Question 22

[Maximum mark: 6]

In the following triangle ABC, $\text{AB} = \sqrt{6}\ \text{cm}$, $\text{AC} = 10\,\text{cm}$ and $\cos \text{B}\hat{\text{A}}\text{C} = \dfrac{1}{5}$.

**diagram not to scale**




Find the area of triangle ABC.

## Question 23

[Maximum mark: 19]

The plane $\Pi_1$ has equation $2x + 6y - 2z = 5$.

(a) Verify that the point $A\left(2, \frac{1}{2}, 1\right)$ lies on the plane $\Pi_1$. [1]

The plane $\Pi_2$ is given by $(k^2 - 6)x + (2k + 3)y + pz = q$, where $p$, $q$, $k \in \mathbb{R}$ and $p \neq 0$.

(b) In the case where $p = -6$, $\Pi_2$ is perpendicular to $\Pi_1$ and $A$ lies on $\Pi_2$.
Find the value of $k$ and the value of $q$. [5]

For parts (c), (d) and (e) it is now given that $\Pi_2$ is parallel to $\Pi_1$ with $k = 3$.

(c) Determine the value of $p$. [2]

It is also given that $q = -\frac{51}{2}$.

The line through $A$ that is perpendicular to $\Pi_1$ meets $\Pi_2$ at the point $B$.

(d) (i) Find the coordinates of $B$.

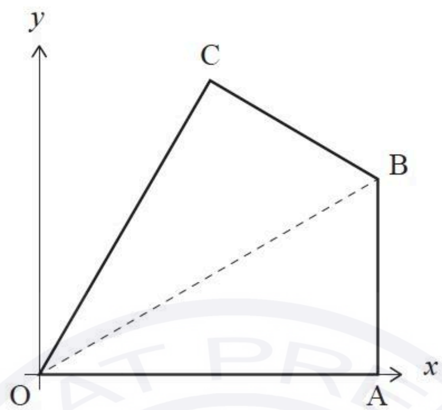
(ii) Hence, show that the perpendicular distance between $\Pi_1$ and $\Pi_2$ is $\sqrt{11}$. [7]

(e) Find the equation of a third parallel plane $\Pi_3$ which is also a perpendicular distance of $\sqrt{11}$ from $\Pi_1$. [4]

## Question 24

[Maximum mark: 7]

Quadrilateral OABC is shown on the following set of axes.




OABC is symmetrical about [OB].

A has coordinates $(6, 0)$ and C has coordinates $\left(3, 3\sqrt{3}\right)$.

(a) (i) Write down the coordinates of the midpoint of [AC].

(ii) Hence or otherwise, find the equation of the line passing through the points O and B. [4]

(b) Given that [OA] is perpendicular to [AB], find the area of the quadrilateral OABC. [3]

## Question 25

[Maximum mark: 4]

Solve $\tan(2x - 5°) = 1$ for $0° \le x \le 180°$.

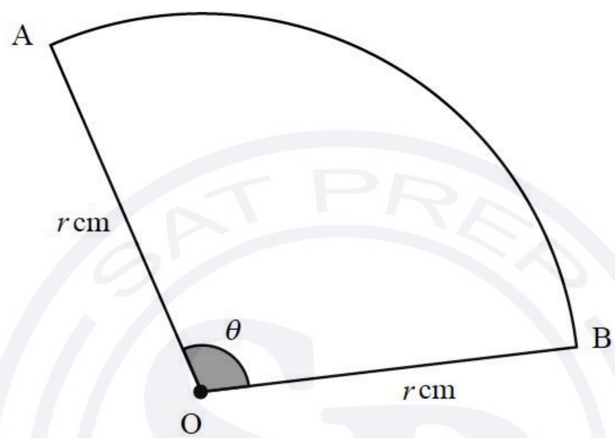
## Question 26

[Maximum mark: 8]

Points $\mathrm{A}$ and $\mathrm{B}$ lie on the circumference of a circle of radius $r\,\mathrm{cm}$ with centre at $\mathrm{O}$.

The sector $\mathrm{OAB}$ is shown on the following diagram. The angle $\mathrm{A\hat{O}B}$ is denoted as $\theta$ and is measured in radians.

