

Subject - Math AA(Higher Level)
Topic - Geometry and Trigonometry
Year - May 2021 - Nov 2024
Paper -2
Questions

Question 1

[Maximum mark: 19]

(a) Show that $\cot 2\theta = \frac{1 - \tan^2 \theta}{2 \tan \theta}$. [1]

(b) Verify that $x = \tan \theta$ and $x = -\cot \theta$ satisfy the equation $x^2 + (2 \cot 2\theta)x - 1 = 0$. [7]

(c) Hence, or otherwise, show that the exact value of $\tan \frac{\pi}{12} = 2 - \sqrt{3}$. [5]

(d) Using the results from parts (b) and (c) find the exact value of $\tan \frac{\pi}{24} - \cot \frac{\pi}{24}$.
Give your answer in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$. [6]

Question 2

[Maximum mark: 5]

Two ships, A and B, are observed from an origin O. Relative to O, their position vectors at time t hours after midday are given by

$$\mathbf{r}_A = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + t \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$
$$\mathbf{r}_B = \begin{pmatrix} 7 \\ -3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 12 \end{pmatrix}$$

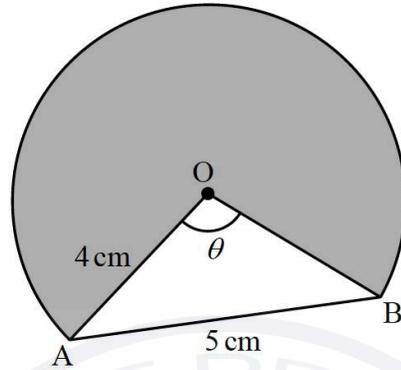
where distances are measured in kilometres.

Find the minimum distance between the two ships.

Question 3

[Maximum mark: 6]

The following diagram shows part of a circle with centre O and radius 4 cm .



Chord AB has a length of 5 cm and $\angle AOB = \theta$.

(a) Find the value of θ , giving your answer in radians.

[3]

(b) Find the area of the shaded region.

[3]

Question 4

[Maximum mark: 21]

Three points $A(3, 0, 0)$, $B(0, -2, 0)$ and $C(1, 1, -7)$ lie on the plane Π_1 .

- (a) (i) Find the vector \vec{AB} and the vector \vec{AC} .
- (ii) Hence find the equation of Π_1 , expressing your answer in the form $ax + by + cz = d$, where $a, b, c, d \in \mathbb{Z}$. [7]

Plane Π_2 has equation $3x - y + 2z = 2$.

- (b) The line L is the intersection of Π_1 and Π_2 . Verify that the vector equation of L can be written as $\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. [2]

(c) The plane Π_3 is given by $2x - 2z = 3$. The line L and the plane Π_3 intersect at the point P .

- (i) Show that at the point P , $\lambda = \frac{3}{4}$.
- (ii) Hence find the coordinates of P . [3]
- (d) The point $B(0, -2, 0)$ lies on L .
- (i) Find the reflection of the point B in the plane Π_3 .
- (ii) Hence find the vector equation of the line formed when L is reflected in the plane Π_3 . [9]

Question 5

[Maximum mark: 5]

Consider the planes Π_1 and Π_2 with the following equations.

$$\Pi_1: 3x + 2y + z = 6$$

$$\Pi_2: x - 2y + z = 4$$

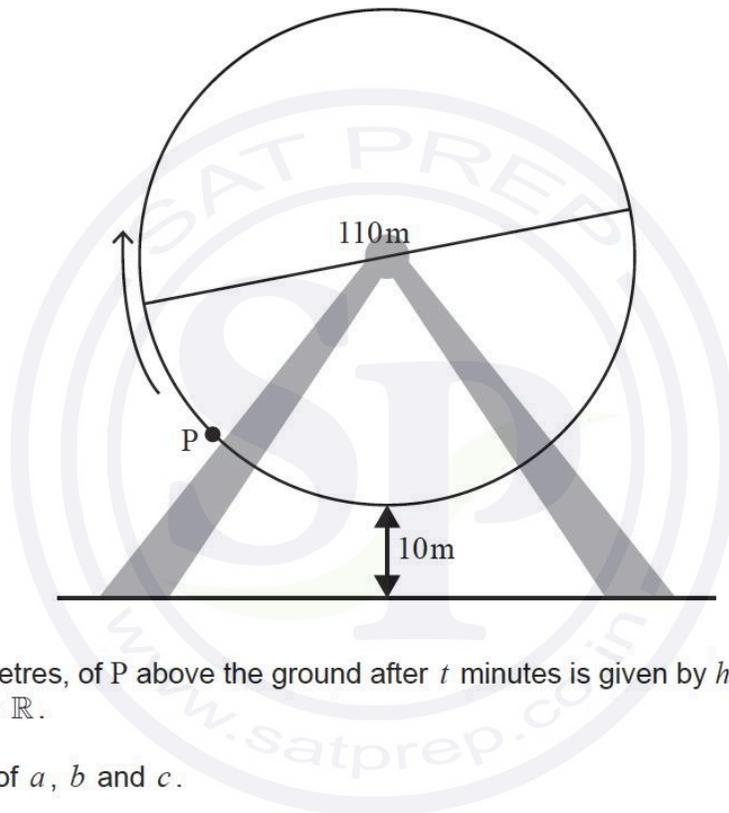
- (a) Find a Cartesian equation of the plane Π_3 which is perpendicular to Π_1 and Π_2 and passes through the origin $(0, 0, 0)$. [3]
- (b) Find the coordinates of the point where Π_1 , Π_2 and Π_3 intersect. [2]

Question 6

[Maximum mark: 5]

A Ferris wheel with diameter 110 metres rotates at a constant speed. The lowest point on the wheel is 10 metres above the ground, as shown on the following diagram. P is a point on the wheel. The wheel starts moving with P at the lowest point and completes one revolution in 20 minutes.

