Subject – Math AA(Higher Level)
Topic - Geometry and Trigonometry
Year - May 2021 – Nov 2024
Paper -1
Answers

Question 1

recognition that the angle between the normal and the line is $60°$ (seen anywhere) ***R1***

attempt to use the formula for the scalar product ***M1***

$$\cos 60° = \frac{\left|\begin{pmatrix}2\\1\\2\end{pmatrix}\bullet\begin{pmatrix}1\\-2\\p\end{pmatrix}\right|}{\sqrt{9}\times\sqrt{1+4+p^2}}$$ ***A1***

$$\frac{1}{2} = \frac{|2p|}{3\sqrt{5+p^2}}$$ ***A1***

$$3\sqrt{5+p^2} = 4|p|$$

attempt to square both sides ***M1***

$$9\left(5+p^2\right) = 16p^2 \Rightarrow 7p^2 = 45$$

$p = \pm 3\sqrt{\frac{5}{7}}$ (or equivalent) ***A1A1***

Total [7 marks]

Question 2

(a) (i) $\frac{-1+1}{2}=0=3-3$ **A1**

the point $(-1,0,3)$ lies on L_1. **AG**

(ii) attempt to set equal to a parameter or rearrange cartesian form ***(M1)***

$\frac{x+1}{2}=y=3-z=\lambda \Rightarrow x=2\lambda-1,\ y=\lambda,\ z=3-\lambda$ OR $\frac{x+1}{2}=\frac{y-0}{1}=\frac{z-3}{-1}$

correct direction vector $\begin{pmatrix}2\\1\\-1\end{pmatrix}$ or equivalent seen in vector form ***(A1)***

$r=\begin{pmatrix}-1\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$ (or equivalent) **A1**

Note: Award ***A0*** if $r=$ is omitted.

[4 marks]

(b) attempt to use the scalar product formula ***(M1)***

$\begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}a\\1\\-1\end{pmatrix}=(\pm)\sqrt{6}\sqrt{a^2+2}\cos 45^\circ$ ***(A1)(A1)***

Note: Award ***A1*** for LHS and ***A1*** for RHS

$2a+2=\frac{(\pm)\sqrt{6}\sqrt{a^2+2}\sqrt{2}}{2}\left(\Rightarrow 2a+2=(\pm)\sqrt{3}\sqrt{a^2+2}\right)$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS

$4a^2+8a+4=3a^2+6$

$a^2+8a-2=0$ ***A1***

attempt to solve their quadratic ***M1***

$a=\frac{-8\pm\sqrt{64+8}}{2}=\frac{-8\pm\sqrt{72}}{2}\left(=-4\pm3\sqrt{2}\right)$ ***A1***

[8 marks]

(c) **METHOD 1**

attempt to equate the parametric forms of L_1 and L_2 ***(M1)***

$$\begin{cases} 2\lambda - 1 = ta \\ \lambda = 1 + t \\ 3 - \lambda = 2 - t \end{cases}$$ ***A1***

attempt to solve equations by eliminating λ or t ***(M1)***

$2 + 2t - 1 = ta \Rightarrow 1 = t(a-2)$ or $2\lambda - 1 = (\lambda - 1)a \Rightarrow a - 1 = \lambda(a-2)$

Solutions exist unless $a - 2 = 0$

$k = 2$ ***A1***

Note: This ***A1*** is independent of the following marks.

$t = \dfrac{1}{a-2}$ or $\lambda = \dfrac{a-1}{a-2}$ ***A1***

A has coordinates $\left(\dfrac{a}{a-2}, 1 + \dfrac{1}{a-2}, 2 - \dfrac{1}{a-2}\right) \left(= \left(\dfrac{a}{a-2}, \dfrac{a-1}{a-2}, \dfrac{2a-5}{a-2}\right)\right)$ ***A2***

Note: Award ***A1*** for any two correct coordinates seen or final answer in vector form.

METHOD 2

no unique point of intersection implies direction vectors of L_1 and L_2 parallel

$k = 2$ ***A1***

Note: This ***A1*** is independent of the following marks.

attempt to equate the parametric forms of L_1 and L_2 ***(M1)***

$$\begin{cases} 2\lambda - 1 = ta \\ \lambda = 1 + t \\ 3 - \lambda = 2 - t \end{cases}$$ ***A1***

attempt to solve equations by eliminating λ or t ***(M1)***

$2 + 2t - 1 = ta \Rightarrow 1 = t(a-2)$ or $2\lambda - 1 = (\lambda - 1)a \Rightarrow a - 1 = \lambda(a-2)$

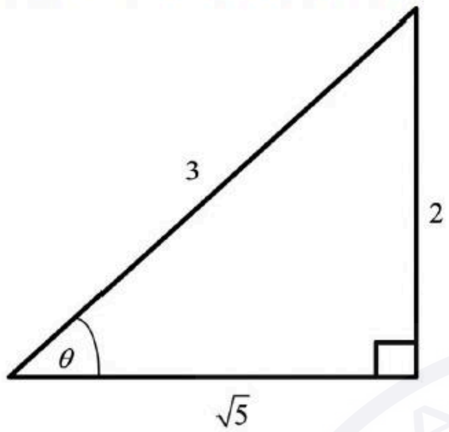
$t = \dfrac{1}{a-2}$ or $\lambda = \dfrac{a-1}{a-2}$ ***A1***

A has coordinates $\left(\dfrac{a}{a-2}, 1 + \dfrac{1}{a-2}, 2 - \dfrac{1}{a-2}\right) \left(= \left(\dfrac{a}{a-2}, \dfrac{a-1}{a-2}, \dfrac{2a-5}{a-2}\right)\right)$ ***A2***

Question 3

METHOD 1

attempt to use a right angled triangle **M1**



correct placement of all three values and θ seen in the triangle **(A1)**

$\cot\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) **R1**

$\cot\theta = -\dfrac{\sqrt{5}}{2}$ **A1**

Note: Award ***M1A1R0A0*** for $\cot\theta = \dfrac{\sqrt{5}}{2}$ seen as the final answer

The ***R1*** should be awarded independently for a negative value only given as a final answer.

[4 marks]

METHOD 2

Attempt to use $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ **M1**

$1 + \cot^2\theta = \dfrac{9}{4}$

$\cot^2\theta = \dfrac{5}{4}$ **(A1)**

$\cot\theta = \pm\dfrac{\sqrt{5}}{2}$

$\cot\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) **R1**

$\cot\theta = -\dfrac{\sqrt{5}}{2}$ **A1**

Note: Award ***M1A1R0A0*** for $\cot\theta = \dfrac{\sqrt{5}}{2}$ seen as the final answer

The ***R1*** should be awarded independently for a negative value only given as a final answer.

METHOD 3

$\sin\theta = \frac{2}{3}$

attempt to use $\sin^2\theta + \cos^2\theta = 1$ ***M1***

$\frac{4}{9} + \cos^2\theta = 1$

$\cos^2\theta = \frac{5}{9}$ ***(A1)***

$\cos\theta = \pm\frac{\sqrt{5}}{3}$

$\cos\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cos\theta = -\frac{\sqrt{5}}{3}$

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer

The ***R1*** should be awarded independently for a negative value only given as a final answer.

Question 4

(a)

Note: Do not award the final ***A1*** for proofs which work from both sides to find a common expression other than $2\sin x\cos x - 2\sin^2 x$.

