Subject – Math AA(Higher Level)
Topic - Function
Year - May 2021 – Nov 2022
Paper -2
Answers

Question 1

METHOD 1

sketching the graph of $y = \frac{x^2}{x-3}$ ($y = x + 3 + \frac{9}{x-3}$) ***M1***

the (oblique) asymptote has a gradient equal to 1

and so the maximum value of m is 1 ***R1***

consideration of a straight line steeper than the horizontal line joining

$(-3,0)$ and $(0,0)$ ***M1***

so $m > 0$ ***R1***

hence $0 < m \le 1$ ***A1***

METHOD 2

attempting to eliminate y to form a quadratic equation in x ***M1***

$x^2 = m(x^2 - 9)$

$\Rightarrow (m-1)x^2 - 9m = 0$ ***A1***

EITHER

attempting to solve $-4(m-1)(-9m) < 0$ for m ***M1***

OR

attempting to solve $x^2 < 0$ *ie* $\frac{9m}{m-1} < 0$ $(m \ne 1)$ for m ***M1***

THEN

$\Rightarrow 0 < m < 1$ ***A1***

a valid reason to explain why $m = 1$ gives no solutions *eg* if $m = 1$,

$(m-1)x^2 - 9m = 0 \Rightarrow -9 = 0$ and so $0 < m \le 1$ ***R1***

Total [5 marks]

Question 2

(a) attempt to solve $4x^2 - 1 = 0$ e.g. by factorising $4x^2 - 1$ ***(M1)***

$p = \frac{1}{2}, q = -\frac{1}{2}$ or vice versa ***A1***

[2 marks]

(b) attempt to use quotient rule or product rule ***(M1)***

EITHER

$$f'(x) = \frac{3(4x^2 - 1) - 8x(3x + 2)}{(4x^2 - 1)^2} \left(= \frac{-12x^2 - 16x - 3}{(4x^2 - 1)^2} \right)$$ ***A1A1***

Note:	Award ***A1*** for each term in the numerator with correct signs, provided correct denominator is seen.

OR

$$f'(x) = -8x(3x + 2)(4x^2 - 1)^{-2} + 3(4x^2 - 1)^{-1}$$ ***A1A1***

Note:	Award ***A1*** for each term.

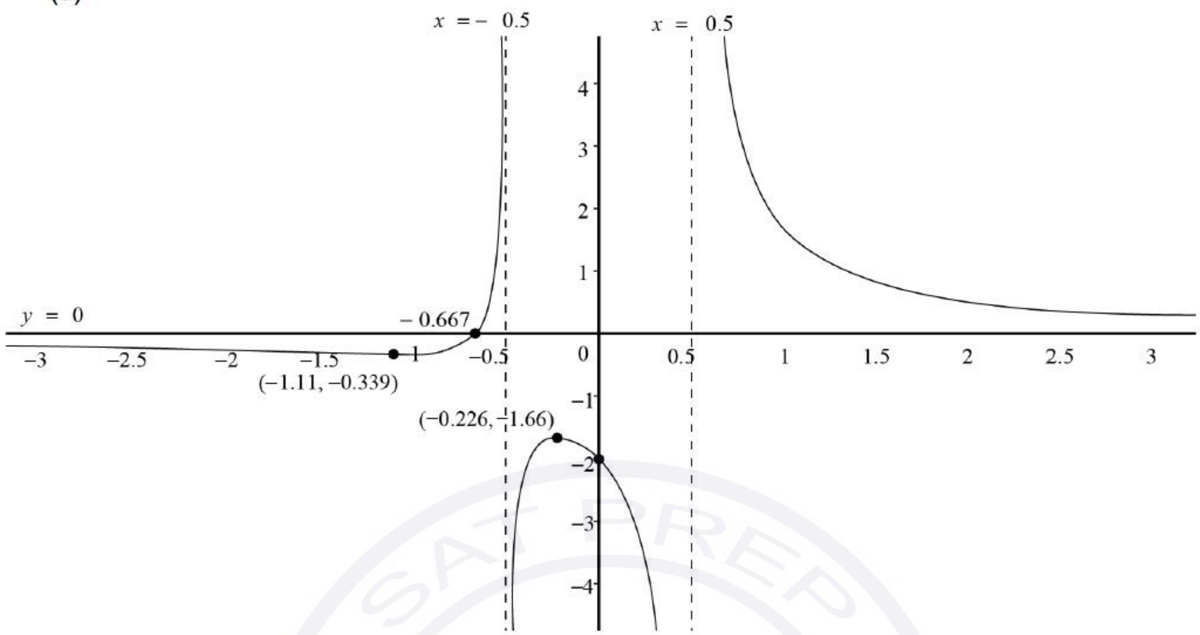
[3 marks]

