Subject – Math AA(Higher Level)
Topic - Geometry and Trigonometry
Year - May 2021 – Nov 2022
Paper -2
Questions

Question 1

[Maximum mark: 19]

(a) Show that $\cot 2\theta = \dfrac{1 - \tan^2\theta}{2\tan\theta}$. [1]

(b) Verify that $x = \tan\theta$ and $x = -\cot\theta$ satisfy the equation $x^2 + (2\cot 2\theta)x - 1 = 0$. [7]

(c) Hence, or otherwise, show that the exact value of $\tan\dfrac{\pi}{12} = 2 - \sqrt{3}$. [5]

(d) Using the results from parts (b) and (c) find the exact value of $\tan\dfrac{\pi}{24} - \cot\dfrac{\pi}{24}$.
Give your answer in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$. [6]

Question 2

[Maximum mark: 5]

Two ships, A and B, are observed from an origin O. Relative to O, their position vectors at time t hours after midday are given by

$$\boldsymbol{r}_{\mathrm{A}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{B}} = \begin{pmatrix} 7 \\ -3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 12 \end{pmatrix}$$

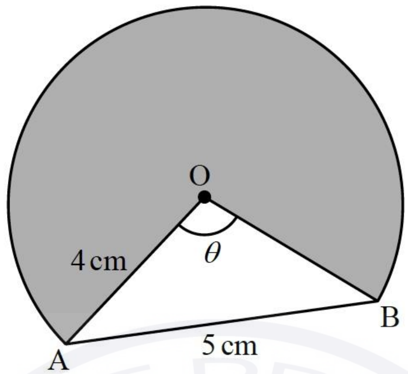
where distances are measured in kilometres.

Find the minimum distance between the two ships.

Question 3

[Maximum mark: 6]

The following diagram shows part of a circle with centre O and radius $4\,\mathrm{cm}$.



Chord AB has a length of $5\,\mathrm{cm}$ and $\mathrm{A\hat{O}B} = \theta$.

(a) Find the value of θ, giving your answer in radians. [3]

(b) Find the area of the shaded region. [3]

Question 4

[Maximum mark: 21]

Three points $\mathrm{A}(3, 0, 0)$, $\mathrm{B}(0, -2, 0)$ and $\mathrm{C}(1, 1, -7)$ lie on the plane Π_1.

(a) (i) Find the vector $\overrightarrow{\mathrm{AB}}$ and the vector $\overrightarrow{\mathrm{AC}}$.

(ii) Hence find the equation of Π_1, expressing your answer in the form $ax + by + cz = d$, where $a, b, c, d \in \mathbb{Z}$. [7]

Plane Π_2 has equation $3x - y + 2z = 2$.

(b) The line L is the intersection of Π_1 and Π_2. Verify that the vector equation of L can be written as $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. [2]

(c) The plane Π_3 is given by $2x - 2z = 3$. The line L and the plane Π_3 intersect at the point P.

(i) Show that at the point P, $\lambda = \frac{3}{4}$.

(ii) Hence find the coordinates of P. [3]

(d) The point $\mathrm{B}(0, -2, 0)$ lies on L.

(i) Find the reflection of the point B in the plane Π_3.

(ii) Hence find the vector equation of the line formed when L is reflected in the plane Π_3. [9]

Question 5

[Maximum mark: 5]

Consider the planes Π_1 and Π_2 with the following equations.

$\Pi_1 :\ 3x + 2y + z = 6$

$\Pi_2 :\ x - 2y + z = 4$

(a) Find a Cartesian equation of the plane Π_3 which is perpendicular to Π_1 and Π_2 and passes through the origin $(0, 0, 0)$. [3]

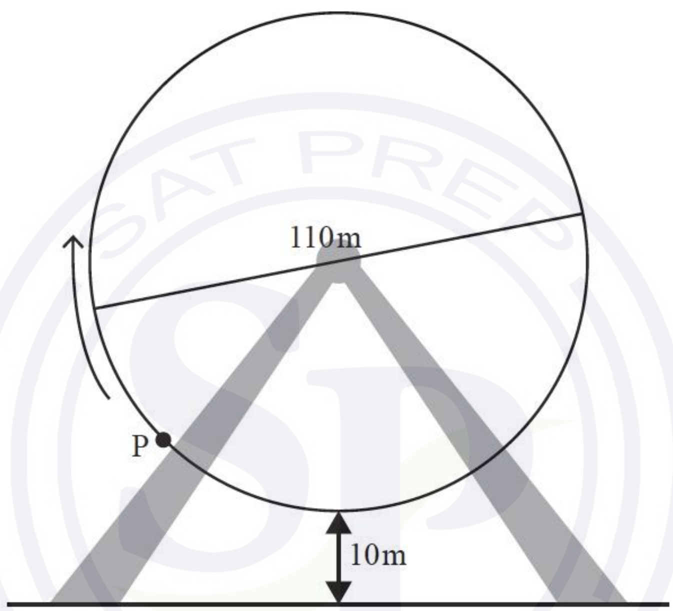
(b) Find the coordinates of the point where Π_1, Π_2 and Π_3 intersect. [2]

