Subject – Math AA(Higher Level)
Topic - Geometry and Trigonometry
Year - May 2021 – Nov 2022
Paper -2
Answers

Question 1

(a) stating the relationship between cot and tan and stating the identity for $\tan 2\theta$ ***M1***

$\cot 2\theta = \frac{1}{\tan 2\theta}$ and $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$ ***AG***

[1 mark]

(b)

attempting to substitute $\tan\theta$ for x and using the result from (a) ***M1***

$\text{LHS} = \tan^2\theta + 2\tan\theta\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$\tan^2\theta + 1 - \tan^2\theta - 1 = 0 (= \text{RHS})$ ***A1***

so $x = \tan\theta$ satisfies the equation ***AG***

attempting to substitute $-\cot\theta$ for x and using the result from (a) ***M1***

$\text{LHS} = \cot^2\theta - 2\cot\theta\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$= \frac{1}{\tan^2\theta} - \left(\frac{1-\tan^2\theta}{\tan^2\theta}\right) - 1$ ***A1***

$\frac{1}{\tan^2\theta} - \frac{1}{\tan^2\theta} + 1 - 1 = 0 (= \text{RHS})$ ***A1***

so $x = -\cot\theta$ satisfies the equation ***AG***

7 marks

(c) ***METHOD 1***

$x = \tan\frac{\pi}{12}$ and $x = -\cot\frac{\pi}{12}$ are roots of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$ ***R1***

Note: Award ***R1*** if only $x = \tan\frac{\pi}{12}$ is stated as a root of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$.

$x^2 + 2\sqrt{3}x - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$x = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ $\left(-\cot\frac{\pi}{12} < 0\right)$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

METHOD 2

attempting to substitute $\theta = \frac{\pi}{12}$ into the identity for $\tan 2\theta$ ***M1***

$$\tan\frac{\pi}{6} = \frac{2\tan\frac{\pi}{12}}{1 - \tan^2\frac{\pi}{12}}$$

$\tan^2\frac{\pi}{12} + 2\sqrt{3}\tan\frac{\pi}{12} - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$\tan\frac{\pi}{12} = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

[5 marks]

(d) $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$ is the sum of the roots of $x^2 + \left(2\cot\frac{\pi}{12}\right)x - 1 = 0$ ***R1***

$\tan\frac{\pi}{24} - \cot\frac{\pi}{24} = -2\cot\frac{\pi}{12}$ ***A1***

$= \frac{-2}{2 - \sqrt{3}}$ ***A1***

attempting to rationalise **their** denominator ***(M1)***

$= -4 - 2\sqrt{3}$ ***A1A1***

[6 marks]

Total [19 marks]

Question 2

attempting to find $r_B - r_A$ for example **(M1)**

$$r_B - r_A = \begin{pmatrix} 3 \\ -6 \end{pmatrix} + t\begin{pmatrix} -5 \\ 4 \end{pmatrix}$$

attempting to find $|r_B - r_A|$ **M1**

distance $d(t) = \sqrt{(3-5t)^2 + (4t-6)^2}\left(= \sqrt{41t^2 - 78t + 45}\right)$ **A1**

using a graph to find the $d-$ coordinate of the local minimum **M1**

the minimum distance between the ships is $2.81\ (\text{km})\left(= \frac{11\sqrt{41}}{41}\ (\text{km})\right)$ **A1**

Total [5 marks]

Question 3

(a) ***METHOD 1***

attempt to use the cosine rule ***(M1)***

$\cos\theta = \frac{4^2 + 4^2 - 5^2}{2 \times 4 \times 4}$ (or equivalent) ***A1***

$\theta = 1.35$ ***A1***

[3 marks]

METHOD 2

attempt to split triangle AOB into two congruent right triangles ***(M1)***

$\sin\left(\frac{\theta}{2}\right) = \frac{2.5}{4}$ ***A1***

$\theta = 1.35$ ***A1***

[3 marks]

(b) attempt to find the area of the shaded region ***(M1)***

$\frac{1}{2} \times 4 \times 4 \times (2\pi - 1.35\ldots)$ ***A1***

$= 39.5\ (\text{cm}^2)$ ***A1***

[3 marks]