(c) attempt to find the local min point on $y = f'(x)$ OR solve $f''(x) = 0$ ***(M1)***

$x = -1.60$ ***A1***

[2 marks]

(d)



A1A1A1A1A1

[5 marks]

e) $x=-\frac{2}{3}(=-0.667)$ ***A1***

(oblique asymptote has) gradient $\frac{4}{3}(=1.33)$ ***(A1)***

appropriate method to find complete equation of oblique asymptote ***M1***

$$
\begin{array}{r}
\frac{4}{3}x-\frac{8}{9} \\
3x+2\overline{)4x^2+0x-1} \\
4x^2+\frac{8}{3}x \\
\hline
-\frac{8}{3}x-1 \\
-\frac{8}{3}x-\frac{16}{9} \\
\hline
\frac{7}{9}
\end{array}
$$

$y=\frac{4}{3}x-\frac{8}{9}(=1.33x-0.889)$ ***A1***

Note: Do not award the final ***A1*** if the answer is not given as an equation.

[4 marks]

f) attempting to find at least one critical value $(x=-0.568729...,x=1.31872...)$ ***(M1)***

$-\frac{2}{3}<x<-0.569$ OR $-0.5<x<0.5$ OR $x>1.32$ ***A1A1A1***

Note: Only penalize once for use of $\le$ rather than $<$.

[4 marks]
Total [20 marks]

Question 3

(a) **EITHER**

$f(-x) = \arcsin\left(\frac{(-x)^2 - 1}{(-x)^2 + 1}\right) = \arcsin\left(\frac{x^2 - 1}{x^2 + 1}\right) = f(x)$ ***R1***

OR

a sketch graph of $y = f(x)$ with line symmetry in the y-axis indicated ***R1***

THEN

so $f(x)$ is an even function ***AG***

[1 mark]

(b) as $x \to \pm\infty$, $f(x) \to \arcsin 1 \left(\to \frac{\pi}{2}\right)$ ***A1***

so the horizontal asymptote is $y = \frac{\pi}{2}$ ***A1***

[2 marks]

(c) (i) attempting to use the quotient rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)$ ***M1***

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)=\frac{2x\left(x^2+1\right)-2x\left(x^2-1\right)}{\left(x^2+1\right)^2}\quad\left(=\frac{4x}{\left(x^2+1\right)^2}\right)$$ ***A1***

attempting to use the chain rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin\left(\frac{x^2-1}{x^2+1}\right)\right)$ ***M1***

let $u=\frac{x^2-1}{x^2+1}$ and so $y=\arcsin u$ and $\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{1}{\sqrt{1-u^2}}$

$$f'(x)=\frac{1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}\times\frac{4x}{\left(x^2+1\right)^2}$$ ***M1***

$$=\frac{4x}{\sqrt{\left(x^2+1\right)^2-\left(x^2-1\right)^2}}\times\frac{1}{\left(x^2+1\right)}$$ ***A1***

$$=\frac{4x}{\sqrt{4x^2}}\times\frac{1}{\left(x^2+1\right)}$$ ***A1***

$$=\frac{2x}{\sqrt{x^2}\left(x^2+1\right)}$$ ***AG***

(ii) $f'(x)=\frac{2x}{|x|\left(x^2+1\right)}$

EITHER

for $x<0$, $|x|=-x$ ***(A1)***

so $f'(x)=-\frac{2}{x^2+1}$ ***A1***

OR

$|x|>0$ and $x^2+1>0$ ***A1***

$2x<0,\ x<0$ ***A1***

THEN

$f'(x)<0$ ***R1***

Note: Award ***R1*** for stating that in $f'(x)$, the numerator is negative, and the denominator is positive.

so f is decreasing for $x<0$ ***AG***

Note: Do not accept a graphical solution.