diagram not to scale



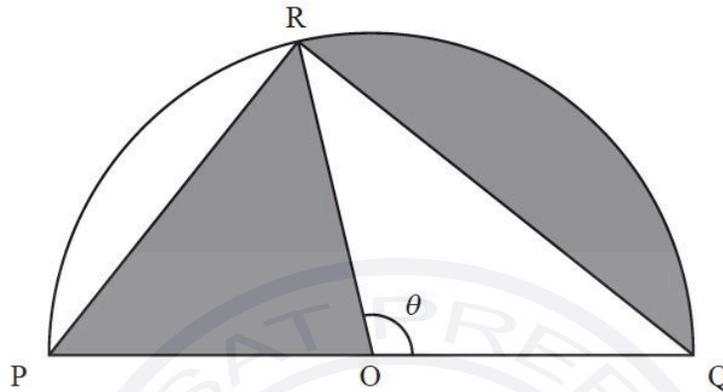
The height, h metres, of P above the ground after t minutes is given by $h(t) = a \cos(bt) + c$, where $a, b, c \in \mathbb{R}$.

Find the values of a , b and c .

Question 7

[Maximum mark: 6]

The following diagram shows a semicircle with centre O and radius r . Points P , Q and R lie on the circumference of the circle, such that $PQ = 2r$ and $\widehat{ROQ} = \theta$, where $0 < \theta < \pi$.



(a) Given that the areas of the two shaded regions are equal, show that $\theta = 2 \sin \theta$. [5]

(b) Hence determine the value of θ . [1]

Question 8

[Maximum mark: 5]

Consider a triangle ABC , where $AC = 12$, $CB = 7$ and $\widehat{BAC} = 25^\circ$.

Find the smallest possible perimeter of triangle ABC .

Question 9

[Maximum mark: 9]

Consider the vectors \mathbf{a} and \mathbf{b} such that $\mathbf{a} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$ and $|\mathbf{b}| = 15$.

(a) Find the possible range of values for $|\mathbf{a} + \mathbf{b}|$. [2]

Consider the vector \mathbf{p} such that $\mathbf{p} = \mathbf{a} + \mathbf{b}$.

(b) Given that $|\mathbf{a} + \mathbf{b}|$ is a minimum, find \mathbf{p} . [2]

Consider the vector \mathbf{q} such that $\mathbf{q} = \begin{pmatrix} x \\ y \end{pmatrix}$, where $x, y \in \mathbb{R}^+$.

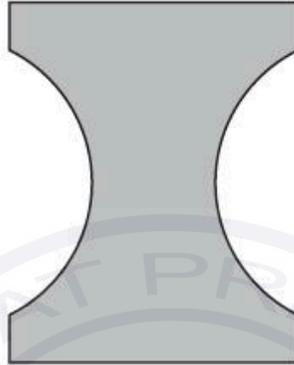
(c) Find \mathbf{q} such that $|\mathbf{q}| = |\mathbf{b}|$ and \mathbf{q} is perpendicular to \mathbf{a} . [5]

Question 10

[Maximum mark: 6]

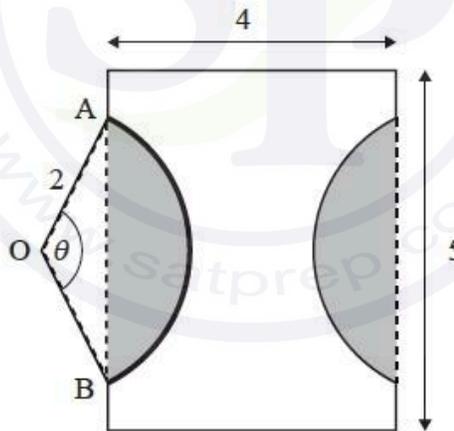
A company is designing a new logo. The logo is created by removing two equal segments from a rectangle, as shown in the following diagram.

diagram not to scale



The rectangle measures 5 cm by 4 cm. The points A and B lie on a circle, with centre O and radius 2 cm, such that $\angle AOB = \theta$, where $0 < \theta < \pi$. This information is shown in the following diagram.

diagram not to scale



- (a) Find the area of one of the shaded segments in terms of θ . [3]
- (b) Given that the area of the logo is 13.4 cm^2 , find the value of θ . [3]

Question 11

[Maximum mark: 20]

Two airplanes, A and B , have position vectors with respect to an origin O given respectively by

$$\mathbf{r}_A = \begin{pmatrix} 19 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -6 \\ 2 \\ 4 \end{pmatrix}$$

$$\mathbf{r}_B = \begin{pmatrix} 1 \\ 0 \\ 12 \end{pmatrix} + t \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$$

where t represents the time in minutes and $0 \leq t \leq 2.5$.

Entries in each column vector give the displacement east of O , the displacement north of O and the distance above sea level, all measured in kilometres.

- (a) Find the three-figure bearing on which airplane B is travelling. [2]
- (b) Show that airplane A travels at a greater speed than airplane B . [2]
- (c) Find the acute angle between the two airplanes' lines of flight. Give your answer in degrees. [4]

The two airplanes' lines of flight cross at point P .

- (d) (i) Find the coordinates of P .
- (ii) Determine the length of time between the first airplane arriving at P and the second airplane arriving at P . [7]

Let $D(t)$ represent the distance between airplane A and airplane B for $0 \leq t \leq 2.5$.

- (e) Find the minimum value of $D(t)$. [5]

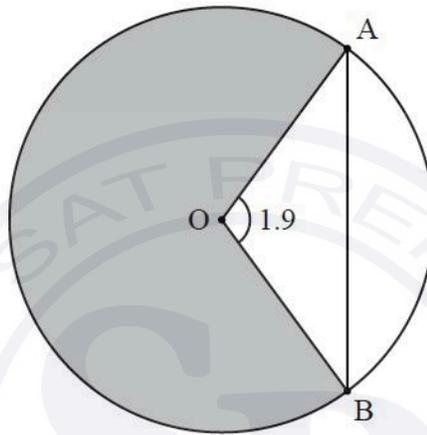
Question 12

[Maximum mark: 6]

The following diagram shows a circle with centre O and radius 5 metres.