Total [6 marks]

Question 4

(a) (i) attempts to find either $\vec{AB}$ or $\vec{AC}$ **(M1)**

$\vec{AB} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$ and $\vec{AC} = \begin{pmatrix} -2 \\ 1 \\ -7 \end{pmatrix}$ **A1**

(ii) **METHOD 1**

attempts to find $\vec{AB} \times \vec{AC}$ **(M1)**

$\vec{AB} \times \vec{AC} = \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix}$ **A1**

EITHER

equation of plane is of the form $14x - 21y - 7z = d$ $(2x - 3y - z = d)$ **(A1)**

substitutes a valid point e.g $(3,0,0)$ to obtain a value of d **M1**

$d = 42$ $(d = 6)$

OR

attempts to use $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ **(M1)**

$\boldsymbol{r} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} \left(\boldsymbol{r} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} = 42 \right)$ **A1**

$\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \left(\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = 6 \right)$

THEN

$14x - 21y - 7z = 42$ $(2x - 3y - z = 6)$ **A1**

METHOD 2

equation of plane is of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -7 \end{pmatrix}$ **A1**

attempts to form equations for x, y, z in terms of their parameters **(M1)**

$x = 3 - 3s - 2t$, $y = -2s + t$, $z = -7t$

A1

eliminates at least one of their parameters **(M1)**

for example, $2x - 3y = 6 - 7t (\Rightarrow 2x - 3y = 6 + z)$

$2x - 3y - z = 6$ **A1**

[7 marks]

(b) **METHOD 1**

substitutes $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ into their Π_1 and Π_2 (given) ***(M1)***

$\Pi_1 : 2\lambda - 3(-2+\lambda) - (-\lambda) = 6$ and $\Pi_2 : 3\lambda - (-2+\lambda) + 2(-\lambda) = 2$ ***A1***

Note: Award ***(M1)A0*** for correct verification using a specific value of λ.

so the vector equation of L can be written as $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ ***AG***

METHOD 2
EITHER

attempts to find $\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ ***M1***

$= \begin{pmatrix} -7 \\ -7 \\ 7 \end{pmatrix}$

OR

$\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = (2-3+1) = 0$ and $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = (3-1-2) = 0$ ***M1***

THEN

substitutes $(0, -2, 0)$ into Π_1 and Π_2

$\Pi_1 : 2(0) - 3(-2) - (0) = 6$ and $\Pi_2 : 3(0) - (-2) + 2(0) = 2$ ***A1***

so the vector equation of L can be written as $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ ***AG***

METHOD 3

attempts to solve $2x - 3y - z = 6$ and $3x - y + 2z = 2$ ***(M1)***

for example, $x = -\lambda, y = -2 - \lambda, z = \lambda$ ***A1***

Note: Award **A1** for substituting $x = 0$ (or $y = -2$ or $z = 0$) into Π_1 and Π_2 and solving simultaneously. For example, solving $-3y - z = 6$ and $-y + 2z = 2$ to obtain $y = -2$ and $z = 0$.

so the vector equation of L can be written as $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ ***AG***

[2 marks]

(c) (i) substitutes the equation of L into the equation of Π_3 ***(M1)***

$2\lambda + 2\lambda = 3 \Rightarrow 4\lambda = 3$ ***A1***

$\lambda = \frac{3}{4}$ ***AG***

(ii) P has coordinates $\left(\frac{3}{4}, -\frac{5}{4}, -\frac{3}{4}\right)$ ***A1***

[3 marks]

(d) (i) normal to Π_3 is $\boldsymbol{n} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ ***(A1)***

Note: May be seen or used anywhere.

considers the line normal to Π_3 passing through $\mathrm{B}(0, -2, 0)$ ***(M1)***