METHOD 1 (LHS to RHS)

attempt to use double angle formula for $\sin 2x$ or $\cos 2x$ ***M1***

LHS= $2\sin x\cos x + \cos 2x - 1$ OR

$\sin 2x + 1 - 2\sin^2 x - 1$ OR

$2\sin x\cos x + 1 - 2\sin^2 x - 1$

$= 2\sin x\cos x - 2\sin^2 x$ ***A1***

$\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$ =RHS ***AG***

METHOD 2 (RHS to LHS)

RHS $= 2\sin x\cos x - 2\sin^2 x$

attempt to use double angle formula for $\sin 2x$ or $\cos 2x$ ***M1***

$= \sin 2x + 1 - 2\sin^2 x - 1$ ***A1***

$= \sin 2x + \cos 2x - 1 =$ LHS ***AG***

[2 marks]

(b) attempt to factorise ***M1***

$(\cos x - \sin x)(2\sin x + 1) = 0$ ***A1***

recognition of $\cos x = \sin x \Rightarrow \frac{\sin x}{\cos x} = \tan x = 1$ OR $\sin x = -\frac{1}{2}$ ***(M1)***

one correct reference angle seen anywhere, accept degrees ***(A1)***

$\frac{\pi}{4}$ OR $\frac{\pi}{6}$ (accept $-\frac{\pi}{6}, \frac{7\pi}{6}$)

Note: This ***(M1)(A1)*** is independent of the previous ***M1A1.***

$x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{4}, \frac{5\pi}{4}$ ***A2***

Note: Award ***A1*** for any two correct (radian) answers.
Award ***A1A0*** if additional values given with the four correct (radian) answers. Award ***A1A0*** for four correct answers given in degrees.

[6 marks]

Total [8 marks]

Question 5

Continued from function

(b) let $\alpha = \arctan p$ and $\beta = \arctan q$ ***M1***

$p = \tan\alpha$ and $q = \tan\beta$ ***(A1)***

$\tan(\alpha+\beta) = \dfrac{p+q}{1-pq}$ ***A1***

$\alpha+\beta = \arctan\left(\dfrac{p+q}{1-pq}\right)$ ***A1***

so $\arctan p + \arctan q \equiv \arctan\left(\dfrac{p+q}{1-pq}\right)$ where $p,q>0$ and $pq<1$ ***AG***

[4 marks]

(c)

$\dfrac{\pi}{4} = \arctan 1$ (or equivalent) ***A1***

$\arctan\left(\dfrac{x}{x+1}\right) + \arctan 1 = \arctan\left(\dfrac{\dfrac{x}{x+1}+1}{1-\dfrac{x}{x+1}(1)}\right)$ ***A1***

$= \arctan\left(\dfrac{\dfrac{x+x+1}{x+1}}{\dfrac{x+1-x}{x+1}}\right)$ ***A1***

$= \arctan(2x+1)$ ***AG***

[3 marks]

Question 6

(a)

setting at least two components of l_1 and l_2 equal **M1**

$$3+2\lambda=2+\mu \quad (1)$$
$$2-2\lambda=-\mu \quad (2)$$
$$-1+2\lambda=4+\mu \quad (3)$$

attempt to solve two of the equations eg. adding (1) and (2) **M1**

gives a contradiction (no solution), eg $5=2$ **R1**

so l_1 and l_2 do not intersect **AG**

(b)

l_1 and l_2 are parallel (as $\begin{pmatrix}2\\-2\\2\end{pmatrix}$ is a multiple of $\begin{pmatrix}1\\-1\\1\end{pmatrix}$)

let A be $(3,2,-1)$ on l_1 and let B be $(2,0,4)$ on l_2

Attempt to find vector $\vec{\text{AB}}\left(=\begin{pmatrix}-1\\-2\\5\end{pmatrix}\right)$ ***(M1)***

Distance required is $\dfrac{\left|\boldsymbol{v}\times\vec{\text{AB}}\right|}{|\boldsymbol{v}|}$ ***M1***

$$=\frac{1}{\sqrt{3}}\left|\begin{pmatrix}1\\-1\\1\end{pmatrix}\times\begin{pmatrix}-1\\-2\\5\end{pmatrix}\right|$$ ***(A1)***

$$=\frac{1}{\sqrt{3}}\left|\begin{pmatrix}3\\6\\3\end{pmatrix}\right|$$ ***A1***

minimum distance is $\sqrt{18}\left(=3\sqrt{2}\right)$ ***A1***

[5 marks]

OR

let A be $(3,2,-1)$ on l_1 and let B be $(2+\mu,-\mu,4+\mu)$ on l_2 ***(M1)***

(or let A be $(2,0,4)$ on l_2 and let B be $(3+2\lambda,2-2\lambda,-1+2\lambda)$ on l_1)

$\vec{AB}=\begin{pmatrix}-1\\-2\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\1\end{pmatrix}$ $(\mu\in\mathbb{R})$ (or $\vec{AB}=\begin{pmatrix}2\lambda+1\\-2\lambda+2\\2\lambda-5\end{pmatrix}$) ***A1***

$\begin{pmatrix}\mu-1\\-\mu-2\\\mu+5\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$ (or $\begin{pmatrix}2\lambda+1\\-2\lambda+2\\2\lambda-5\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$) ***M1***

$\mu=-2$ or $\lambda=1$ ***A1***

minimum distance is $\sqrt{18}\left(=3\sqrt{2}\right)$ ***A1***

[5 marks]
Total [8 marks]

Question 7

METHOD 1

attempt to use the cosine rule to find the value of x ***(M1)***

$$100 = x^2 + 4x^2 - 2(x)(2x)\left(\frac{3}{4}\right)$$ ***A1***

$$2x^2 = 100$$

$x^2 = 50$ OR $x = \sqrt{50}\ \left(= 5\sqrt{2}\right)$ ***A1***

attempt to find $\sin \hat{C}$ (seen anywhere) ***(M1)***

$\sin^2 \hat{C} + \left(\frac{3}{4}\right)^2 = 1$ or $x^2 + 3^2 = 4^2$ OR right triangle with side 3 and hypotenuse 4

$$\sin \hat{C} = \frac{\sqrt{7}}{4}$$ ***(A1)***

Note: The marks for finding $\sin \hat{C}$ may be awarded independently of the first three marks for finding x.

correct substitution into the area formula using their value of x (or x^2) and their value of $\sin \hat{C}$ ***(M1)***

$A = \frac{1}{2} \times 5\sqrt{2} \times 10\sqrt{2} \times \frac{\sqrt{7}}{4}$ or $A = \frac{1}{2} \times \sqrt{50} \times 2\sqrt{50} \times \frac{\sqrt{7}}{4}$

$$A = \frac{25\sqrt{7}}{2}$$ ***A1***

METHOD 2

attempt to find the height, h, of the triangle in terms of x ***(M1)***

$h^2+\left(\frac{3}{4}x\right)^2=x^2$ OR $h^2+\left(\frac{5}{4}x\right)^2=10^2$ OR $h=\frac{\sqrt{7}}{4}x$ ***A1***

equating their expressions for either h^2 or h ***(M1)***

$x^2-\left(\frac{3}{4}x\right)^2=10^2-\left(\frac{5}{4}x\right)^2$ OR $\sqrt{100-\frac{25}{16}x^2}=\frac{\sqrt{7}}{4}x$ (or equivalent) ***A1***

$x^2=50$ OR $x=\sqrt{50}$ $\left(=5\sqrt{2}\right)$ ***A1***

correct substitution into the area formula using their value of x (or x^2) ***(M1)***

$A=\frac{1}{2}\times 2\sqrt{50}\times\frac{\sqrt{7}}{4}\sqrt{50}$ OR $A=\frac{1}{2}\left(2\times 5\sqrt{2}\right)\left(\frac{\sqrt{7}}{4}5\sqrt{2}\right)$

$A=\frac{25\sqrt{7}}{2}$ ***A1***

Total [7 marks]

Question 8

METHOD 1

use of $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$ on the LHS **(M1)**

$|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \sin^2\theta$ **A1**

$= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 (1 - \cos^2\theta)$ **M1**

$= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \cos^2\theta$ OR $= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (|\boldsymbol{a}||\boldsymbol{b}|\cos\theta)^2$ **A1**

$= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$ **AG**

METHOD 2

use of $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ on the RHS **(M1)**

$= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \cos^2\theta$ **A1**

$= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 (1 - \cos^2\theta)$ **M1**

$= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \sin^2\theta$ OR $= (|\boldsymbol{a}||\boldsymbol{b}|\sin\theta)^2$ **A1**

$= |\boldsymbol{a} \times \boldsymbol{b}|^2$ **AG**

Question 9

attempt to use $\cos^2 x = 1 - \sin^2 x$ ***M1***

$2\sin^2 x - 5\sin x + 2 = 0$ ***A1***

EITHER

attempting to factorise ***M1***

$(2\sin x - 1)(\sin x - 2)$ ***A1***

OR

attempting to use the quadratic formula ***M1***

$\sin x = \frac{5 \pm \sqrt{5^2 - 4 \times 2 \times 2}}{4} \left(= \frac{5 \pm 3}{4} \right)$ ***A1***

THEN

$\sin x = \frac{1}{2}$ ***(A1)***

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ ***A1A1***

Total [7 marks]

Question 10

(a) **METHOD 1**

attempt to write all LHS terms over a common denominator of $x-1$ ***(M1)***

$$2x-3-\frac{6}{x-1}=\frac{2x(x-1)-3(x-1)-6}{x-1} \text{ OR } \frac{(2x-3)(x-1)}{x-1}-\frac{6}{x-1}$$

$$=\frac{2x^2-2x-3x+3-6}{x-1} \text{ OR } \frac{2x^2-5x+3}{x-1}-\frac{6}{x-1}$$ ***A1***

$$=\frac{2x^2-5x-3}{x-1}$$ ***AG***

[2 marks]

METHOD 2

attempt to use algebraic division on RHS ***(M1)***

correctly obtains quotient of $2x-3$ and remainder -6 ***A1***

$=2x-3-\frac{6}{x-1}$ as required. ***AG***

[2 marks]

(b) consider the equation $\frac{2\sin^2 2\theta - 5\sin 2\theta - 3}{\sin 2\theta - 1} = 0$ ***(M1)***

$\Rightarrow 2\sin^2 2\theta - 5\sin 2\theta - 3 = 0$

EITHER

attempt to factorise in the form $(2\sin 2\theta + a)(\sin 2\theta + b)$ ***(M1)***

Note: Accept any variable in place of $\sin 2\theta$.

$(2\sin 2\theta + 1)(\sin 2\theta - 3) = 0$

OR

attempt to substitute into quadratic formula ***(M1)***

$\sin 2\theta = \frac{5 \pm \sqrt{49}}{4}$

THEN

$\sin 2\theta = -\frac{1}{2}$ or $\sin 2\theta = 3$ ***(A1)***

Note: Award ***A1*** for $\sin 2\theta = -\frac{1}{2}$ only.

one of $\frac{7\pi}{6}$ OR $\frac{11\pi}{6}$ (accept 210 or 330) ***(A1)***

$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}$ (must be in radians) ***A1***

Note: Award ***A0*** if additional answers given.

[5 marks]

Total [7 marks]

Question 11

(a)

vector product of the two normals $= \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$ (or equivalent) ***A1***

$\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** if "$r =$" is missing. Subsequent marks may still be awarded.

Attempt to substitute $(1+\lambda, -2+5\lambda, 3\lambda)$ in Π_3 ***M1***

$-9(1+\lambda)+3(-2+5\lambda)-2(3\lambda)=32$

$-15=32$, a contradiction ***R1***

hence the three planes do not intersect ***AG***

[4 marks]

(b) (i) $\Pi_1: 2+2+0=4$ and $\Pi_2: 1+4+0=5$ ***A1***

(ii) **METHOD 1**

attempt to find the vector product of the two normals ***M1***

$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$ ***A1***

$\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ ***A1A1***

METHOD 2

attempt to eliminate a variable from $\prod_1$ and $\prod_2$ **M1**

$3x - z = 3$ OR $3y - 5z = -6$ OR $5x - y = 7$

Let $x = t$

substituting $x = t$ in $3x - z = 3$ to obtain

$z = -3 + 3t$ and $y = 5t - 7$ (for all three variables in parametric form) **A1**

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ -7 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$ **A1A1**

(c) **METHOD 1**

the line connecting L and $\prod_3$ is given by L_1

attempt to substitute position and direction vector to form L_1 ***(M1)***

$$\boldsymbol{s} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix}$$ **A1**

substitute $(1 - 9t, -2 + 3t, -2t)$ in $\prod_3$ **M1**

$-9(1 - 9t) + 3(-2 + 3t) - 2(-2t) = 32$

$94t = 47 \Rightarrow t = \frac{1}{2}$ **A1**

attempt to find distance between $(1, -2, 0)$ and their point $\left(-\frac{7}{2}, -\frac{1}{2}, -1\right)$ ***(M1)***

$$= \left| \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right| = \frac{1}{2}\sqrt{(-9)^2 + 3^2 + (-2)^2}$$

$$= \frac{\sqrt{94}}{2}$$ **A1**

[6 marks]

METHOD 2

unit normal vector equation of $\prod_3$ is given by $\dfrac{\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}}{\sqrt{81+9+4}}$ ***(M1)***

$=\dfrac{32}{\sqrt{94}}$ ***A1***

let $\prod_4$ be the plane parallel to $\prod_3$ and passing through P,

then the normal vector equation of $\prod_4$ is given by

$\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\0\end{pmatrix}=-15$ ***M1***

unit normal vector equation of $\prod_4$ is given by

$\dfrac{\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}}{\sqrt{81+9+4}}=\dfrac{-15}{\sqrt{94}}$ ***A1***

distance between the planes is $\dfrac{32}{\sqrt{94}}-\dfrac{-15}{\sqrt{94}}$ ***(M1)***

$=\dfrac{47}{\sqrt{94}}\left(=\dfrac{\sqrt{94}}{2}\right)$ ***A1***

[6 marks]

Total [15 marks]

Question 12

(a) $(f \circ g)(x) = f(2x)$ **(A1)**

$f(2x) = \sqrt{3}\sin 2x + \cos 2x$ **A1**

[2 marks]

(b) $\sqrt{3}\sin 2x + \cos 2x = 2\cos 2x$

$\sqrt{3}\sin 2x = \cos 2x$

recognizing to use $\tan$ or $\cot$ **M1**

$\tan 2x = \frac{1}{\sqrt{3}}$ OR $\cot 2x = \sqrt{3}$ (values may be seen in right triangle) **(A1)**

$\left(\arctan\left(\frac{1}{\sqrt{3}}\right) =\right) \frac{\pi}{6}$ (seen anywhere) (accept degrees) **(A1)**

$2x = \frac{\pi}{6}, \frac{7\pi}{6}$

$x = \frac{\pi}{12}, \frac{7\pi}{12}$ **A1A1**

Question 13

determines $\frac{\pi}{4}$ (or $45°$) as the first quadrant (reference) angle **(A1)**

attempts to solve $\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}$ **(M1)**

Note: Award ***M1*** for attempting to solve $\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4},\frac{7\pi}{4}(,\dots)$

$\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}\Rightarrow x<0$ and so $\frac{\pi}{4}$ is rejected **(R1)**

$\frac{x}{2}+\frac{\pi}{3}=2\pi-\frac{\pi}{4}\left(=\frac{7\pi}{4}\right)$ **A1**

$x=\frac{17\pi}{6}$ (must be in radians) **A1**

[5 marks]

Question 14

(a) $\cos^2 x-3\sin^2 x=0$

valid attempt to reduce equation to one involving one trigonometric function **(M1)**

$\frac{\sin^2 x}{\cos^2 x}=\frac{1}{3}$ OR $1-\sin^2 x-3\sin^2 x=0$ OR $\cos^2 x-3\left(1-\cos^2 x\right)=0$

OR $\cos 2x-1+\cos 2x=0$

correct equation **(A1)**

$\tan^2 x=\frac{1}{3}$ OR $\cos^2 x=\frac{3}{4}$ OR $\sin^2 x=\frac{1}{4}$ OR $\cos 2x=\frac{1}{2}$

$\tan x=\pm\frac{1}{\sqrt{3}}$ OR $\cos x=\pm\frac{\sqrt{3}}{2}$ OR $\sin x=(\pm)\frac{1}{2}$ OR $2x=\frac{\pi}{3}\left(,\frac{5\pi}{3}\right)$ **(A1)**

$x=\frac{\pi}{6},\ x=\frac{5\pi}{6}$ **A1A1**

(b) (i) attempt to use the chain rule (may be evidenced by at least one $\cos x \sin x$ term) ***(M1)***

$f'(x) = -2\cos x \sin x - 6\sin x \cos x \left(= -8\sin x \cos x = -4\sin 2x\right)$ ***A1***

(ii) valid attempt to solve their $f'(x) = 0$ ***(M1)***

at least 2 correct x-coordinates (may be seen in coordinates) ***(A1)***

$x = 0,\ x = \frac{\pi}{2},\ x = \pi$

Note:	Accept additional correct solutions outside the domain. Award ***A0*** if any additional incorrect solutions are given.