[9 marks]

(d) $x = \arcsin\left(\frac{y^2-1}{y^2+1}\right)$ **M1**

$\sin x = \frac{y^2-1}{y^2+1} \Rightarrow y^2 \sin x + \sin x = y^2 - 1$ **A1**

$y^2 = \frac{1+\sin x}{1-\sin x}$ **A1**

domain of g is $x \in \mathbb{R}, x \geq 0$ and so the range of g^{-1} must be $y \in \mathbb{R}, y \geq 0$

hence the positive root is taken (or the negative root is rejected) **R1**

Note: The ***R1*** is dependent on the above ***A1***.

so $\left(g^{-1}(x) =\right)\sqrt{\frac{1+\sin x}{1-\sin x}}$ **A1**

Note: The final ***A1*** is not dependent on ***R1*** mark.

[5 marks]

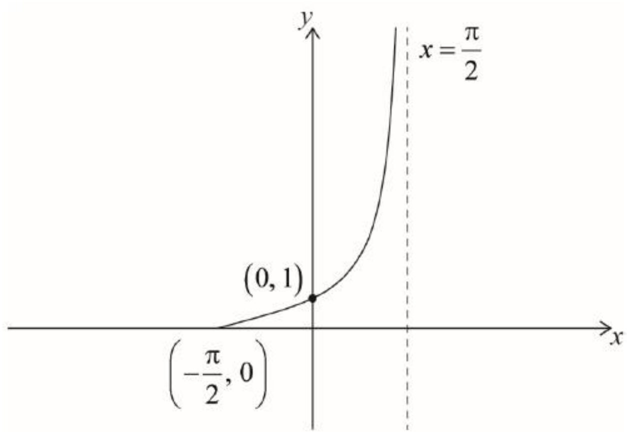
(e) domain is $-\frac{\pi}{2} \leq x < \frac{\pi}{2}$ **A1**

Note: Accept correct alternative notations, for example, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Accept $[-1.57, 1.57[$ if correct to 3 s.f.

[1 mark]

(f)



A1A1A1

Note: ***A1*** for correct domain and correct range and y-intercept at $y = 1$ ***A1*** for asymptotic behaviour $x \to \frac{\pi}{2}$ ***A1*** for $x = \frac{\pi}{2}$ Coordinates are not required. Do not accept $x = 1.57$ or other inexact values.

[3 marks]
Total [21 marks]

Question 4

(a) $100 = A_0 e^0$ ***A1***

$A_0 = 100$ ***AG***

[1 mark]

(b) correct substitution of values into exponential equation ***(M1)***

$50 = 100e^{-5730k}$ OR $e^{-5730k} = \frac{1}{2}$

EITHER

$-5730k = \ln\frac{1}{2}$ ***A1***

$\ln\frac{1}{2} = -\ln 2$ OR $-\ln\frac{1}{2} = \ln 2$ ***A1***

OR

$e^{5730k} = 2$ ***A1***

$5730k = \ln 2$ ***A1***

THEN

$k = \frac{\ln 2}{5730}$ ***AG***

Note: There are many different ways of showing that $k = \frac{\ln 2}{5730}$ which involve showing different steps. Award full marks for at least two correct algebraic steps seen.

[3 marks]

(c) if 25% of the carbon-14 has decayed, 75% remains ie, 75 units remain

$75 = 100e^{-\frac{\ln 2}{5730}t}$ ***(A1)***

EITHER

using an appropriate graph to attempt to solve for t ***(M1)***

OR

manipulating logs to attempt to solve for t ***(M1)***

$\ln 0.75 = -\frac{\ln 2}{5730}t$

$t = 2378.164...$

THEN

$t = 2380$ (years) (correct to the nearest 10 years) ***A1***

[3 marks]
Total [7 marks]

Question 5

(a) attempting to find the vertex *(M1)*

$x=1$ OR $y=-5$ OR $f(x)=6(x-1)^2-5$

range is $y \geq -5$ ***A1***

[2 marks]

(b) **METHOD 1**

$(g \circ f)(x) = -(6x^2-12x+1)+c$ $\left(=-\left(6(x-1)^2-5\right)+c\right)$ *(A1)*

EITHER

relating to the range of f OR attempting to find $g(-5)$ *(M1)*

$5+c \leq 0$ *(A1)*

OR

attempting to find the discriminant of $(g \circ f)(x)$ *(M1)*

$144+24(c-1) \leq 0$ $(120+24c \leq 0)$ *(A1)*

THEN

$c \leq -5$ ***A1***

[4 marks]