$\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ ***A1***

EITHER

finding the point on the normal line that intersects Π_3

attempts to solve simultaneously with plane $2x - 2z = 3$ ***(M1)***

$4\mu + 4\mu = 3$

$\mu = \frac{3}{8}$ ***A1***

point is $\left(\frac{3}{4}, -2, -\frac{3}{4}\right)$

OR

$$\left(\begin{pmatrix} 2\mu \\ -2 \\ -2\mu \end{pmatrix} - \begin{pmatrix} \frac{3}{4} \\ -\frac{5}{4} \\ -\frac{3}{4} \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = 0$$ ***(M1)***

$$4\mu - \frac{3}{2} + 4\mu - \frac{3}{2} = 0$$

$$\mu = \frac{3}{8}$$ ***A1***

OR

attempts to find the equation of the plane parallel to Π_3 containing B' $(x - z = 3)$ and solve simultaneously with L ***(M1)***

$$2\mu' + 2\mu' = 3$$

$$\mu' = \frac{3}{4}$$ ***A1***

THEN

so, another point on the reflected line is given by

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \frac{3}{4}\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$ ***(A1)***

$$\Rightarrow \mathrm{B}'\left(\frac{3}{2}, -2, -\frac{3}{2}\right)$$ ***A1***

(ii) **EITHER**

attempts to find the direction vector of the reflected line using their P and B' ***(M1)***

$$\overrightarrow{\mathrm{PB}'} = \begin{pmatrix} \frac{3}{4} \\ -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix}$$

OR

attempts to find their direction vector of the reflected line using a vector approach ***(M1)***

$$\overrightarrow{\mathrm{PB}'} = \overrightarrow{\mathrm{PB}} + \overrightarrow{\mathrm{BB}'} = -\frac{3}{4}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

THEN

$$\boldsymbol{r} = \begin{pmatrix} \frac{3}{2} \\ -2 \\ -\frac{3}{2} \end{pmatrix} + \lambda\begin{pmatrix} \frac{3}{4} \\ -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix}$$ (or equivalent) ***A1***

Note: Award ***A0*** for either '$\boldsymbol{r} =$' or '$\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$' not stated. Award ***A0*** for '$L' =$'.

[9 marks]
Total [21 marks]

Question 5

(a) attempt to find a vector perpendicular to Π_1 and Π_2 using a cross product ***(M1)***

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = (2-(-2))\boldsymbol{i} + (1-3)\boldsymbol{j} + (-6-2)\boldsymbol{k}$$

$$= \begin{pmatrix} 4 \\ -2 \\ -8 \end{pmatrix} \left(= 2\begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} \right)$$ ***(A1)***

equation is $4x - 2y - 8z = 0 (\Rightarrow 2x - y - 4z = 0)$ ***A1***

[3 marks]

(b) attempt to solve 3 simultaneous equations in 3 variables ***(M1)***

$\left(\frac{41}{21}, -\frac{10}{21}, \frac{23}{21} \right) (= (1.95, -0.476, 1.10))$ ***A1***

[2 marks]
Total [5 marks]

Question 6

Amplitude is $\frac{110}{2} = 55$ *(A1)*

$a = -55$ ***A1***

$c = 65$ ***A1***

$\frac{2\pi}{b} = 20$ OR $-55\cos(20b) + 65 = 10$ *(M1)*

$b = \frac{\pi}{10}(= 0.314)$ ***A1***

Total [5 marks]

Question 7

(a) attempt to find the area of either shaded region in terms of r and θ ***(M1)***

Note*:* Do not award ***M1*** if they have only copied from the booklet and not applied to the shaded area.

Area of segment $=\frac{1}{2}r^2\theta-\frac{1}{2}r^2\sin\theta$ ***A1***

Area of triangle $=\frac{1}{2}r^2\sin(\pi-\theta)$ ***A1***

correct equation in terms of θ only ***(A1)***

$\theta-\sin\theta=\sin(\pi-\theta)$

$\theta-\sin\theta=\sin\theta$ ***A1***

$\theta=2\sin\theta$ ***AG***

Note*:* Award a maximum of ***M1A1A0A0A0*** if a candidate uses degrees (i.e., $\frac{1}{2}r^2\sin(180°-\theta)$), even if later work is correct.

Note*:* If a candidate directly states that the area of the triangle is $\frac{1}{2}r^2\sin\theta$, award a maximum of ***M1A1A0A1A1***.