Points A and B lie on the circle and $\hat{AOB} = 1.9$ radians.

diagram not to scale



(a) Find the length of the chord $[AB]$.

[3]

(b) Find the area of the shaded sector.

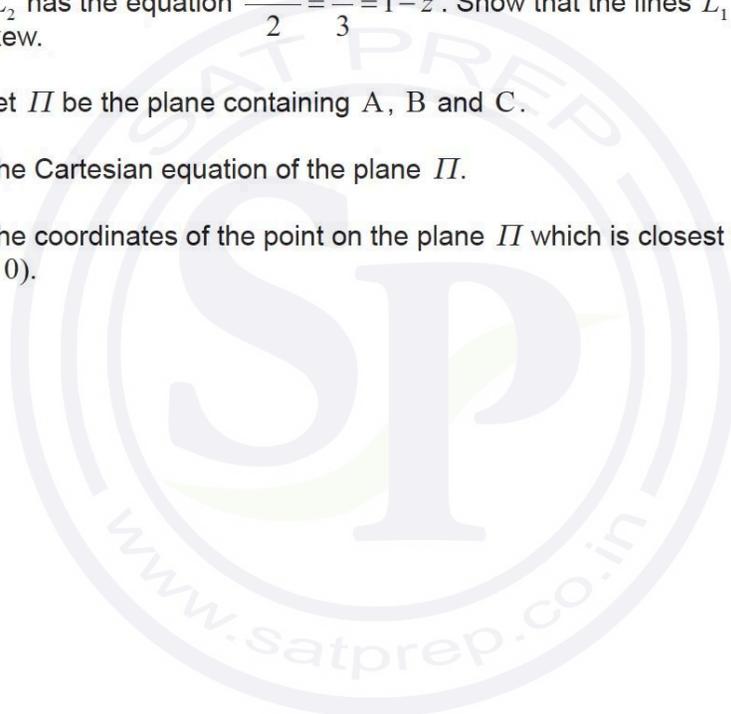
[3]

Question 13

[Maximum mark: 22]

Consider the points $A(1, 2, 3)$, $B(k, -2, 1)$ and $C(5, 0, 2)$, where $k \in \mathbb{R}$.

- (a) Write down \vec{AB} and \vec{AC} . [2]
- (b) Given that the points A , B and C lie on a straight line, show that $k = 9$. [1]
- (c) For $k = 9$, let L_1 be the line passing through A , B and C .
- (i) Find a vector equation of the line L_1 .
- (ii) Line L_2 has the equation $\frac{x-1}{2} = \frac{y}{3} = 1-z$. Show that the lines L_1 and L_2 are skew. [10]
- (d) For $k \neq 9$, let Π be the plane containing A , B and C .
- (i) Find the Cartesian equation of the plane Π .
- (ii) Find the coordinates of the point on the plane Π which is closest to the origin $(0, 0, 0)$. [9]

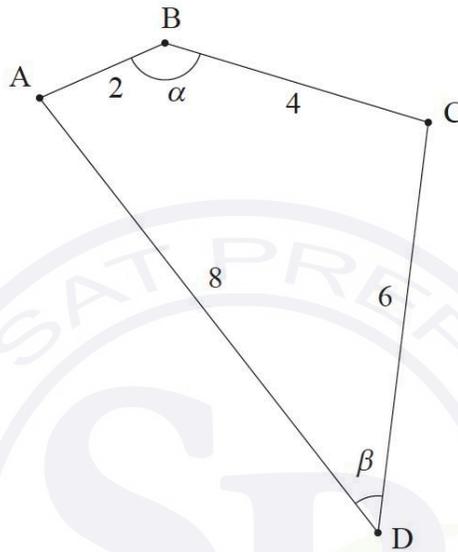


Question 14

[Maximum mark: 8]

Consider a quadrilateral $ABCD$ such that $AB = 2$, $BC = 4$, $CD = 6$ and $DA = 8$, as shown in the following diagram. Let $\alpha = \hat{A}BC$ and $\beta = \hat{A}DC$.

diagram not to scale



- (a) (i) Find AC in terms of α .
- (ii) Find AC in terms of β .
- (iii) Hence or otherwise, find an expression for α in terms of β . [4]
- (b) Find the maximum area of the quadrilateral $ABCD$. [4]

Question 15

[Maximum mark: 6]

Consider the vectors $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \left(\cos \frac{1}{n}\right)\mathbf{i} + \left(\sin \frac{1}{n}\right)\mathbf{j}$, where $n \in \mathbb{Z}^+$.

Let θ be the angle between \mathbf{u} and \mathbf{v} .

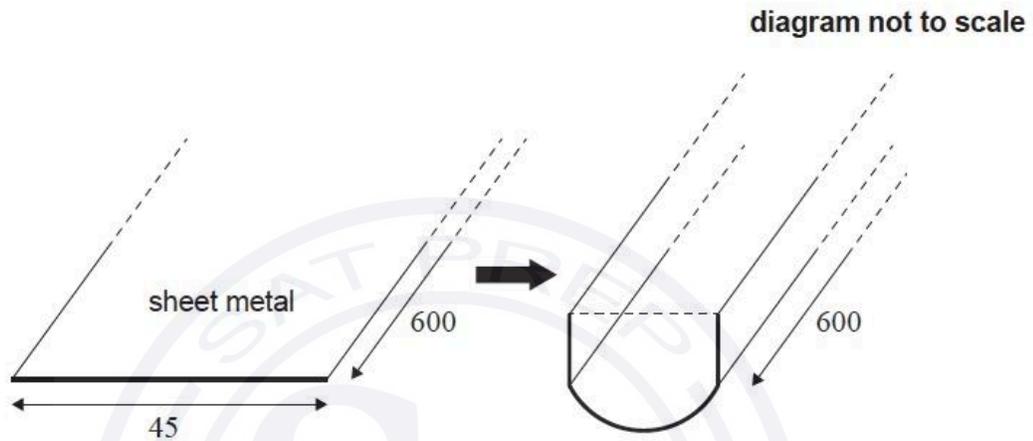
- (a) Find an expression for $\cos \theta$ in terms of n . [3]
- (b) Find the exact value of the limit approached by θ as $n \rightarrow \infty$. [3]