METHOD 2

vertical reflection followed by vertical shift *(M1)*

new vertex is $(1, 5+c)$ *(A1)*

$5+c \leq 0$ *(A1)*

$c \leq -5$ ***A1***

[4 marks]

Total [6 marks]

Question 6

(a) (i)

Note: In part (a), penalise once only, if correct values are given instead of correct coordinates.

attempts to solve $x^2 - x - 12 = 0$ **(M1)**

$(-3,0)$ and $(4,0)$ **A1**

(ii) $\left(0, \frac{4}{5}\right)$ **A1**

[3 marks]

(b) $x = \frac{15}{2}$ **A1**

Note: Award ***A0*** for $x \neq \frac{15}{2}$.

Award ***A1*** in part (b), if $x = \frac{15}{2}$ is seen on their graph in part (d).

[1 mark]

(c) **METHOD 1**

$(ax+b)(2x-15) \equiv x^2 - x - 12$

attempts to expand $(ax+b)(2x-15)$ **(M1)**

$2ax^2 - 15ax + 2bx - 15b \equiv x^2 - x - 12$

$a = \frac{1}{2}$ **A1**

equates coefficients of x **(M1)**

$-1 = -\frac{15}{2} + 2b$

$b = \frac{13}{4}$ **A1**

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

METHOD 2

attempts division on $\dfrac{x^2-x-12}{2x-15}$ ***M1***

$\dfrac{x}{2}+\dfrac{13}{4}+\ldots$ ***M1***

$a=\dfrac{1}{2}$ ***A1***

$b=\dfrac{13}{4}$ ***A1***

$\left(y=\dfrac{x}{2}+\dfrac{13}{4}\right)$

METHOD 3

$a=\dfrac{1}{2}$ ***A1***

$\dfrac{x^2-x-12}{2x-15}\equiv\dfrac{x}{2}+b+\dfrac{c}{2x-15}$ ***M1***

$x^2-x-12\equiv\dfrac{(2x-15)x}{2}+(2x-15)b+c$

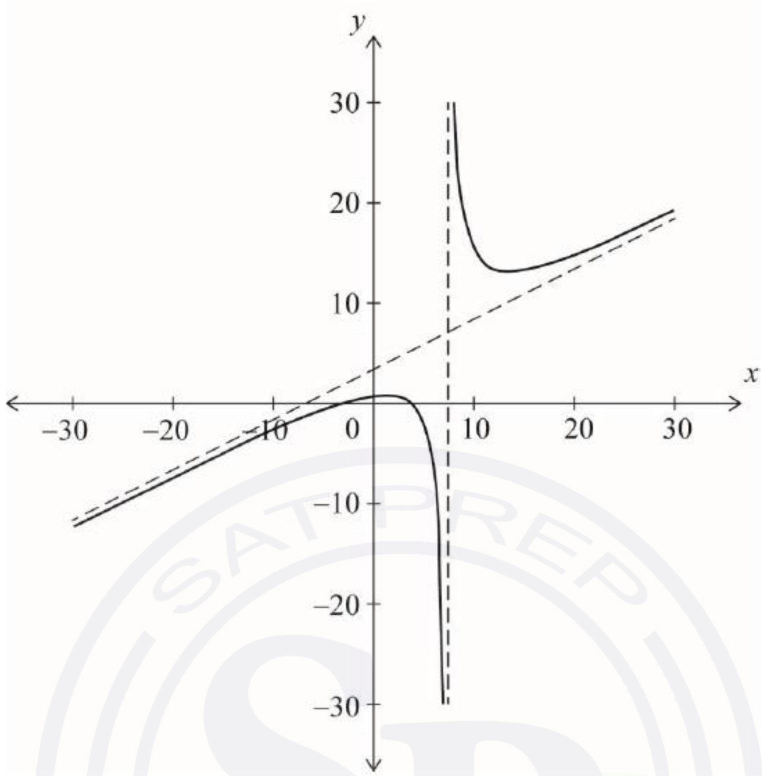
equates coefficients of x: ***(M1)***

$-1=-\dfrac{15}{2}+2b$

$b=\dfrac{13}{4}$ ***A1***

$\left(y=\dfrac{x}{2}+\dfrac{13}{4}\right)$

(d)



two branches with approximately correct shape (for $-30 \le x \le 30$) ***A1***
their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1***
their axes intercepts in approximately the correct positions ***A1***

Note: Points of intersection with the axes and the equations of asymptotes are not required to be labelled.