[5 marks]

(b) $\theta=1.89549...$

$\theta=1.90$ ***A1***

Note: Award ***A0*** if there is more than one solution. Award ***A0*** for an answer in degrees.

[1 mark]

Total [6 marks]

Question 8

EITHER

attempt to use cosine rule **(M1)**

$12^2 + AB^2 - 2 \times 12 \times \cos 25^\circ \times AB = 7^2$ OR $AB^2 - 21.7513...AB + 95 = 0$ **(A1)**

at least one correct value for AB **(A1)**

$AB = 6.05068...$ OR $AB = 15.7007...$

using their smaller value for AB to find minimum perimeter **(M1)**

$12 + 7 + 6.05068...$

OR

attempt to use sine rule **(M1)**

$\frac{\sin B}{12} = \frac{\sin 25^\circ}{7}$ OR $\sin B = 0.724488...$ OR $\hat{B} = 133.573...^\circ$ OR $\hat{B} = 46.4263...^\circ$ **(A1)**

at least one correct value for $\hat{C}$

$\hat{C} = 21.4263...^\circ$ OR $\hat{C} = 108.573...^\circ$ **(A1)**

using their acute value for $\hat{C}$ to find minimum perimeter **(M1)**

$12 + 7 + \sqrt{12^2 + 7^2 - 2 \times 12 \times 7 \cos 21.4263...^\circ}$ OR $12 + 7 + \frac{7 \sin 21.4263...^\circ}{\sin 25^\circ}$

THEN

$25.0506\ldots$

minimum perimeter = 25.1. **A1**

Total [5 marks]

Question 9

(a) $|a| = \sqrt{12^2 + (-5)^2} \; (= 13)$ ***(A1)***

$2 \le |a + b| \le 28$ (accept min 2 and max 28) ***A1***

Note: Award ***(A1)A0*** for 2 and 28 seen with no indication that they are the endpoints of an interval.

[2 marks]

(b) recognition that p or b is a negative multiple of a ***(M1)***

$p = -2\hat{a}$ OR $b = -\frac{15}{13}a \left(= -\frac{15}{13}\begin{pmatrix} 12 \\ -5 \end{pmatrix} \right)$

$p = -\frac{2}{13}\begin{pmatrix} 12 \\ -5 \end{pmatrix} \left(= \begin{pmatrix} -1.85 \\ 0.769 \end{pmatrix} \right)$ ***A1***

[2 marks]

(c) **METHOD 1**

q is perpendicular to $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$

$\Rightarrow q$ is in the direction $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$ ***(M1)***

$q = k\begin{pmatrix} 5 \\ 12 \end{pmatrix}$ ***(A1)***

$(|q| =)\sqrt{(5k)^2 + (12k)^2} = 15$ ***(M1)***

$k = \frac{15}{13}$ ***(A1)***

$q = \frac{15}{13}\begin{pmatrix} 5 \\ 12 \end{pmatrix} \left(= \begin{pmatrix} \frac{75}{13} \\ \frac{180}{13} \end{pmatrix} = \begin{pmatrix} 5.77 \\ 13.8 \end{pmatrix} \right)$ ***A1***

[5 marks]

METHOD 2

$\boldsymbol{q}$ is perpendicular to $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$

attempt to set scalar product $\boldsymbol{q}.\boldsymbol{a} = 0$ OR product of gradients $= -1$ ***(M1)***

$12x - 5y = 0$ ***(A1)***

$(|\boldsymbol{q}| =)\sqrt{x^2 + y^2} = 15$

attempt to solve simultaneously to find a quadratic in x or in y ***(M1)***

$x^2 + \left(\frac{12x}{5}\right)^2 = 15^2$ OR $\left(\frac{5y}{12}\right)^2 + y^2 = 15^2$

$\boldsymbol{q} = \begin{pmatrix} \frac{75}{13} \\ \frac{180}{13} \end{pmatrix} \left(= \begin{pmatrix} 5.77 \\ 13.8 \end{pmatrix} \right)$ ***A1A1***

> **Note:** Award ***A1*** independently for each value. Accept values given as $x = \frac{75}{13}$ and $y = \frac{180}{13}$ or equivalent.