**diagram not to scale**




The perimeter of the sector is $10\,\mathrm{cm}$ and the area of the sector is $6.25\,\mathrm{cm}^2$.

(a) Show that $4r^2 - 20r + 25 = 0$. [4]

(b) Hence, or otherwise, find the value of $r$ and the value of $\theta$. [4]

## Question 27

[Maximum mark: 9]

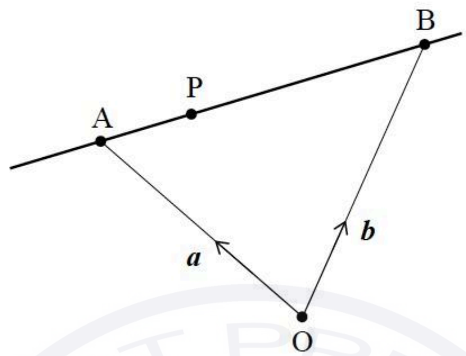
(a) Prove that $\tan\left(\theta - \frac{\pi}{4}\right) \equiv \frac{\sin 2\theta - 1}{\cos 2\theta}$, where $\theta \neq \frac{(2n+1)\pi}{4}$, $n \in \mathbb{Z}$. [6]

(b) Hence, or otherwise, solve $\frac{\sin x - 1}{\cos x} = \sqrt{3}$ for $0 \leq x \leq 2\pi$. [3]

## Question 28

[Maximum mark: 8]

The following diagram shows two points $\mathrm{A}$ and $\mathrm{B}$ such that $\overrightarrow{\mathrm{OA}} = \boldsymbol{a}$, $\overrightarrow{\mathrm{OB}} = \boldsymbol{b}$.




The point $\mathrm{P}$ lies on $(\mathrm{AB})$ so that $\overrightarrow{\mathrm{AP}} = \lambda\,\overrightarrow{\mathrm{AB}}$ where $0 < \lambda < 1$.

(a) Show that $\overrightarrow{\mathrm{OP}} = (1-\lambda)\boldsymbol{a} + \lambda\boldsymbol{b}$. [1]

It is given that $|\boldsymbol{a}| = 1$, $|\boldsymbol{b}| = 2$ and $\boldsymbol{a}\cdot\boldsymbol{b} = \frac{1}{4}$.

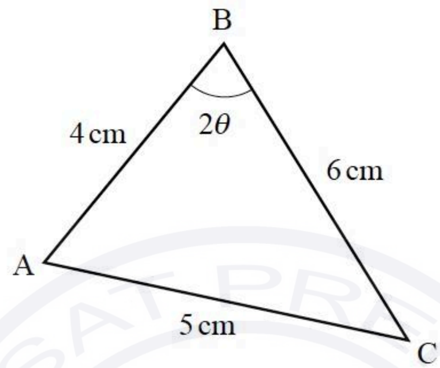
(b) In the case that $\overrightarrow{\mathrm{OP}}$ is perpendicular to $\overrightarrow{\mathrm{AB}}$, find the value of $\lambda$. [7]

## Question 29

[Maximum mark: 6]

The following diagram shows triangle ABC, where $AB = 4\text{ cm}$, $BC = 6\text{ cm}$, $AC = 5\text{ cm}$ and $A\hat{B}C = 2\theta$.

**diagram not to scale**




Find the exact value of $\cos\theta$, giving your answer in the form $\frac{p\sqrt{2}}{q}$, where $p, q \in \mathbb{Z}^+$.

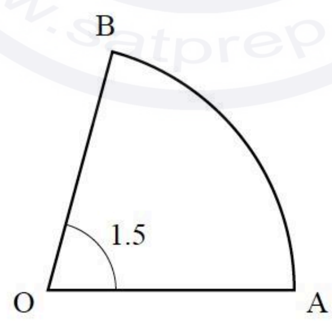
## Question 30

[Maximum mark: 5]

Points A and B lie on a circle with centre O and radius $r$ cm, where $A\hat{O}B = 1.5$ radians.

This is shown on the following diagram.

**diagram not to scale**




The area of sector OAB is $48\text{ cm}^2$.

(a) Find the value of $r$. [3]

(b) Hence, find the perimeter of sector OAB. [2]