Question 16

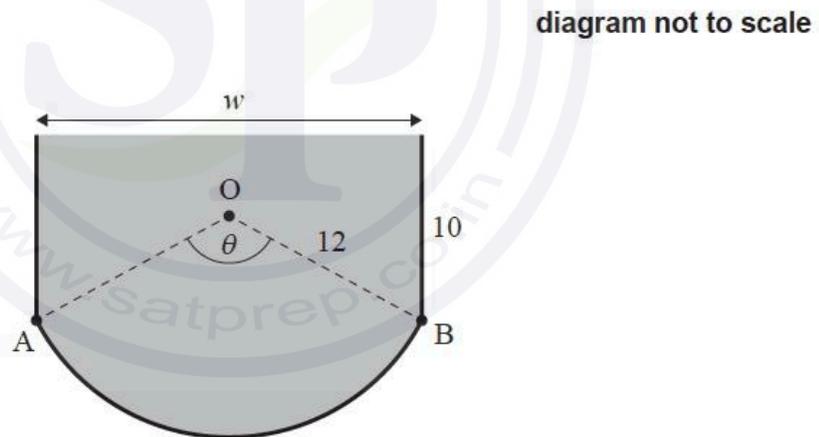
[Maximum mark: 15]

An engineer is designing a gutter to catch rainwater from the roof of a house.

The gutter will be open at the top and is made by folding a piece of sheet metal 45 cm wide and 600 cm long.



The cross-section of the gutter is shaded in the following diagram.



The height of both vertical sides is 10 cm. The width of the gutter is w cm.

Arc AB lies on the circumference of a circle with centre O and radius 12 cm.

Let $\widehat{AOB} = \theta$ radians, where $0 < \theta < \pi$.

(a) Show that $\theta = 2.08$, correct to three significant figures. [3]

(b) Find the area of the cross-section of the gutter. [7]

In a storm, the total volume, in cm^3 , of rainwater that enters the gutter can be modelled by a function $R(t)$, where t is the time, in seconds, since the start of the storm.

It was determined that the rate at which rainwater entered the gutter could be modelled by

$$R'(t) = 50 \cos\left(\frac{2\pi t}{5}\right) + 3000, \quad t \geq 0.$$

During any 60-second period, if the volume of rainwater entering the gutter is greater than the volume of the gutter, it will overflow.

(c) Determine whether the gutter overflowed in this storm. Justify your answer. [5]

Question 17

[Maximum mark: 7]

The angle between a line and a plane is α , where $\alpha \in \mathbb{R}$, $0 < \alpha < \frac{\pi}{2}$.

The equation of the line is $\frac{x-1}{3} = \frac{y+2}{2} = 5-z$, and the equation of the plane is $4x + (\cos \alpha)y + (\sin \alpha)z = 1$.

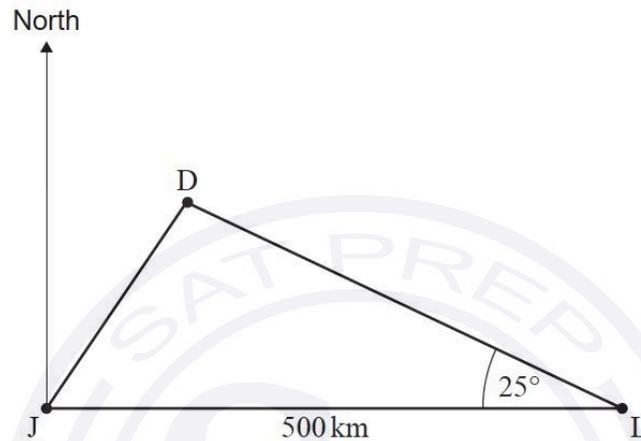
Find the value of α .

Question 18

[Maximum mark: 5]

The cities Lucknow (L), Jaipur (J) and Delhi (D) are represented in the following diagram. Lucknow lies 500 km directly east of Jaipur, and $\angle JLD = 25^\circ$.

diagram not to scale



The bearing of D from J is 034° .

(a) Find \hat{JDL} .

[2]

(b) Find the distance between Lucknow and Delhi.

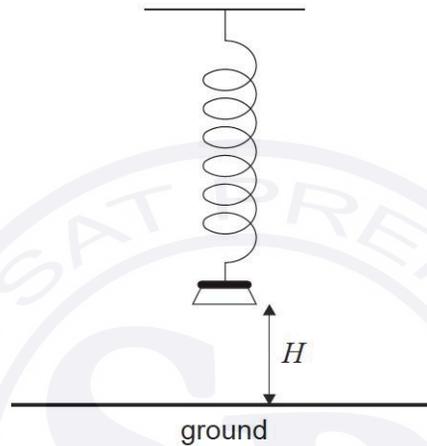
[3]

Question 19

[Maximum mark: 13]

A weight suspended on a spring is pulled down and released, so that it moves up and down vertically.

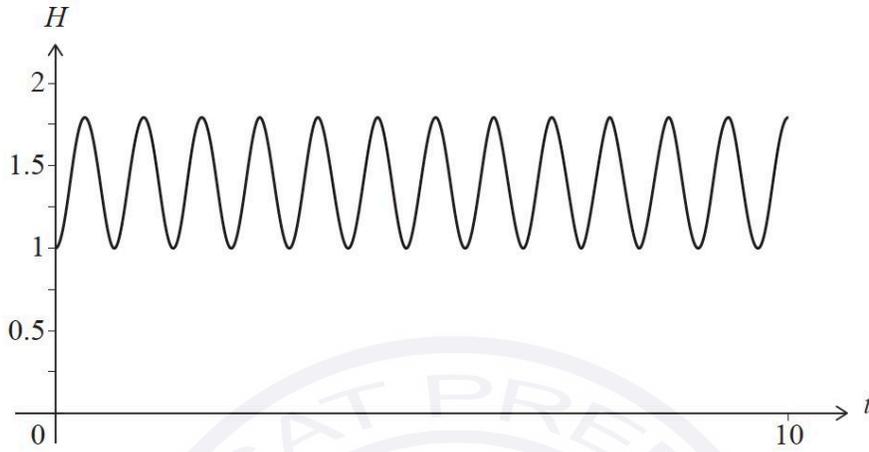
The height, H metres, of the base of the weight above the ground can be modelled by the function $H(t) = a \cos(7.8t) + b$, for $a, b \in \mathbb{R}$ and $0 \leq t \leq 10$, where t is the time in seconds after the weight is released.