[5 marks]

Total [9 marks]

Question 10

(a) valid approach to find area of segment by finding area of sector – area of triangle ***(M1)***

$$\frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$$

$$\frac{1}{2}(2)^2\theta - \frac{1}{2}(2)^2\sin\theta$$ ***(A1)***

area = $2\theta - 2\sin\theta$ ***A1***

[3 marks]

(b) **EITHER**

area of logo = area of rectangle – area of segments ***(M1)***

$5\times4-2\times(2\theta-2\sin\theta)=13.4$ ***(A1)***

OR

area of one segment = $\dfrac{20-13.4}{2}\ (=3.3)$ ***(M1)***

$2\theta-2\sin\theta=3.3$ ***(A1)***

THEN

$\theta=2.35672...$

$\theta=2.36$ (do not accept an answer in degrees) ***A1***

Note: Award ***(M1)(A1)A0*** if there is more than one solution.
Award ***(M1)(A1FT)A0*** if the candidate works in degrees and obtains a final answer of $135.030\ldots$

[3 marks]

Total [6 marks]

Question 11

(a) let ϕ be the required angle (bearing)

EITHER

$\phi = 90° - \arctan\frac{1}{2}$ $(= \arctan 2)$ ***(M1)***

Note: Award ***M1*** for a labelled sketch.

OR

$\cos\phi = \frac{\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}4\\2\end{pmatrix}}{\sqrt{1}\times\sqrt{20}}$ $\left(= 0.4472..., = \frac{1}{\sqrt{5}}\right)$ ***(M1)***

$\phi = \arccos(0.4472...)$

THEN

$063°$ ***A1***

Note: Do not accept $063.4°$ or $63.4°$ or 1.10^c.

[2 marks]

(b) **Method 1**

let $|\boldsymbol{b}_A|$ be the speed of A and let $|\boldsymbol{b}_B|$ be the speed of B

attempts to find the speed of one of A or B ***(M1)***

$|\boldsymbol{b}_A| = \sqrt{(-6)^2 + 2^2 + 4^2}$ or $|\boldsymbol{b}_B| = \sqrt{4^2 + 2^2 + (-2)^2}$

Note: Award ***M0*** for $\lvert\boldsymbol{b}_A\rvert = \sqrt{19^2 + (-1)^2 + 1^2}$ and $\lvert\boldsymbol{b}_B\rvert = \sqrt{1^2 + 0^2 + 12^2}$.

$|\boldsymbol{b}_A| = 7.48...$ $(= \sqrt{56})$ (km min^{-1}) and $|\boldsymbol{b}_B| = 4.89...$ $(= \sqrt{24})$ (km min^{-1}) ***A1***

$|\boldsymbol{b}_A| > |\boldsymbol{b}_B|$ so A travels at a greater speed than B ***AG***

[2 marks]

Method 2

attempts to use $\text{speed} = \dfrac{\text{distance}}{\text{time}}$

$\text{speed}_A = \dfrac{|r_A(t_2) - r_A(t_1)|}{t_2 - t_1}$ and $\text{speed}_B = \dfrac{|r_B(t_2) - r_B(t_1)|}{t_2 - t_1}$ ***(M1)***

for example:

$\text{speed}_A = \dfrac{|r_A(1) - r_A(0)|}{1}$ and $\text{speed}_B = \dfrac{|r_B(1) - r_B(0)|}{1}$

$\text{speed}_A = \dfrac{\sqrt{(-6)^2 + 2^2 + 4^2}}{1}$ and $\text{speed}_B = \dfrac{\sqrt{4^2 + 2^2 + 2^2}}{1}$

$\text{speed}_A = 7.48...\left(2\sqrt{14}\right)$ and $\text{speed}_B = 4.89...\left(\sqrt{24}\right)$ ***A1***

$\text{speed}_A > \text{speed}_B$ so A travels at a greater speed than B ***AG***

[2 marks]