Question 6

[Maximum mark: 5]

A Ferris wheel with diameter 110 metres rotates at a constant speed. The lowest point on the wheel is 10 metres above the ground, as shown on the following diagram. P is a point on the wheel. The wheel starts moving with P at the lowest point and completes one revolution in 20 minutes.

diagram not to scale



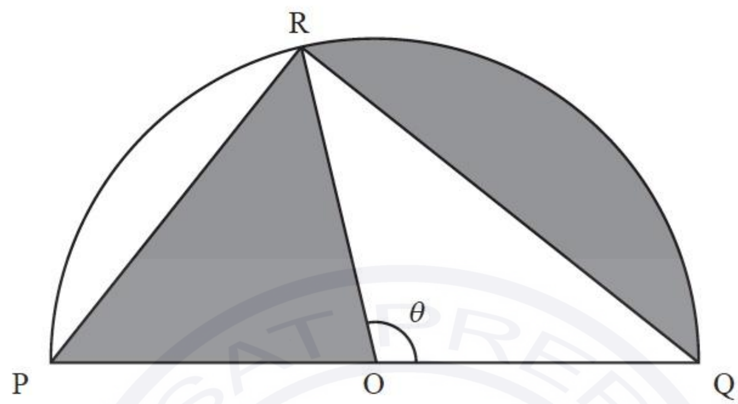
The height, h metres, of P above the ground after t minutes is given by $h(t) = a\cos(bt) + c$, where a, b, $c \in \mathbb{R}$.

Find the values of a, b and c.

Question 7

[Maximum mark: 6]

The following diagram shows a semicircle with centre O and radius r. Points P, Q and R lie on the circumference of the circle, such that $PQ = 2r$ and $R\hat{O}Q = \theta$, where $0 < \theta < \pi$.



(a) Given that the areas of the two shaded regions are equal, show that $\theta = 2\sin\theta$. [5]

(b) Hence determine the value of θ. [1]

Question 8

[Maximum mark: 5]

Consider a triangle ABC, where $AC = 12$, $CB = 7$ and $B\hat{A}C = 25°$.

Find the smallest possible perimeter of triangle ABC.

Question 9

[Maximum mark: 9]

Consider the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$ and $|\boldsymbol{b}| = 15$.

(a) Find the possible range of values for $|\boldsymbol{a} + \boldsymbol{b}|$. [2]

Consider the vector $\boldsymbol{p}$ such that $\boldsymbol{p} = \boldsymbol{a} + \boldsymbol{b}$.

(b) Given that $|\boldsymbol{a} + \boldsymbol{b}|$ is a minimum, find $\boldsymbol{p}$. [2]

Consider the vector $\boldsymbol{q}$ such that $\boldsymbol{q} = \begin{pmatrix} x \\ y \end{pmatrix}$, where $x, y \in \mathbb{R}^+$.

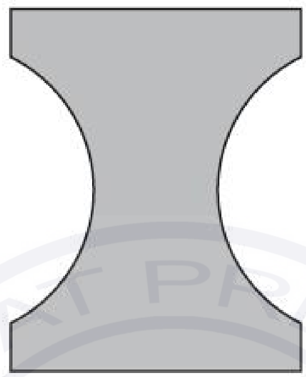
(c) Find $\boldsymbol{q}$ such that $|\boldsymbol{q}| = |\boldsymbol{b}|$ and $\boldsymbol{q}$ is perpendicular to $\boldsymbol{a}$. [5]

Question 10

[Maximum mark: 6]

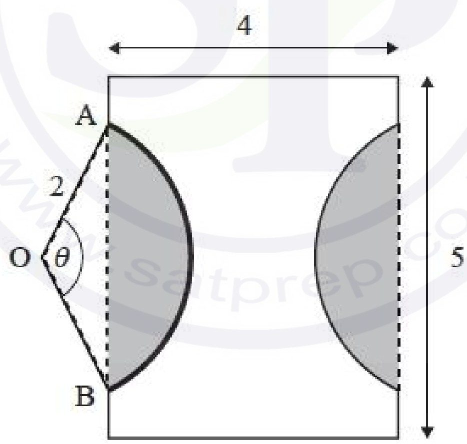
A company is designing a new logo. The logo is created by removing two equal segments from a rectangle, as shown in the following diagram.

diagram not to scale



The rectangle measures $5\,\text{cm}$ by $4\,\text{cm}$. The points A and B lie on a circle, with centre O and radius $2\,\text{cm}$, such that $\hat{AOB} = \theta$, where $0 < \theta < \pi$. This information is shown in the following diagram.

diagram not to scale



(a) Find the area of one of the shaded segments in terms of θ. [3]

(b) Given that the area of the logo is $13.4\,\text{cm}^2$, find the value of θ. [3]

Question 11

[Maximum mark: 20]

Two airplanes, A and B, have position vectors with respect to an origin O given respectively by

$$\boldsymbol{r}_A = \begin{pmatrix} 19 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -6 \\ 2 \\ 4 \end{pmatrix}$$

$$\boldsymbol{r}_B = \begin{pmatrix} 1 \\ 0 \\ 12 \end{pmatrix} + t\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$$

where t represents the time in minutes and $0 \le t \le 2.5$.