- (a) Find the period of the function.

[2]

The weight is released when its base is at a minimum height of 1 metre above the ground, and it reaches a maximum height of 1.8 metres above the ground. The graph of H is shown in the following diagram.



- (b) Find the value of
- (i) a ;
 - (ii) b . [3]
- (c) Find the number of times that the weight reaches its maximum height in the first five seconds of its motion. [2]
- (d) Find the first time that the base of the weight reaches a height of 1.5 metres. [2]

A camera is set to take a picture of the weight at a random time during the first five seconds of its motion.

- (e) Find the probability that the height of the base of the weight is greater than 1.5 metres at the time the picture is taken. [4]

Question 20

[Maximum mark: 8]

Consider the two planes

$$\Pi_1 : 2x - y + 2z = 6$$

$$\Pi_2 : 4x + 3y - z = 2$$

Let L be the line of intersection of Π_1 and Π_2 .

- (a) Verify that a vector equation of L is $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$, where $\lambda \in \mathbb{R}$. [3]
- (b) Find the coordinates of the point P on L that is nearest to the origin. [5]

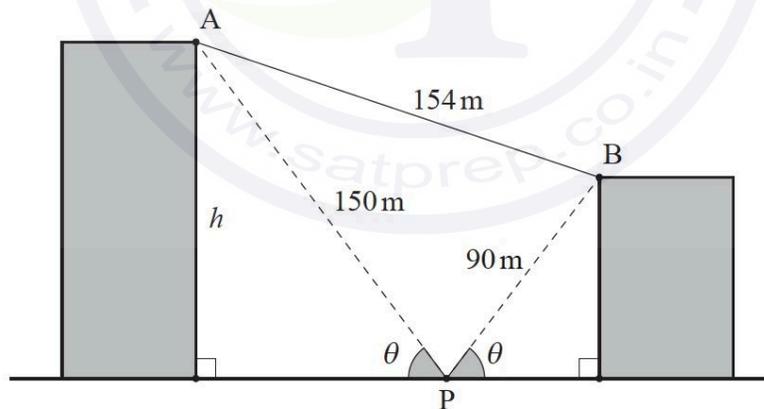
Question 21

[Maximum mark: 6]

The following diagram shows two buildings situated on level ground.

From point P on the ground directly between the two buildings, the angle of elevation to the top of each building is θ .

diagram not to scale



The distance from point P to point A at the top of the taller building is 150 metres.

The distance from point P to point B at the top of the shorter building is 90 metres.

The distance between A and B is 154 metres.

Find the height, h , of the taller building.

Question 22

[Maximum mark: 18]

Line L is given by the vector equation $\mathbf{r}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + s \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ where $s \in \mathbb{R}$.

Line M is given by the vector equation $\mathbf{r}_2 = \begin{pmatrix} 9 \\ 9 \\ 11 \end{pmatrix} + t \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ where $t \in \mathbb{R}$.

(a) Show that lines L and M intersect at a point A and find the position vector of A . [5]

(b) Verify that the lines L and M both lie in the plane Π given by $\mathbf{r} \cdot \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 7$. [3]

Point B has position vector $\begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix}$. A line through B perpendicular to Π intersects Π at point C .

(c) (i) Find the position vector of C .

(ii) Hence, find $|\vec{BC}|$. [7]

(d) Find the reflection of the point B in the plane Π . [3]

Question 23

[Maximum mark: 9]

Three points are given by $A(0, p, 2)$, $B(1, 1, 1)$ and $C(p, 0, 4)$, where p is a positive constant.

(a) Show that $\vec{AB} \times \vec{AC} = \begin{pmatrix} 2-3p \\ -2-p \\ p^2-2p \end{pmatrix}$. [4]

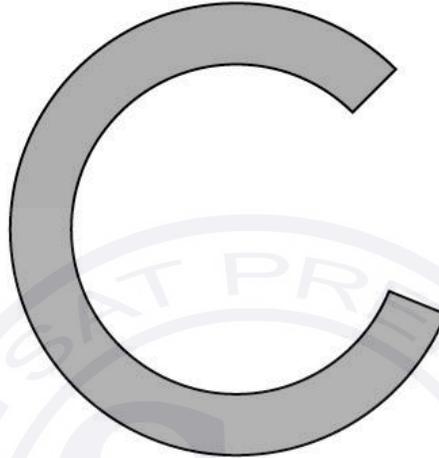
(b) Hence, find the smallest possible value of $|\vec{AB} \times \vec{AC}|^2$. [3]

(c) Hence, find the smallest possible area of triangle ABC . [2]

Question 24

[Maximum mark: 7]

A company is designing a new logo in the shape of a letter “C”.



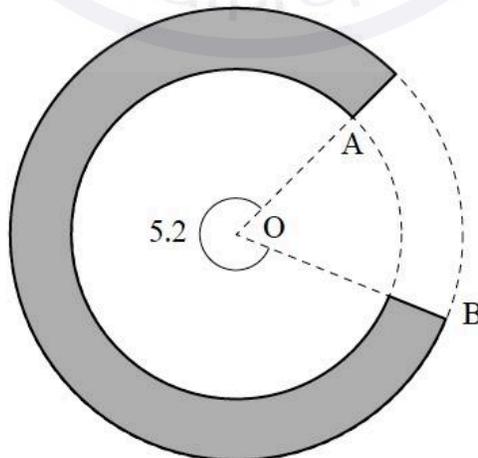
The letter “C” is formed between two circles with centre O .