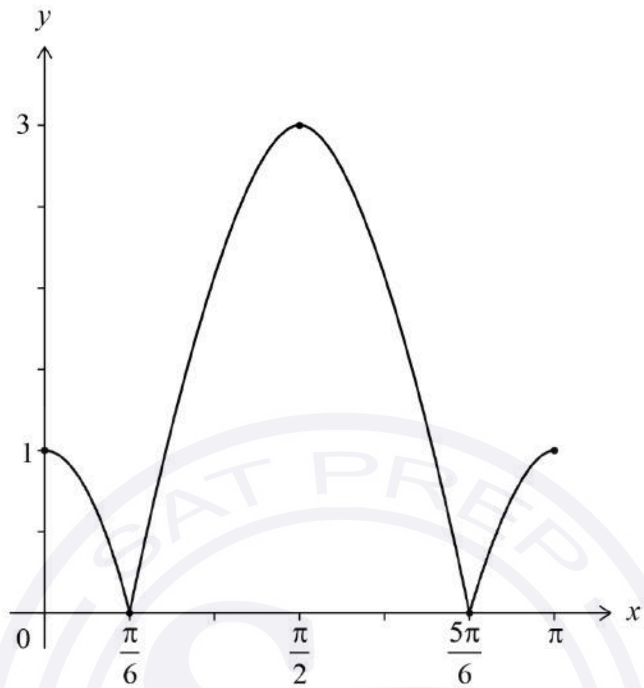
correct coordinates (may be seen in graph for part (c)) ***A1A1A1***

$(0,1),\ (\pi,1),\ \left(\frac{\pi}{2},-3\right)$

Note:	Award a maximum of ***M1A1A1A1A0*** if any additional solutions are given.
Note:	If candidates do not find at least two correct x-coordinates, it is possible to award the appropriate final marks for their correct coordinates, such as ***M1A0A0A1A0.***

[7 marks]

(c)



attempt to reflect the negative part of the graph of f in the x-axis ***M1***

endpoints have coordinates $(0,1)$, $(\pi,1)$ ***A1***

smooth maximum at $\left(\frac{\pi}{2},3\right)$ ***A1***

sharp points (cusps) at x-intercepts $\frac{\pi}{6}$, $\frac{5\pi}{6}$ ***A1***

[4 marks]

(d) considers points of intersection of $y=\left|f(x)\right|$ and $y=1$ on graph or algebraically ***(M1)***

$-\left(\cos^2 x-3\sin^2 x\right)=1$ or $-\left(1-4\sin^2 x\right)=1$ or $-\left(4\cos^2 x-3\right)=1$ or $-\left(2\cos 2x-1\right)=1$

$\tan^2 x=1$ or $\sin^2 x=\frac{1}{2}$ or $\cos^2 x=\frac{1}{2}$ or $\cos 2x=0$ ***(A1)***

$x=\frac{\pi}{4},\frac{3\pi}{4}$ ***(A1)***

For $\left|f(x)\right|>1$

$\frac{\pi}{4}<x<\frac{3\pi}{4}$ ***A1***

[4 marks]

Total [20 marks]

Question 15

(a)

Note: Award a maximum of ***M1A0A0*** if the candidate manipulates both sides of the equation (such as moving terms from one side to the other).

METHOD 1 (working with LHS)

attempting to expand $\left(a^2-1\right)^2$ (do not accept a^4+1 or a^4-1) ***(M1)***

$\text{LHS}=a^2+\frac{a^4-2a^2+1}{4}$ or $\frac{4a^2+a^4-2a^2+1}{4}$ ***A1***

$=\frac{a^4+2a^2+1}{4}$ ***A1***

$=\left(\frac{a^2+1}{2}\right)^2$ $(=\text{RHS})$ ***AG***

Note: Do not award the final A1 if further working contradicts the AG.

METHOD 2 (working with RHS)

attempting to expand $\left(a^2+1\right)^2$ ***(M1)***

$\text{RHS}=\frac{a^4+2a^2+1}{4}$

$=\frac{4a^2+a^4-2a^2+1}{4}$ ***A1***

$=a^2+\frac{a^4-2a^2+1}{4}$ ***A1***

$=a^2+\left(\frac{a^2-1}{2}\right)^2$ (=LHS) ***AG***

Note: Do not award the final A1 if further working contradicts the AG.

[3 marks]

(b) recognise base and height as a and $\left(\frac{a^2-1}{2}\right)$ (may be seen in diagram) ***(M1)***

correct substitution into triangle area formula ***A1***

$$\text{Area} = \frac{a}{2}\left(\frac{a^2-1}{2}\right) \text{ (or equivalent) } \left(= \frac{a\left(a^2-1\right)}{4} = \frac{a^3-a}{4}\right)$$

[2 marks]

Total [5 marks]

Question 16

(a) (i) attempt to find midpoint of A and B ***(M1)***

centre $(-1,3,-2)$ (accept vector notation and/or missing brackets) ***A1***

(ii) attempt to find AB or half of AB or distance between the centre and A (or B) ***(M1)***

$$\frac{\sqrt{4^2+2^2+4^2}}{2} \text{ or } \sqrt{2^2+1^2+2^2}$$

$=3$ ***A1***

[4 marks]

(b) attempt to find the distance between their centre and V
(the perpendicular height of the cone) ***(M1)***

$\sqrt{0^2+4^2+2^2}$ OR $\sqrt{(\text{their slant height})^2-(\text{their radius})^2}$

$=\sqrt{20}\left(=2\sqrt{5}\right)$ ***(A1)***

Volume $=\frac{1}{3}\pi 3^2\sqrt{20}$

$=3\pi\sqrt{20}\left(=6\pi\sqrt{5}\right)$ ***A1***

[3 marks]

Total [7 marks]

Question 17

(a) $2t+1\times 0+0\times(3+t)\quad(=2t)$ (seen anywhere) ***(A1)***

one correct magnitude $\sqrt{1^2+1^2+0^2}, \sqrt{(2t)^2+(3+t)^2}$ ***(A1)***

correct substitution of their magnitudes and scalar product ***M1***

$2t=\sqrt{2}\times\sqrt{(2t)^2+(3+t)^2}\times\cos\frac{\pi}{3}$ OR $\cos\frac{\pi}{3}=\frac{2t}{\sqrt{2}\times\sqrt{5t^2+6t+9}}$

$4t=\sqrt{2\left(4t^2+9+6t+t^2\right)}$ OR $\frac{1}{2}=\frac{2t}{\sqrt{2\left(5t^2+6t+9\right)}}$ (or equivalent) ***A1***

$4t=\sqrt{10t^2+12t+18}$ ***AG***

[4 marks]

(b) correct quadratic equation ***A1***

$16t^2=10t^2+12t+18,\ 6t^2-12t-18=0\ ,\ t^2-2t-3=0$

valid attempt to solve their quadratic set =0 ***(M1)***

$(t+1)(t-3)$ OR $\frac{12\pm\sqrt{(-12)^2-4\times 6\times(-18)}}{12}$ OR $(t-1)^2-4$ ***(A1)***

$t=3$ ***A1***

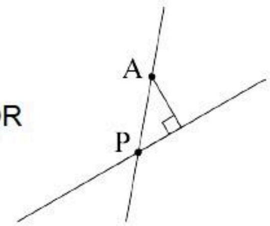
Note: Award ***A0*** if additional answer(s) given.

