Subject – Math AA(Higher Level)
Topic - Function
Year - May 2021 – Nov 2022
Paper -2
Questions

Question 1

[Maximum mark: 5]

Consider the graphs of $y = \dfrac{x^2}{x-3}$ and $y = m(x+3)$, $m \in \mathbb{R}$.

Find the set of values for m such that the two graphs have no intersection points.

Question 2

[Maximum mark: 20]

The function f is defined by $f(x) = \dfrac{3x+2}{4x^2-1}$, for $x \in \mathbb{R}$, $x \neq p$, $x \neq q$.

(a) Find the value of p and the value of q. [2]

(b) Find an expression for $f'(x)$. [3]

The graph of $y = f(x)$ has exactly one point of inflexion.

(c) Find the x-coordinate of the point of inflexion. [2]

(d) Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$, showing the values of any axes intercepts, the coordinates of any local maxima and local minima, and giving the equations of any asymptotes. [5]

The function g is defined by $g(x) = \dfrac{4x^2-1}{3x+2}$, for $x \in \mathbb{R}$, $x \neq -\dfrac{2}{3}$.

(e) Find the equations of all the asymptotes on the graph of $y = g(x)$. [4]

(f) By considering the graph of $y = g(x) - f(x)$, or otherwise, solve $f(x) < g(x)$ for $x \in \mathbb{R}$. [4]

Question 3

[Maximum mark: 21]

A function f is defined by $f(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$.

(a) Show that f is an even function. [1]

(b) By considering limits, show that the graph of $y = f(x)$ has a horizontal asymptote and state its equation. [2]

(c) (i) Show that $f'(x) = \frac{2x}{\sqrt{x^2}\left(x^2+1\right)}$ for $x \in \mathbb{R}$, $x \neq 0$.

(ii) By using the expression for $f'(x)$ and the result $\sqrt{x^2} = |x|$, show that f is decreasing for $x < 0$. [9]

A function g is defined by $g(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$, $x \geq 0$.

(d) Find an expression for $g^{-1}(x)$, justifying your answer. [5]

(e) State the domain of g^{-1}. [1]

(f) Sketch the graph of $y = g^{-1}(x)$, clearly indicating any asymptotes with their equations and stating the values of any axes intercepts. [3]

Question 4

[Maximum mark: 7]

All living plants contain an isotope of carbon called carbon-14. When a plant dies, the isotope decays so that the amount of carbon-14 present in the remains of the plant decreases. The time since the death of a plant can be determined by measuring the amount of carbon-14 still present in the remains.

The amount, A, of carbon-14 present in a plant t years after its death can be modelled by $A = A_0 e^{-kt}$ where $t \geq 0$ and A_0, k are positive constants.

At the time of death, a plant is defined to have 100 units of carbon-14.

(a) Show that $A_0 = 100$. [1]

The time taken for half the original amount of carbon-14 to decay is known to be 5730 years.

(b) Show that $k = \frac{\ln 2}{5730}$. [3]

(c) Find, correct to the nearest 10 years, the time taken after the plant's death for 25% of the carbon-14 to decay. [3]

Question 5

[Maximum mark: 6]

The functions f and g are defined for $x \in \mathbb{R}$ by $f(x) = 6x^2 - 12x + 1$ and $g(x) = -x + c$, where $c \in \mathbb{R}$.

(a) Find the range of f. [2]

(b) Given that $(g \circ f)(x) \leq 0$ for all $x \in \mathbb{R}$, determine the set of possible values for c. [4]

Question 6

Consider the function $f\left(x\right) = \dfrac{x^2 - x - 12}{2x - 15}$, $x \in \mathbb{R}$, $x \neq \dfrac{15}{2}$.

(a) Find the coordinates where the graph of f crosses the

(i) x-axis;

(ii) y-axis. [3]

(b) Write down the equation of the vertical asymptote of the graph of f. [1]

(c) The oblique asymptote of the graph of f can be written as $y = ax + b$ where a, $b \in \mathbb{Q}$.

Find the value of a and the value of b. [4]

(d) Sketch the graph of f for $-30 \leq x \leq 30$, clearly indicating the points of intersection with each axis and any asymptotes. [3]

(e) (i) Express $\dfrac{1}{f\left(x\right)}$ in partial fractions.