The point A lies on the circumference of the inner circle with radius r cm, where $r < 10$.

The point B lies on the circumference of the outer circle with radius 10 cm.

The reflex angle \widehat{AOB} is 5.2 radians. The letter “C” is shown by the shaded area in the following diagram.

diagram not to scale



- (a) Show that the area of the "C" is given by $260 - 2.6r^2$. [2]

The area of the "C" is 64 cm^2 .

- (b) (i) Find the value of r .
(ii) Find the perimeter of the "C". [5]

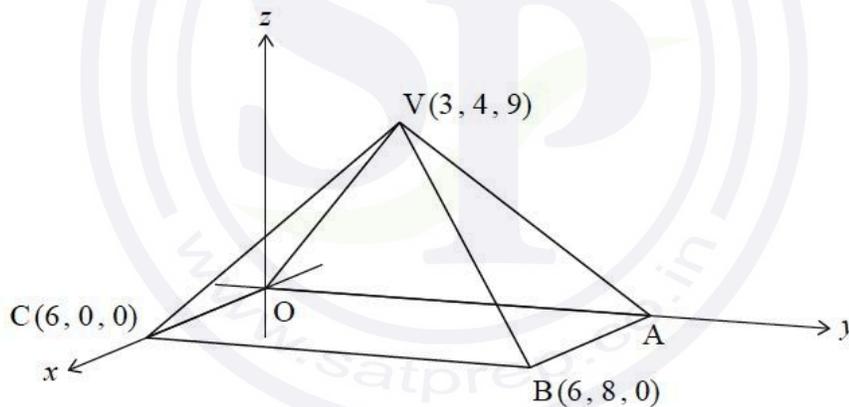
Question 25

[Maximum mark: 6]

The following diagram shows a pyramid with vertex V and rectangular base $OABC$.

Point B has coordinates $(6, 8, 0)$, point C has coordinates $(6, 0, 0)$ and point V has coordinates $(3, 4, 9)$.

diagram not to scale



- (a) Find BV . [2]
(b) Find the size of \widehat{BVC} . [4]

Question 26

[Maximum mark: 17]

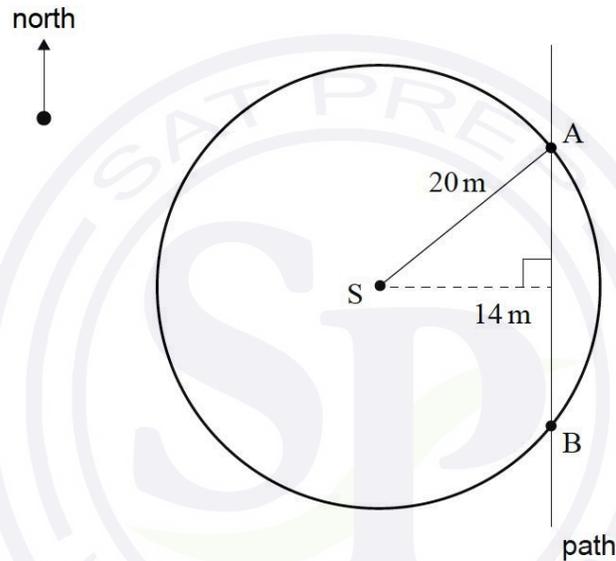
A rotating sprinkler is at a fixed point S .

It waters all points inside and on a circle of radius 20 metres.

Point S is 14 metres from the edge of a path which runs in a north-south direction.

The edge of the path intersects the circle at points A and B .

This information is shown in the following diagram.



- (a) Show that $AB = 28.57$, correct to four significant figures. [3]

The sprinkler rotates at a constant rate of one revolution every 16 seconds.

- (b) Show that the sprinkler rotates through an angle of $\frac{\pi}{8}$ radians in one second. [1]

Let T seconds be the time that $[AB]$ is watered in each revolution.

- (c) Find the value of T . [4]

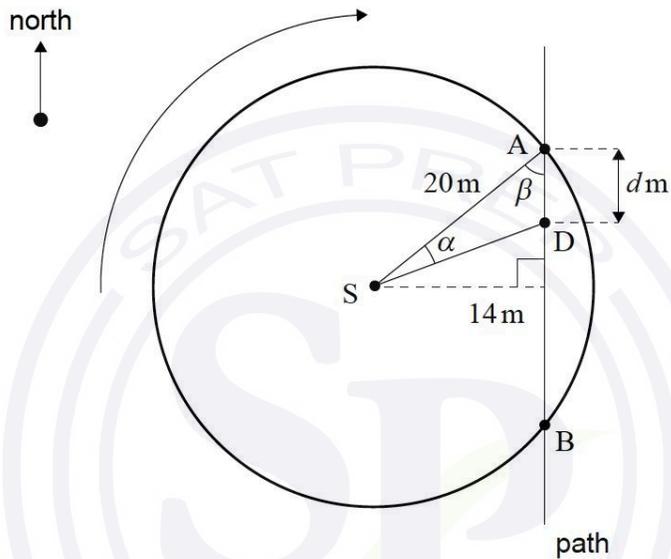
Consider one clockwise revolution of the sprinkler.

At $t = 0$, the water crosses the edge of the path at A.

At time t seconds, the water crosses the edge of the path at a movable point D which is a distance d metres south of point A.

Let $\alpha = \widehat{ASD}$ and $\beta = \widehat{SAB}$, where α, β are measured in radians.

This information is shown in the following diagram.



(d) Write down an expression for α in terms of t . [1]

It is known that $\beta = 0.7754$ radians, correct to four significant figures.

(e) By using the sine rule in $\triangle ASD$, show that the distance, d , at time t , can be modelled by

$$d(t) = \frac{20 \sin\left(\frac{\pi t}{8}\right)}{\sin\left(2.37 - \frac{\pi t}{8}\right)}. \quad [3]$$

A turtle walks south along the edge of the path.

At time t seconds, the turtle's distance, g metres south of A , can be modelled by

$$g(t) = 0.05t^2 + 1.1t + 18, \text{ where } t \geq 0.$$

(f) At $t = 0$, state how far south the turtle is from A . [1]

Let w represent the distance between the turtle and point D at time t seconds.