[4 marks]

continue…

(c) **METHOD 1**

recognizing shortest distance from A is perpendicular to L_1 *(M1)*

OR  $\sin\frac{\pi}{3} = \frac{x}{\left|\overrightarrow{PA}\right|}$

$\left|\overrightarrow{PA}\right| = \sqrt{6^2+6^2}$ $\left(=\sqrt{72},\ 6\sqrt{2}\right)$ (seen anywhere) *(A1)*

$\frac{\sqrt{3}}{2} = \frac{x}{\sqrt{72}}$ *(A1)*

$x = \frac{\sqrt{216}}{2}$ $\left(=\sqrt{54}\ ,\ 3\sqrt{6}\right)$

shortest distance is $\frac{\sqrt{216}}{2}\left(=\sqrt{54}\ ,\ 3\sqrt{6}\right)$ **A1**

[4 marks]

METHOD 2

recognition that the distance required is $\frac{\left|\boldsymbol{v}\times\overrightarrow{PA}\right|}{|\boldsymbol{v}|}$ *(M1)*

$= \frac{1}{\sqrt{2}}\left|\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}6\\0\\6\end{pmatrix}\right|$ *(A1)*

$= \frac{1}{\sqrt{2}}\left|\begin{pmatrix}6\\-6\\-6\end{pmatrix}\right|$ *(A1)*

shortest distance is $\sqrt{54}\ \left(=3\sqrt{6}\right)$ **A1**

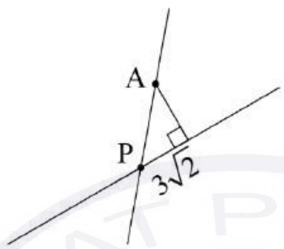
[4 marks]

METHOD 3

recognition that the base of the triangle is $\dfrac{\left|\boldsymbol{v}\cdot\overrightarrow{\mathrm{PA}}\right|}{|\boldsymbol{v}|}$ *(M1)*

$$\frac{1}{\sqrt{2}}\left|\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}6\\0\\6\end{pmatrix}\right|$$

$=\dfrac{6}{\sqrt{2}}\left(=3\sqrt{2}\right)$ OR



(A1)

$\left|\overrightarrow{\mathrm{PA}}\right|=\sqrt{6^2+6^2}\quad\left(=\sqrt{72},\ 6\sqrt{2}\right)$ (seen anywhere) *(A1)*

Note: The value of $\left|\overrightarrow{\mathrm{PA}}\right|=\sqrt{6^2+6^2}$ may be seen as part of the working of their shortest distance, $d=\sqrt{\left|\overrightarrow{\mathrm{PA}}\right|^2-b^2}=\sqrt{\left(\sqrt{72}\right)^2-\left(3\sqrt{2}\right)^2}$

shortest distance is $\sqrt{54}\left(=3\sqrt{6}\right)$ **A1**

[4 marks]

METHOD 4

Let B be a general point on L_1 $(\lambda, 8+\lambda, -3)$ such that AB is perpendicular to L_1

attempt to find vector $\overrightarrow{\mathrm{AB}}$ OR $\left|\overrightarrow{\mathrm{AB}}\right|$ (the shortest distance from A to L_1) ***(M1)***

$$\overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{OB}} - \overrightarrow{\mathrm{OA}} = \overrightarrow{\mathrm{OP}} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} - \overrightarrow{\mathrm{OA}}\left(=\begin{pmatrix}0\\8\\-3\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}-\begin{pmatrix}6\\8\\3\end{pmatrix}=\overrightarrow{\mathrm{AP}}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}\right)\ (\lambda\in\mathbb{R})$$

$$\overrightarrow{\mathrm{AB}} = \begin{pmatrix}-6\\0\\-6\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix} \text{ OR } \left|\overrightarrow{\mathrm{AB}}\right| = \sqrt{(\lambda-6)^2+(8+\lambda-8)^2+(-3-3)^2}$$ ***A1***

$$\left|\overrightarrow{\mathrm{AB}}\right| = \sqrt{(\lambda-6)^2+\lambda^2+(-6)^2}\ \left(=\sqrt{2\lambda^2-12\lambda+72}\right)$$

EITHER

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\left|\overrightarrow{\mathrm{AB}}\right|^2\right)=0 \Rightarrow 4\lambda-12=0 \Rightarrow \lambda=3$$ ***A1***

OR

$\left|\overrightarrow{\mathrm{AB}}\right| = \sqrt{2(\lambda-3)^2+54}$ to obtain $\lambda = 3$ ***A1***

OR

$$\begin{pmatrix}-6+\lambda\\\lambda\\-6\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}=0 \Rightarrow -6+\lambda+\lambda=0 \Rightarrow \lambda=3$$ ***A1***

THEN

shortest distance is $\sqrt{54}\ \left(=3\sqrt{6}\right)$ ***A1***

[4 marks]

(d) attempt to find the vector product of two direction vectors *(M1)*

$$\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}6\\0\\6\end{pmatrix}$$

$$\boldsymbol{n}=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$$ (or any scalar multiple of this) (accept $\boldsymbol{n} = <1, -1, -1>$ or equivalent) ***A1***

Note: Award ***A0*** for a final answer given in coordinate form.

[2 marks]

Continue …

(e) substituting their x into volume formula and equating **(M1)**

$$\frac{1}{3}\pi\left(3\sqrt{6}\right)^2 h = 90\sqrt{3}\pi$$

$h = 5\sqrt{3}$ (seen anywhere) **A1**

recognition that the position vector of vertex is given by $\vec{OA} + \mu\boldsymbol{n}$ OR $\vec{OA} + h \times \hat{\boldsymbol{n}}$ **(M1)**

$$\begin{pmatrix}6\\8\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\-1\end{pmatrix} \text{ OR } (6+\mu, 8-\mu, 3-\mu)$$

EITHER

recognition that $\mu|\boldsymbol{n}| = h$ (where μ is a parameter) **(M1)**

$\mu|\boldsymbol{n}| = 5\sqrt{3}$ OR $\sqrt{\mu^2 + (-\mu)^2 + (-\mu)^2} = 5\sqrt{3}$ OR $3\mu^2 = 75$ $\left(\Rightarrow \sqrt{3}\mu = 5\sqrt{3}\right)$

$\mu = \pm 5$ (accept $\mu = 5$) **(A1)**

OR

attempt to find cone's height vector $h \times \hat{\boldsymbol{n}}$ **(M1)**

$$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\-1\end{pmatrix}$$ **(A1)**

$$5\sqrt{3} \times \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\-1\end{pmatrix}$$

THEN

$$= \begin{pmatrix}6\\8\\3\end{pmatrix} \pm 5\begin{pmatrix}1\\-1\\-1\end{pmatrix}\left(= \begin{pmatrix}6\\8\\3\end{pmatrix} \pm \begin{pmatrix}5\\-5\\-5\end{pmatrix}\right)$$

vertex = $(11, 3, -2)$ and $(1, 13, 8)$ (accept position vectors) **A1A1**

Note: Award a maximum of ***(M0)A0(M1)(M1)(A1)A1A1FT*** for $\left(\frac{39}{4}, \frac{17}{4}, -\frac{3}{4}\right)$ and $\left(\frac{9}{4}, \frac{47}{4}, \frac{27}{4}\right)$ obtained using $x = \left|\vec{PA}\right|$ from part (c).