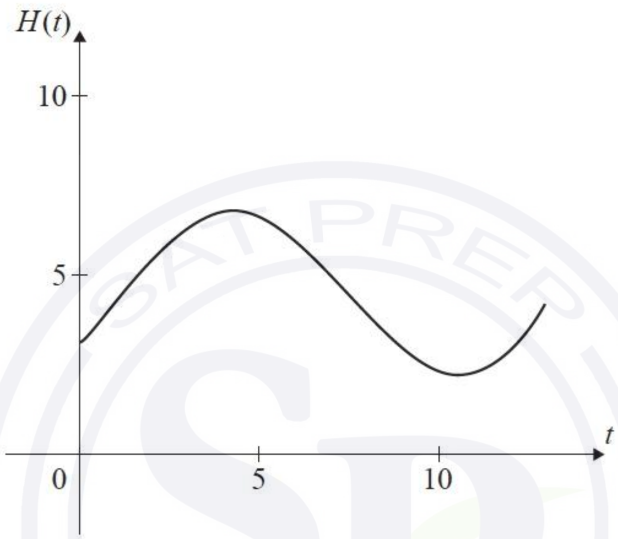
(ii) Hence find the exact value of $\displaystyle\int_0^3 \frac{1}{f\left(x\right)}\,\mathrm{d}x$, expressing your answer as a single logarithm. [7]

Question 7

[Maximum mark: 15]

The height of water, in metres, in Dungeness harbour is modelled by the function $H(t) = a\sin\left(b(t-c)\right) + d$, where t is the number of hours after midnight, and a, b, c and d are constants, where $a > 0$, $b > 0$ and $c > 0$.

The following graph shows the height of the water for 13 hours, starting at midnight.



The first high tide occurs at 04:30 and the next high tide occurs 12 hours later. Throughout the day, the height of the water fluctuates between $2.2\,\text{m}$ and $6.8\,\text{m}$.

All heights are given correct to one decimal place.

(a) Show that $b = \frac{\pi}{6}$. [1]

(b) Find the value of a. [2]

(c) Find the value of d. [2]

(d) Find the smallest possible value of c. [3]

(e) Find the height of the water at 12:00. [2]

(f) Determine the number of hours, over a 24-hour period, for which the tide is higher than 5 metres. [3]

A fisherman notes that the water height at nearby Folkestone harbour follows the same sinusoidal pattern as that of Dungeness harbour, with the exception that high tides (and low tides) occur 50 minutes earlier than at Dungeness.

(g) Find a suitable equation that may be used to model the tidal height of water at Folkestone harbour. [2]

Question 8

[Maximum mark: 6]

Consider the function $f(x)=2^x-\frac{1}{2^x}$, $x\in\mathbb{R}$.

(a) Show that f is an odd function. [2]

The function g is given by $g(x)=\frac{x-1}{x^2-2x-3}$, where $x\in\mathbb{R}$, $x\neq-1$, $x\neq 3$.

(b) Solve the inequality $f(x)\geq g(x)$. [4]

Question 9

[Maximum mark: 15]

A scientist conducted a nine-week experiment on two plants, A and B, of the same species. He wanted to determine the effect of using a new plant fertilizer. Plant A was given fertilizer regularly, while Plant B was not.

The scientist found that the height of Plant A, $h_A\,\mathrm{cm}$, at time t weeks can be modelled by the function $h_A(t)=\sin(2t+6)+9t+27$, where $0\leq t\leq 9$.

The scientist found that the height of Plant B, $h_B\,\mathrm{cm}$, at time t weeks can be modelled by the function $h_B(t)=8t+32$, where $0\leq t\leq 9$.

(a) Use the scientist's models to find the initial height of

(i) Plant B;

(ii) Plant A correct to three significant figures. [3]

(b) Find the values of t when $h_A(t)=h_B(t)$. [3]

(c) For $t>6$, prove that Plant A was always taller than Plant B. [3]

(d) For $0\leq t\leq 9$, find the total amount of time when the rate of growth of Plant B was greater than the rate of growth of Plant A. [6]

Question 10

[Maximum mark: 7]

The population of a town t years after 1 January 2014 can be modelled by the function

$$P(t) = 15\,000\mathrm{e}^{kt}\text{, where } k < 0 \text{ and } t \geq 0.$$

It is known that between 1 January 2014 and 1 January 2022 the population decreased by 11%.

Use this model to estimate the population of this town on 1 January 2041.