(g) (i) Use the expressions for $g(t)$ and $d(t)$ to write down an expression for w in terms of t .

(ii) Hence find when and where on the path the water first reaches the turtle. [4]

Question 27

[Maximum mark: 20]

Consider the non-zero vectors \mathbf{u} and \mathbf{v} . Let θ be the angle between \mathbf{u} and \mathbf{v} .

(a) Using the definitions of $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ in terms of $|\mathbf{u}|$, $|\mathbf{v}|$ and θ , show that $(\mathbf{u} \cdot \mathbf{v})^2 + |\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2 |\mathbf{v}|^2$. [2]

A triangle ABC has vertices $A(0, 1, 2)$, $B(p, q, 3)$ and $C(3, 2, 1)$, $p, q \in \mathbb{Q}$.

The vectors \mathbf{u} and \mathbf{v} are defined as $\mathbf{u} = \vec{AB}$ and $\mathbf{v} = \vec{AC}$.

It is given that $\mathbf{u} \cdot \mathbf{v} = 3$ and the area of triangle ABC is $\sqrt{6}$.

(b) (i) Find the value of $|\mathbf{u} \times \mathbf{v}|$.
(ii) Hence, or otherwise, find the value of $|\mathbf{u}|$.
(iii) Hence, or otherwise, find the possible values of p and the corresponding values of q . [13]

Consider a new point D , the vector \mathbf{w} is defined as $\mathbf{w} = \vec{CD}$.

It is given that $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} = 0$ and the area of triangle ACD is 5 square units.

(c) Assuming that $p = 1$, find the possible vectors for \mathbf{w} . [5]

Question 28

[Maximum mark: 16]

Sule Skerry and Rockall are small islands in the Atlantic Ocean, in the same time zone.

On a given day, the height of water in metres at Sule Skerry is modelled by the function $H(t) = 1.63 \sin(0.513(t - 8.20)) + 2.13$, where t is the number of hours after midnight.

The following graph shows the height of the water for 15 hours, starting at midnight.

At low tide the height of the water is 0.50 m. At high tide the height of the water is 3.76 m.

All heights are given correct to two decimal places.



- (a) The length of time between the first low tide and the first high tide is 6 hours and m minutes. Find the value of m to the nearest integer. [3]
- (b) Between two consecutive high tides, determine the length of time, in hours, for which the height of the water is less than 1 metre. [2]
- (c) Find the rate of change of the height of the water when $t = 13$, giving your answer in metres per hour. [2]

On the same day, the height of water at the second island, Rockall, is modelled by the function $h(t) = a \sin(b(t - c)) + d$, where t is the number of hours after midnight, and $a, b, c, d > 0$.

The first low tide occurs at 02:41 when the height of the water is 0.40 m.

The first high tide occurs at 09:02 when the height of the water is 2.74 m.

(d) Find the values of a, b, c and d . [7]

When $t = T$, the height of the water at Sule Skerry is the same as the height of the water at Rockall for the first time.

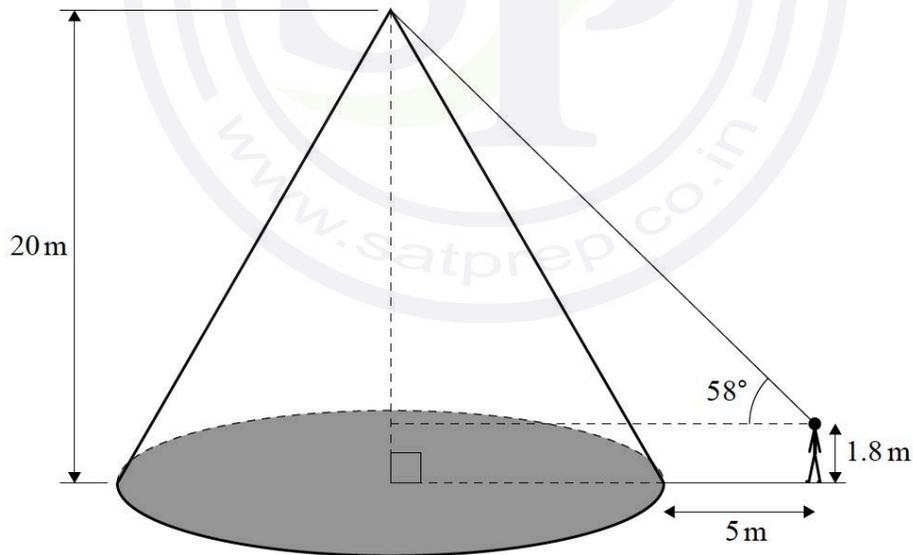
(e) Find the value of T . [2]

Question 29

[Maximum mark: 5]

A monument is in the shape of a right cone with a vertical height of 20 metres. Oliver stands 5 metres from the base of the monument. His eye level is 1.8 metres above the ground and the angle of elevation from Oliver's eye level to the vertex of the cone is 58° , as shown on the following diagram.

diagram not to scale



(a) Find the radius of the base of the cone. [3]

(b) Find the volume of the monument. [2]

Question 30

[Maximum mark: 18]

A line L is defined by $L: -\frac{x}{2} + 1 = y + 4 = \frac{z}{3}$.