[7 marks]

Total [21 marks]

Question 18

(a) recognition that period is $4m$ OR substitution of a point on f (except the origin) ***(M1)***

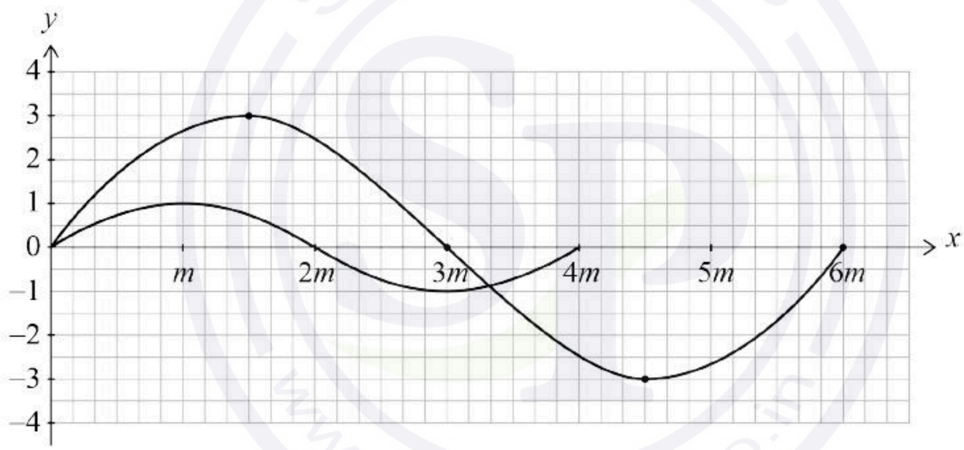
$4m = \frac{2\pi}{q}$ OR $1 = \sin qm$

$m = \frac{\pi}{2q}$ ***A1***

[2 marks]

(b) horizontal scale factor is $\frac{3}{2}$ (seen anywhere) ***(A1)***

Note: This ***(A1)*** may be earned by seeing a period of $6m$, half period of $3m$ or the correct x-coordinate of the maximum/minimum point.



A1A1A1

Note: Curve must be an approximate sinusoidal shape (sine or cosine).
Only in this case, award the following:
A1 for correct amplitude.
A1 for correct domain.
A1 for correct max and min points **and** correct x-intercepts.

[4 marks]

Total [6 marks]

Question 19

$1-2\sin^2 x=\sin x$ **_A1_**

$2\sin^2 x+\sin x-1=0$

valid attempt to solve quadratic **_(M1)_**

$(2\sin x-1)(\sin x+1)$ OR $\frac{-1\pm\sqrt{1-4(2)(-1)}}{2(2)}$

recognition to solve for $\sin x$ **_(M1)_**

$\sin x=\frac{1}{2}$ OR $\sin x=-1$

any correct solution from $\sin x=-1$ **_A1_**

any correct solution from $\sin x=\frac{1}{2}$ **_A1_**

Note: The previous two marks may be awarded for degree or radian values, irrespective of domain.

$x=-\frac{\pi}{2},\frac{\pi}{6},\frac{5\pi}{6}$ **_A1_**

Note: If no working shown, award no marks for a final value(s).
Award ***A0*** for $-\frac{\pi}{2},\frac{\pi}{6},\frac{5\pi}{6}$ if additional values also given.

Total [6 marks]

Question 20

(a) $\overrightarrow{OM} = \boldsymbol{a} + k\boldsymbol{c}$ ***A1***

$\overrightarrow{MC} = (1-k)\boldsymbol{c} - \boldsymbol{a}$ ***A1***

[2 marks]

(b) attempts to expand their dot product $\overrightarrow{OM} \cdot \overrightarrow{MC} = (\boldsymbol{a} + k\boldsymbol{c}) \cdot ((1-k)\boldsymbol{c} - \boldsymbol{a})$ ***M1***

$= (1-2k)(\boldsymbol{a} \cdot \boldsymbol{c}) - |\boldsymbol{a}|^2 + k(1-k)|\boldsymbol{c}|^2$ (or equivalent)

uses $|\boldsymbol{c}| = 2|\boldsymbol{a}|$ and $\boldsymbol{a} \cdot \boldsymbol{c} = 2|\boldsymbol{a}|^2 \cos\theta$ ***M1***

$= 2(1-2k)|\boldsymbol{a}|^2 \cos\theta - |\boldsymbol{a}|^2 + 4k(1-k)|\boldsymbol{a}|^2$

$= 2(1-2k)|\boldsymbol{a}|^2 \cos\theta - (1-2k)^2 |\boldsymbol{a}|^2$ ***A1***

$|\boldsymbol{a}|^2 (1-2k)(2\cos\theta - (1-2k)) = 0$ ***AG***

[3 marks]

(c) attempts to solve $|\boldsymbol{a}|^2 (1-2k)(2\cos\theta - (1-2k)) = 0$ for k ***(M1)***

$k = \frac{1}{2}$ or $k = \frac{1}{2} - \cos\theta$ $\left(|\boldsymbol{a}|^2 > 0\right)$

Note: Award ***(M1)*** for their '$k =$' or their '$\cos\theta =$'. For example, $\cos\theta = \frac{1-2k}{2}$ or equivalent.

as $0 \le k \le 1$, $0 \le \frac{1}{2} - \cos\theta \le 1$

$-\frac{1}{2} \le \cos\theta \le \frac{1}{2}$ ***A1***

$\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$, $\theta \ne \frac{\pi}{2}$ ***A1A1***

($\theta = \frac{\pi}{2}$ corresponds to only one possible position for M when $k = \frac{1}{2}$)

[4 marks]

Total [9 marks]

Question 21

(a) attempts to find perimeter *(M1)*

$\text{arc} + 2 \times \text{radius}$ OR $10 + 4 + 4$

$= 18$ (cm) ***A1***

[2 marks]

(b) $10 = 4\theta$ *(A1)*

$\theta = \frac{10}{4} \left(= \frac{5}{2}, 2.5\right)$ ***A1***

[2 marks]

(c) area $= \frac{1}{2}\left(\frac{10}{4}\right)(4^2)$ $(= 1.25 \times 16)$ *(A1)*

$= 20$ (cm^2) ***A1***

[2 marks]

Total [6 marks]

Question 22

METHOD 1

EITHER

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\left(\sqrt{5^2-1^2}=\right)\sqrt{24}$ ***(A1)***

OR

attempt to use the Pythagorean identity $\cos^2\alpha+\sin^2\alpha=1$ ***(M1)***

$\sin^2 \mathrm{B\hat{A}C}=1-\left(\frac{1}{5}\right)^2$ ***(A1)***

THEN

$\sin \mathrm{B\hat{A}C}=\frac{\sqrt{24}}{5}$ (may be seen in area formula) ***A1***

attempt to use 'Area $=\frac{1}{2}ab\sin C$' (must include their calculated value of $\sin \mathrm{B\hat{A}C}$) ***(M1)***

$=\frac{1}{2}\times 10\times\sqrt{6}\times\frac{\sqrt{24}}{5}$ ***(A1)***

$=12$ (cm^2) ***A1***

[6 marks]

METHOD 2

attempt to find perpendicular height of triangle BAC ***(M1)***

EITHER

$\text{height} = \sqrt{6} \times \sin \mathrm{B\hat{A}C}$

attempt to use the Pythagorean identity $\cos^2 \alpha + \sin^2 \alpha = 1$ ***(M1)***

$\text{height} = \sqrt{6} \times \sqrt{1 - \left(\frac{1}{5}\right)^2}$ ***(A1)***

$= \sqrt{6} \times \frac{\sqrt{24}}{5} \left(= \frac{12}{5}\right)$ ***A1***

OR

$\text{adjacent} = \frac{\sqrt{6}}{5}$ ***(A1)***

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\text{height} = \sqrt{6 - \frac{6}{25}} \left(= \frac{12}{5}\right)$ (may be seen in area formula) ***(A1)***