[3 marks]

(e) (i) attempts to split into partial fractions: *(M1)*

$$\frac{2x-15}{(x+3)(x-4)} \equiv \frac{A}{x+3} + \frac{B}{x-4}$$

$$2x-15 \equiv A(x-4) + B(x+3)$$

$A = 3$ ***A1***

$B = -1$ ***A1***

$$\left(\frac{3}{x+3} - \frac{1}{x-4}\right)$$

(ii) $\int_0^3 \left(\frac{3}{x+3} - \frac{1}{x-4}\right) \mathrm{d}x$

attempts to integrate and obtains two terms involving 'ln' *(M1)*

$= \left[3\ln|x+3| - \ln|x-4|\right]_0^3$ ***A1***

$= 3\ln 6 - \ln 1 - 3\ln 3 + \ln 4$ ***A1***

$= 3\ln 2 + \ln 4 \ (= \ln 8 + \ln 4)$

$= \ln 32 \ (= 5\ln 2)$ ***A1***

Note: The final ***A1*** is dependent on the previous two ***A*** marks.

[7 marks]

Total [18 marks]

Question 7

(a) $12 = \frac{2\pi}{b}$ OR $b = \frac{2\pi}{12}$ ***A1***

$b = \frac{\pi}{6}$ ***AG***

[1 mark]

(b) $a = \frac{6.8-2.2}{2}$ OR $a = \frac{\max - \min}{2}$ *(M1)*

$= 2.3$ (m) ***A1***

[2 marks]

(c) $d = \frac{6.8+2.2}{2}$ OR $d = \frac{\max + \min}{2}$ *(M1)*

$= 4.5$ (m) ***A1***

[2 marks]

(d) **METHOD 1**

substituting $t = 4.5$ and $H = 6.8$ for example into their equation for H ***(A1)***

$$6.8 = 2.3\sin\left(\frac{\pi}{6}(4.5 - c)\right) + 4.5$$

attempt to solve their equation ***(M1)***

$c = 1.5$ ***A1***

METHOD 2

using horizontal translation of $\frac{12}{4}$ ***(M1)***

$4.5 - c = 3$ ***(A1)***

$c = 1.5$ ***A1***

METHOD 3

$$H'(t) = (2.3)\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}(t - c)\right)$$ ***(A1)***

attempts to solve their $H'(4.5) = 0$ for c ***(M1)***

$$(2.3)\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}(4.5 - c)\right) = 0$$

$c = 1.5$ ***A1***

[3 marks]

(e) attempt to find H when $t = 12$ or $t = 0$, graphically or algebraically ***(M1)***

$H = 2.87365...$

$H = 2.87(\text{m})$ ***A1***

[2 marks]

(g) **METHOD 1**

substitutes $t = \frac{11}{3}$ and $H = 6.8$ into their equation for H and attempts to solve for c ***(M1)***

$$6.8 = 2.3\sin\left(\frac{\pi}{6}\left(\frac{11}{3} - c\right)\right) + 4.5 \Rightarrow c = \frac{2}{3}$$

$$H(t) = 2.3\sin\left(\frac{\pi}{6}\left(t - \frac{2}{3}\right)\right) + 4.5$$ ***A1***

METHOD 2

uses their horizontal translation $\left(\frac{12}{4} = 3\right)$ ***(M1)***

$$\frac{11}{3} - c = 3 \Rightarrow c = \frac{2}{3}$$

$$H(t) = 2.3\sin\left(\frac{\pi}{6}\left(t - \frac{2}{3}\right)\right) + 4.5$$ ***A1***

[2 marks]
Total [15 marks]

(f) attempt to solve $5 = 2.3\sin\left(\frac{\pi}{6}(t - 1.5)\right) + 4.5$ ***(M1)***

times are $t = 1.91852...$ and $t = 7.08147...$, $(t = 13.9185..., t = 19.0814...)$ ***(A1)***

total time is $2 \times (7.081... - 1.919...)$

$10.3258...$

$= 10.3$ (hours) ***A1***

Note: Accept 10.