- (a) Find the equation of L , expressing your answer in the form $r = a + \lambda b$, where $\lambda \in \mathbb{R}$. [3]
- (b) Determine the minimum distance from the origin O to the line L . [5]

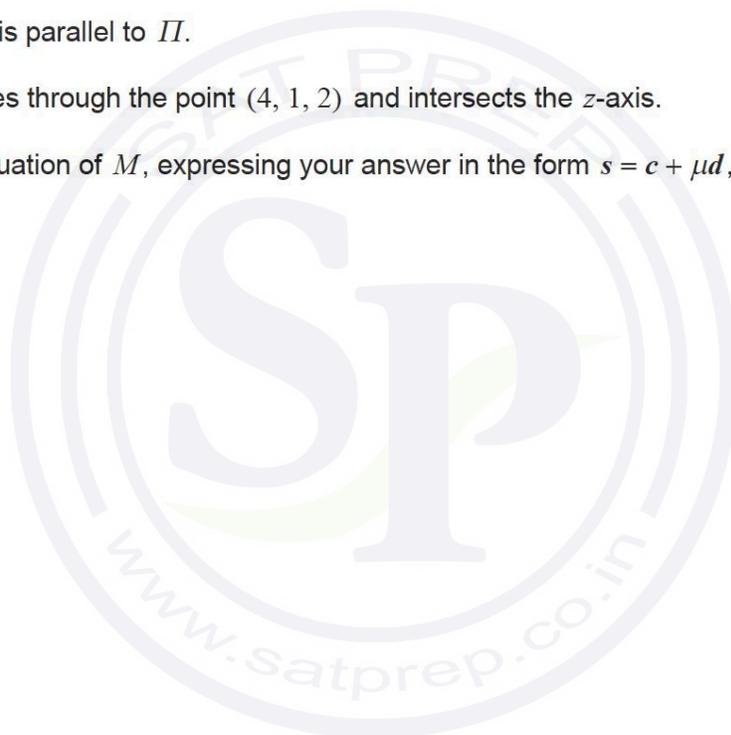
A plane Π is defined by $\Pi: 6x - 3y + 5z = 24$.

- (c) Verify that Π contains L . [3]

A second line M is parallel to Π .

The line M passes through the point $(4, 1, 2)$ and intersects the z -axis.

- (d) Find the equation of M , expressing your answer in the form $s = c + \mu d$, where $\mu \in \mathbb{R}$. [7]



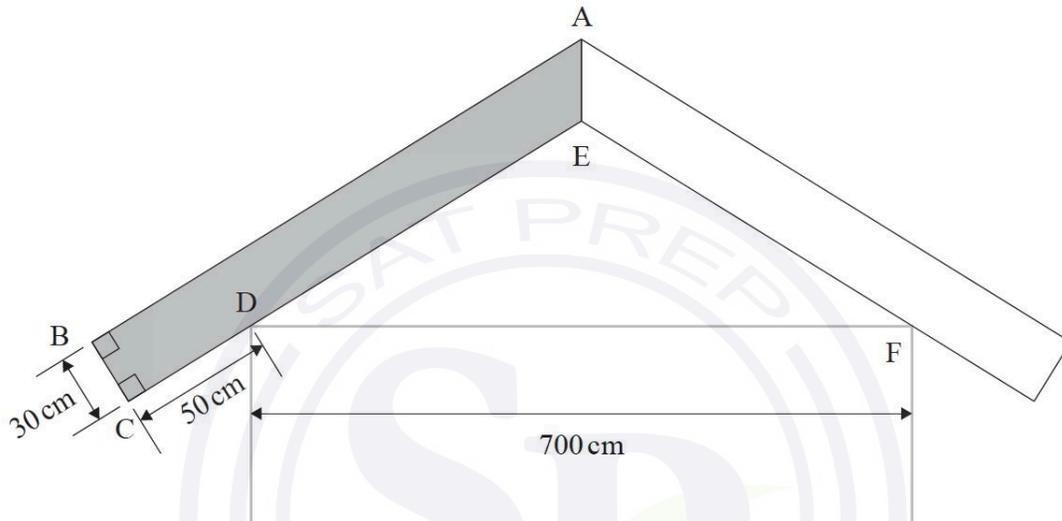
Question 31

A builder requires the lengths of the sides [BA] and [CE].

The builder has the following measurements:

$\hat{A}BC = \hat{B}CE = 90^\circ$, $DC = 50$ cm, $BC = 30$ cm, and $DF = 700$ cm.

diagram not to scale



- (b) Find
- (i) CE ;
 - (ii) BA .

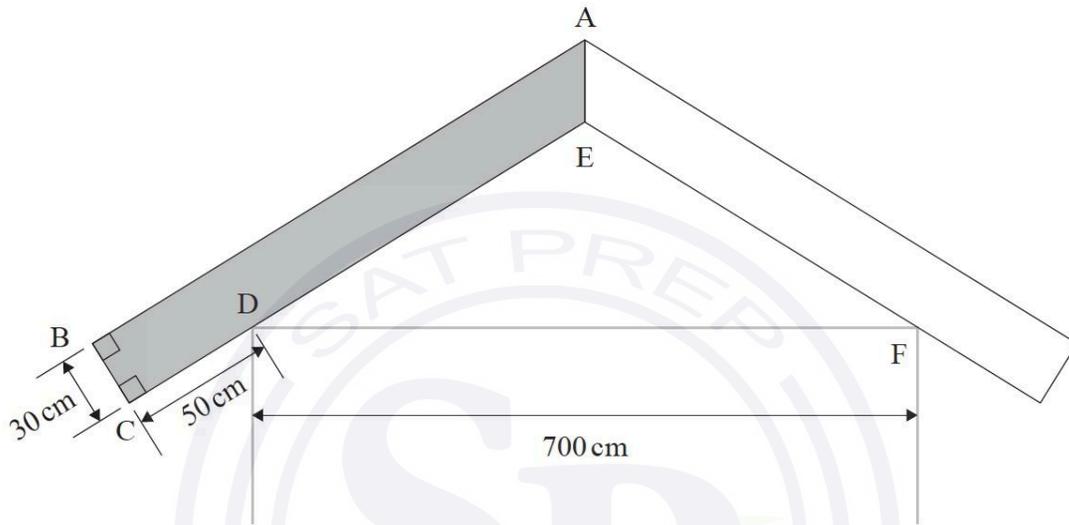
[5]

A builder requires the lengths of the sides [BA] and [CE].

The builder has the following measurements:

$\hat{A}BC = \hat{B}CE = 90^\circ$, $DC = 50$ cm, $BC = 30$ cm, and $DF = 700$ cm.

diagram not to scale



- (b) Find
- (i) CE;
 - (ii) BA.

[5]