Entries in each column vector give the displacement east of O, the displacement north of O and the distance above sea level, all measured in kilometres.

(a) Find the three-figure bearing on which airplane B is travelling. [2]

(b) Show that airplane A travels at a greater speed than airplane B. [2]

(c) Find the acute angle between the two airplanes' lines of flight. Give your answer in degrees. [4]

The two airplanes' lines of flight cross at point P.

(d) (i) Find the coordinates of P.

(ii) Determine the length of time between the first airplane arriving at P and the second airplane arriving at P. [7]

Let $D(t)$ represent the distance between airplane A and airplane B for $0 \le t \le 2.5$.

(e) Find the minimum value of $D(t)$. [5]

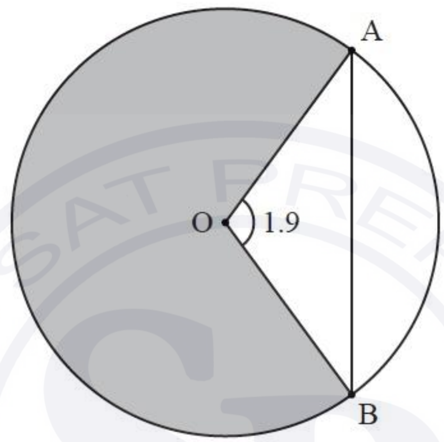
Question 12

[Maximum mark: 6]

The following diagram shows a circle with centre O and radius 5 metres.

Points A and B lie on the circle and $A\hat{O}B = 1.9$ radians.

diagram not to scale



(a) Find the length of the chord [AB]. [3]

(b) Find the area of the shaded sector. [3]

Question 13

[Maximum mark: 22]

Consider the points $\mathrm{A}(1, 2, 3)$, $\mathrm{B}(k, -2, 1)$ and $\mathrm{C}(5, 0, 2)$, where $k \in \mathbb{R}$.

(a) Write down $\vec{\mathrm{AB}}$ and $\vec{\mathrm{AC}}$. [2]

(b) Given that the points A, B and C lie on a straight line, show that $k = 9$. [1]

(c) For $k = 9$, let L_1 be the line passing through A, B and C.

(i) Find a vector equation of the line L_1.

(ii) Line L_2 has the equation $\dfrac{x-1}{2} = \dfrac{y}{3} = 1 - z$. Show that the lines L_1 and L_2 are skew. [10]

(d) For $k \neq 9$, let Π be the plane containing A, B and C.

(i) Find the Cartesian equation of the plane Π.

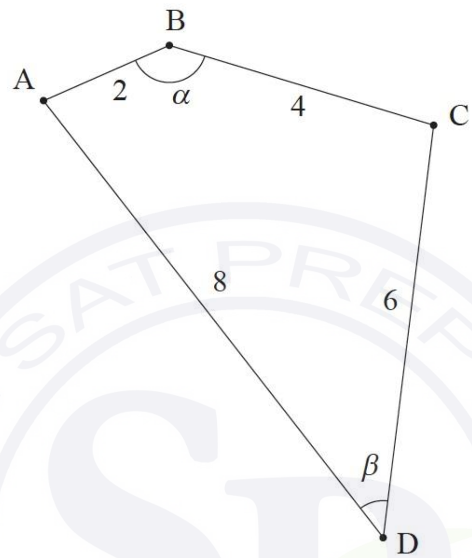
(ii) Find the coordinates of the point on the plane Π which is closest to the origin $(0, 0, 0)$. [9]

Question 14

[Maximum mark: 8]

Consider a quadrilateral ABCD such that $\mathrm{AB}=2$, $\mathrm{BC}=4$, $\mathrm{CD}=6$ and $\mathrm{DA}=8$, as shown in the following diagram. Let $\alpha=\mathrm{A\hat{B}C}$ and $\beta=\mathrm{A\hat{D}C}$.

diagram not to scale



(a) (i) Find AC in terms of α.

(ii) Find AC in terms of β.

(iii) Hence or otherwise, find an expression for α in terms of β. [4]

(b) Find the maximum area of the quadrilateral ABCD. [4]

Question 15

[Maximum mark: 6]

Consider the vectors $\boldsymbol{u}=\boldsymbol{i}+\boldsymbol{j}$ and $\boldsymbol{v}=\left(\cos\frac{1}{n}\right)\boldsymbol{i}+\left(\sin\frac{1}{n}\right)\boldsymbol{j}$, where $n\in\mathbb{Z}^{+}$.

Let θ be the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(a) Find an expression for $\cos\theta$ in terms of n. [3]

(b) Find the exact value of the limit approached by θ as $n\to\infty$. [3]