THEN

attempt to use 'Area $= \frac{1}{2} \text{base} \times \text{height}$' (must include their calculated value for height) ***(M1)***

$= \frac{1}{2} \times 10 \times \frac{12}{5}$

$= 12$ (cm^2) ***A1***

[6 marks]

Question 23

(a) $2\times2+6\times\frac{1}{2}-2\times1=5$ ***A1***

[1 mark]

(b) $2\left(k^2-6\right)+6\left(2k+3\right)+12(=0)$ ***(A1)***

equating their scalar product of the direction normals to zero ***(M1)***

$2\left(k^2-6\right)+6\left(2k+3\right)+12=0$

$k^2-6+6k+9+6=0$ OR $(k+3)^2=0$

$k=-3$ ***A1***

attempt to substitute k, p and coordinates of A into Π_2 ***(M1)***

$q=3\times2-3\times\frac{1}{2}-6\times1$

$q=-\frac{3}{2}$ ***A1***

[5 marks]

(c) attempt to equate a pair of ratios or equate vector product to zero vector **(M1)**

$$\begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix} = \mu \begin{pmatrix} 3 \\ 9 \\ p \end{pmatrix} \Rightarrow \mu = \frac{2}{3} \text{ OR } \frac{2}{-2} = \frac{3}{p} \text{ OR } \frac{6}{-2} = \frac{9}{p} \text{ OR } 6p + 18 = 0 \text{ OR } -6 - 2p = 0$$

$p = -3$ **A1**

[2 marks]

(d) (i) attempt to find the vector equation of the line through A perpendicular to Π_1 **(M1)**

$$(\boldsymbol{r} =)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{1}{2} \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix}$$

attempt to substitute their vector equation into Π_2 **(M1)**

$$3(2 + 2\lambda) + 9\left(\frac{1}{2} + 6\lambda\right) - 3(1 - 2\lambda)\left(= -\frac{51}{2}\right)$$

$6 + 6\lambda + \frac{9}{2} + 54\lambda - 3 + 6\lambda = -\frac{51}{2}$ (or equivalent) **(A1)**

$66\lambda = -\frac{51}{2} - 6 - \frac{9}{2} + 3$ (or equivalent)

$\lambda = -\frac{1}{2}$ **A1**

$\left(1, -\frac{5}{2}, 2\right)$ **A1**

(ii) distance AB or $\left|\lambda\begin{pmatrix}2\\6\\-2\end{pmatrix}\right|$ ***(M1)***

$\sqrt{(2-1)^2+\left(\frac{1}{2}+\frac{5}{2}\right)^2+(1-2)^2}$ OR $\sqrt{1+9+1}$ ***A1***

$=\sqrt{11}$ ***AG***

[7 marks]

(e) Valid method to find a point C on Π_3 using $\overrightarrow{AC}=\overrightarrow{BA}$ or $\overrightarrow{BC}=2\overrightarrow{BA}$ or A as the midpoint of BC. ***(M1)***

$\lambda=\frac{1}{2}$, $\overrightarrow{OC}=\begin{pmatrix}2\\\frac{1}{2}\\1\end{pmatrix}+\begin{pmatrix}1\\3\\-1\end{pmatrix}$, $\overrightarrow{OC}=\begin{pmatrix}1\\-\frac{5}{2}\\2\end{pmatrix}+\begin{pmatrix}2\\6\\-2\end{pmatrix}$, $\overrightarrow{OA}=\frac{1}{2}\left(\overrightarrow{OB}+\overrightarrow{OC}\right)$

point on Π_3 is $\left(3,\frac{7}{2},0\right)$ ***A1***

attempt to substitute their $\left(3,\frac{7}{2},0\right)$ into $\Pi_3: x+3y-z=d$ (or equivalent) ***(M1)***

$1\times3+3\times\frac{7}{2}-1\times0\left(=\frac{27}{2}\right)$

$\Pi_3: x+3y-z=\frac{27}{2}$ $(2x+6y-2z=27)$ (or equivalent) ***A1***

[4 marks]

Total [19 marks]

Question 24

(a) (i) $\left(\frac{9}{2}, \frac{3\sqrt{3}}{2}\right)$ (accept $x = \frac{9}{2}$ and $y = \frac{3\sqrt{3}}{2}$) ***A1***

(ii) **METHOD 1**

using $m = \frac{\text{change in } y}{\text{change in } x}$ with their midpoint OR gradient perpendicular to AC

OR $m = \tan 30^{\circ}$ ***(M1)***

$m = \frac{\sqrt{3}}{3}$ ***(A1)***

$y = \frac{\sqrt{3}}{3}x$ OR $y - \frac{3\sqrt{3}}{2} = \frac{\sqrt{3}}{3}\left(x - \frac{9}{2}\right)$ (must be written as an equation) ***A1***

METHOD 2

attempt to find the vector equation of the line using a direction and a point on the line ***(M1)***

direction vector is $\begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix}$ or equivalent/parallel vectors ***(A1)***

$\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix}$ OR $\boldsymbol{r} = \lambda \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix}$ OR $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix} + \lambda \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix}$ (or equivalent) ***A1***

Note: Vector equation must be in the form $\boldsymbol{r} =$ or $\begin{pmatrix} x \\ y \end{pmatrix} =$.

Allow equivalent parametric forms such as $x = \frac{9}{2}t$, $y = \frac{3\sqrt{3}}{2}t$.

[4 marks]

(b) substituting $x = 6$ into their equation *(M1)*

so at B $y = 2\sqrt{3}$ *(A1)*

area of triangle $\text{OAB} = \frac{1}{2} \times 6 \times 2\sqrt{3} = 6\sqrt{3}$

area of quadrilateral $\text{OABC} = 12\sqrt{3}$ ***A1***

[3 marks]

Total [7 marks]

Question 25

$\tan^{-1} 1 = 45°$ or equivalent *(A1)*

attempt to equate $2x - 5°$ to their reference angle *(M1)*

ote: Do not accept $2x - 5° = 1$.

$2x - 5° = 45°, (225°)$

$x = 25°, 115°$ ***A1A1***

Note: Do not award the final ***A1*** if any additional solutions are seen.

[4 marks]

Question 26

(a) $2r + r\theta = 10$ **A1**

$\frac{1}{2}r^2\theta = 6.25$ **A1**

attempt to eliminate θ to obtain an equation in r **M1**

correct intermediate equation in r **A1**

$10 - 2r = \frac{25}{2r}$ OR $\frac{10}{r} - 2 = \frac{25}{2r^2}$ OR $\frac{1}{2}r^2\left(\frac{10}{r} - 2\right) = 6.25$ OR $12.5 + 2r^2 = 10r$

$4r^2 - 20r + 25 = 0$ **AG**

[4 marks]

(b) attempt to solve quadratic by factorizing or use of formula or completing the square ***(M1)***

$(2r-5)^2 = 0$ OR $r = \frac{20 \pm \sqrt{(-20)^2 - 4(4)(25)}}{2(4)}\left(= \frac{20 \pm \sqrt{400-400}}{8}\right)$

$r = \frac{5}{2}$ **A1**

attempt to substitute their value of r into their perimeter or area equation ***(M1)***

$\theta = \frac{10 - 2\left(\frac{5}{2}\right)}{\left(\frac{5}{2}\right)}$ or $\theta = \frac{25}{2\left(\frac{5}{2}\right)^2}$

$\theta = 2$ **A1**

[4 marks]

Total [8 marks]

Question 27

(a) **METHOD 1**

attempt to use identity $\tan(A-B) \equiv \dfrac{\tan A - \tan B}{1+\tan A \tan B}$ ***M1***