[3 marks]

Question 8

(a) attempt to replace x with $-x$ **_M1_**

$$f(-x)=2^{-x}-\frac{1}{2^{-x}}$$

EITHER

$$=\frac{1}{2^{x}}-2^{x}=-f(x)$$ **_A1_**

OR

$$=-\left(2^{x}-\frac{1}{2^{x}}\right)(=-f(x))$$ **_A1_**

> **Note:** Award ***M1A0*** for a graphical approach including evidence that **either** the graph is invariant after rotation by 180° about the origin **or** the graph is invariant after a reflection in the y-axis and then in the x-axis (or vice versa).

so f is an odd function **_AG_**

[2 marks]

(b) attempt to find at least one intersection point ***(M1)***

$x=-1.26686..., x=0.177935..., x=3.06167...$

$x=-1.27, x=0.178, x=3.06$

$-1.27\le x<-1$, **_A1_**

$0.178\le x<3$, **_A1_**

$x\ge 3.06$ **_A1_**

[4 marks]

Total [6 marks]

Question 9

(a) (i) 32 (cm) **A1**

(ii) $h_A(0) = \sin(6) + 27$ ***(M1)***

$= 26.7205...$

$= 26.7$ (cm) **A1**

[3 marks]

(b) attempts to solve $h_A(t) = h_B(t)$ for t ***(M1)***

$t = 4.0074..., 4.7034..., 5.88332...$

$t = 4.01, 4.70, 5.88$ (weeks) **A2**

[3 marks]

(c) $h_A(t) - h_B(t) = \sin(2t+6) + t - 5$ **A1**

EITHER

for $t > 6$, $t - 5 > 1$ **A1**

and as $\sin(2t+6) \geq -1 \Rightarrow h_A(t) - h_B(t) > 0$ **R1**

OR

the minimum value of $\sin(2t+6) = -1$ **R1**

so for $t > 6$, $h_A(t) - h_B(t) = t - 6 > 0$ **A1**

THEN

hence for $t > 6$, Plant A was always taller than Plant B **AG**

[3 marks]

(d) recognises that $h_A'(t)$ and $h_B'(t)$ are required ***(M1)***

attempts to solve $h_A'(t) = h_B'(t)$ for t ***(M1)***

$t = 1.18879\ldots$ and $2.23598\ldots$ OR $4.33038\ldots$ and $5.37758\ldots$ OR $7.47197\ldots$ and $8.51917\ldots$ ***(A1)***

Note: Award full marks for $t = \frac{4\pi}{3} - 3, \frac{5\pi}{3} - 3, \left(\frac{7\pi}{3} - 3, \frac{8\pi}{3} - 3, \frac{10\pi}{3} - 3, \frac{11\pi}{3} - 3\right)$.
Award subsequent marks for correct use of these exact values.

$1.18879\ldots < t < 2.23598\ldots$ OR $4.33038\ldots < t < 5.37758\ldots$ OR
$7.47197\ldots < t < 8.51917\ldots$ ***(A1)***

attempts to calculate the total amount of time ***(M1)***

$3(2.2359\ldots - 1.1887\ldots)\ \left(= 3\left(\left(\frac{5\pi}{3} - 3\right) - \left(\frac{4\pi}{3} - 3\right)\right)\right)$

$= 3.14\ (= \pi)$ (weeks) ***A1***

[6 marks]

Total [15 marks]

Question 10

recognition that initial population is 15000 (seen anywhere) **(A1)**

$P(0)=15000$ OR 0.11×15000 OR 0.89×15000

population after 11% decrease is $15000\times0.89(=13350)$ **(A1)**

recognizing that $t=8$ on 1 January 2022 (seen anywhere) **(A1)**

substitution of their value of t for 1 January 2022 and their value of $P(8)$ into the model **(M1)**

$15000\times0.89=15000\mathrm{e}^{8k}$ OR $13350=15000\mathrm{e}^{8k}$

$k=\dfrac{\ln 0.89}{8}\ (-0.014566)$ **(A1)**

substitution of $t=2041-2014(=27)$ and their value for k into the model **(M1)**

$P(27)=15000e^{-0.0145...\times27}$

$10122.3\ldots$

$P(27)=10100\ (10122)$ **A1**

Total [7 marks]