$\tan\left(\theta - \dfrac{\pi}{4}\right) = \dfrac{\tan\theta - 1}{1+\tan\theta}$ ***A1***

attempt to write their RHS in terms of $\sin\theta$ and $\cos\theta$ ***M1***

$\dfrac{\dfrac{\sin\theta}{\cos\theta} - 1}{1+\dfrac{\sin\theta}{\cos\theta}}$ OR $\dfrac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta}$

multiply through by the conjugate of the denominator $\dfrac{\cos\theta - \sin\theta}{\cos\theta - \sin\theta}$ ***M1***

$= \dfrac{-\sin^2\theta - \cos^2\theta + 2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$ ***A1***

$= \dfrac{-\left(\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta\right)}{\cos^2\theta - \sin^2\theta}$

$= \dfrac{-(1-\sin 2\theta)}{\cos^2\theta - \sin^2\theta}$ ***A1***

$= \dfrac{\sin 2\theta - 1}{\cos 2\theta}$ ***AG***

METHOD 2

attempt to write $\tan\left(\theta - \frac{\pi}{4}\right)$ in terms of sin and cos: ***M1***

$$\left(\tan\left(\theta - \frac{\pi}{4}\right)\right) = \frac{\sin\left(\theta - \frac{\pi}{4}\right)}{\cos\left(\theta - \frac{\pi}{4}\right)}$$

attempt to use both sin and cos addition formulae: ***M1***

$$= \frac{\sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}}{\cos\theta\cos\frac{\pi}{4} + \sin\theta\sin\frac{\pi}{4}}$$

$$= \frac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta}$$ ***A1***

multiply through by the conjugate of the denominator $\frac{\cos\theta - \sin\theta}{\cos\theta - \sin\theta}$ ***M1***

$$= \frac{-\sin^2\theta - \cos^2\theta + 2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$ ***A1***

$$= \frac{-\left(\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta\right)}{\cos^2\theta - \sin^2\theta}$$

$$= \frac{-(1 - \sin 2\theta)}{\cos^2\theta - \sin^2\theta}$$ ***A1***

$$= \frac{\sin 2\theta - 1}{\cos 2\theta}$$ ***AG***

METHOD 3

attempt to use given identity $\tan(A-B) \equiv \dfrac{\tan A - \tan B}{1+\tan A \tan B}$ ***M1***

$$\tan\left(\theta - \frac{\pi}{4}\right) = \frac{\tan\theta - 1}{1+\tan\theta}$$ ***A1***

multiply through by the conjugate of the denominator $\dfrac{1-\tan\theta}{1-\tan\theta}$ ***M1***

$$\frac{(\tan\theta - 1)}{(1+\tan\theta)} \cdot \frac{(1-\tan\theta)}{(1-\tan\theta)}$$ ***A1***

$$\frac{2\tan\theta - \tan^2\theta - 1}{1-\tan^2\theta}\left(= \frac{2\tan\theta - \sec^2\theta}{1-\tan^2\theta}\right)$$

attempt to write their RHS in terms of $\sin\theta$ and $\cos\theta$ ***M1***

$$= \frac{2\dfrac{\sin\theta}{\cos\theta} - \dfrac{1}{\cos^2\theta}}{1 - \dfrac{\sin^2\theta}{\cos^2\theta}}$$

$$= \frac{2\sin\theta\cos\theta - 1}{\cos^2\theta - \sin^2\theta}$$ ***A1***

$$= \frac{\sin 2\theta - 1}{\cos 2\theta}$$ ***AG***

[6 marks]

(b) recognition that $x = 2\theta \Rightarrow \theta = \frac{x}{2}$ **M1**

$\tan\left(\frac{x}{2} - \frac{\pi}{4}\right) = \sqrt{3}$

$\frac{x}{2} - \frac{\pi}{4} = \frac{\pi}{3}, \left(\frac{4\pi}{3}\right)$ **A1**

$x = \frac{7\pi}{6}$ **A1**

Note: Award ***A0*** if extra solutions outside the domain are seen.

[3 marks]

Total [9 marks]

Question 28

(a) $\overrightarrow{\mathrm{OP}} = \boldsymbol{a} + \lambda(\boldsymbol{b} - \boldsymbol{a})$ **A1**

$= (1-\lambda)\boldsymbol{a} + \lambda\boldsymbol{b}$ **AG**

[1 mark]

(b) **METHOD 1**

recognition that $\overrightarrow{\mathrm{OP}} \cdot \overrightarrow{\mathrm{AB}} = 0$ (may be seen anywhere) ***(M1)***

$[(1-\lambda)\boldsymbol{a} + \lambda\boldsymbol{b}].[\boldsymbol{b} - \boldsymbol{a}](=0)$ **A1**

attempt to multiply out scalar product **M1**

$(1-\lambda)\boldsymbol{a}.\boldsymbol{b} + \lambda\boldsymbol{b}.\boldsymbol{b} - (1-\lambda)\boldsymbol{a}.\boldsymbol{a} - \lambda\boldsymbol{b}.\boldsymbol{a}(=0)$ ***(A1)***

attempt to substitute for $\boldsymbol{a}.\boldsymbol{b}$ and $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$ ***(M1)***

$\frac{1}{4}(1-\lambda) + 4\lambda - (1-\lambda) - \frac{\lambda}{4}(=0)$ ***(A1)***

$1 - \lambda + 16\lambda - 4 + 4\lambda - \lambda = 0$

$18\lambda - 3 = 0$

$\lambda = \frac{1}{6}$ **A1**

METHOD 2

$\cos AOB = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{1}{8}$ ***A1***

attempt to use cosine rule to find AB ***M1***

$|AB|^2 = 1^2 + 2^2 - 2(1)(2)\left(\frac{1}{8}\right)$

$AB = \frac{3\sqrt{2}}{2}$ ***A1***

attempt to apply Pythagoras' Theorem twice: ***M1***

$|OP|^2 + \left(\frac{3\sqrt{2}}{2}\lambda\right)^2 = 1$ and

$|OP|^2 + \left(\frac{3\sqrt{2}}{2}(1-\lambda)\right)^2 = 4$ ***A1***

attempt to solve simultaneously: ***M1***

$\frac{9}{2}(1-\lambda)^2 - \frac{9}{2}\lambda^2 = 3$

$\lambda = \frac{1}{6}$ ***A1***

[7 marks]

Total [8 marks]

Question 29

attempt to substitute into cosine rule **(M1)**

$$\left(\cos 2\theta =\right)\frac{4^2+6^2-5^2}{2\times 4\times 6} \text{ OR } 5^2=4^2+6^2-2\times 4\times 6\times \cos 2\theta$$

$$\left(\cos 2\theta =\right)\frac{27}{48}\left(=\frac{9}{16}\right)$$ **(A1)**

attempt to use $\cos 2\theta = 2\cos^2\theta - 1$ **(M1)**

$$\cos^2\theta = \frac{1+\frac{27}{48}}{2}\left(=\frac{1+\frac{9}{16}}{2}\right)$$

$$\cos^2\theta = \frac{75}{96}\left(=\frac{25}{32}\right)$$ **A1**

$$\cos\theta = (\pm)\sqrt{\frac{75}{96}}\left(=\sqrt{\frac{25}{32}}=\frac{5}{\sqrt{32}}\right)$$ **(A1)**

$$=\frac{5}{4\sqrt{2}}$$

$$=\frac{5\sqrt{2}}{8}\quad (p=5, q=8)$$ **A1**

te: The final answer must be positive.

[6 marks]

Question 30

(a) $\frac{1}{2}r^2\theta = 48$ OR $\frac{1}{2}r^2(1.5) = 48$ ***(A1)***

attempt to solve their equation to find r or r^2 ***(M1)***

Note: To award the ***M1***, candidate's equation must include r^2 and $\theta = 1.5$, and they must attempt to isolate r^2 or r.

$r^2 = 64$

$r = 8$ (cm) ***A1***

[3 marks]

(b) evidence of summing the two radii and the arc length ***(M1)***

perimeter $= 2r + r\theta$

$= 16 + 8(1.5)$

$= 28$ (cm) ***A1***

[2 marks]

Total [5 marks]