(c) attempts to use the angle between two direction vectors formula ***(M1)***

$\cos\theta = \dfrac{(-6)(4) + (2)(2) + (4)(-2)}{\sqrt{(-6)^2 + 2^2 + 4^2}\sqrt{4^2 + 2^2 + (-2)^2}}$ ***(A1)***

$\cos\theta = -0.7637... \left(= -\dfrac{7}{\sqrt{84}}\right)$ or $\theta = \arccos(-0.7637...)$ $(= 2.4399...)$

attempts to find the acute angle $180° - \theta$ using their value of θ ***(M1)***

$= 40.2°$ ***A1***

[4 marks]

(d) (i) for example, sets $\boldsymbol{r}_A(t_1) = \boldsymbol{r}_B(t_2)$ and forms at least two equations ***(M1)***

$19 - 6t_1 = 1 + 4t_2$

$-1 + 2t_1 = 2t_2$

$1 + 4t_1 = 12 - 2t_2$

Note: Award ***M0*** for equations involving t only.

EITHER

attempts to solve the system of equations for one of t_1 or t_2 ***(M1)***

$t_1 = 2$ or $t_2 = \frac{3}{2}$ ***A1***

OR

attempts to solve the system of equations for t_1 and t_2 ***(M1)***

$t_1 = 2$ and $t_2 = \frac{3}{2}$ ***A1***

THEN

substitutes their t_1 or t_2 value into the corresponding $\boldsymbol{r}_A$ or $\boldsymbol{r}_B$ ***(M1)***

$\mathrm{P}(7,3,9)$ ***A1***

Note: Accept $\vec{\mathrm{OP}} = \begin{pmatrix} 7 \\ 3 \\ 9 \end{pmatrix}$. Accept 7 km east of O, 3 km north of O and 9 km above sea level.

(ii) attempts to find the value of $t_1 - t_2$ ***(M1)***

$t_1 - t_2 = 2 - \frac{3}{2}$

0.5 minutes (30 seconds) ***A1***

[7 marks]

(e) **EITHER**

attempts to find $r_B - r_A$ ***(M1)***

$$r_B - r_A = \begin{pmatrix} -18 \\ 1 \\ 11 \end{pmatrix} + t\begin{pmatrix} 10 \\ 0 \\ -6 \end{pmatrix}$$

attempts to find their $D(t)$ ***(M1)***

$$D(t) = \sqrt{(10t-18)^2 + 1 + (11-6t)^2}$$ ***A1***

OR

attempts to find $r_A - r_B$ ***(M1)***

$$r_A - r_B = \begin{pmatrix} 18 \\ -1 \\ -11 \end{pmatrix} + t\begin{pmatrix} -10 \\ 0 \\ 6 \end{pmatrix}$$

attempts to find their $D(t)$ ***(M1)***

$$D(t) = \sqrt{(18-10t)^2 + (-1)^2 + (-11+6t)^2}$$ ***A1***

Note: Award ***M0M0A0*** for expressions using two different time parameters.

THEN

either attempts to find the local minimum point of $D(t)$ or attempts to find the value of t such that $D'(t) = 0$ (or equivalent) ***(M1)***

$$t = 1.8088... \left(= \frac{123}{68}\right)$$

$$D(t) = 1.01459...$$

minimum value of $D(t)$ is $1.01 \left(= \frac{\sqrt{1190}}{34}\right)$ (km) ***A1***

[5 marks]

Note: Award ***M0*** for attempts at the shortest distance between two lines.

Total [20 marks]

Question 12

(a) **EITHER**

uses the cosine rule *(M1)*

$AB^2 = 5^2 + 5^2 - 2\times5\times5\times\cos1.9$ *(A1)*

OR

uses right-angled trigonometry *(M1)*

$\dfrac{\frac{AB}{2}}{5} = \sin0.95$ *(A1)*

OR

uses the sine rule *(M1)*

$\alpha = \frac{1}{2}(\pi - 1.9)(= 0.6207...)$

$\dfrac{AB}{\sin1.9} = \dfrac{5}{\sin0.6207...}$ *(A1)*

THEN

$AB = 8.1341...$

$AB = 8.13$ (m) ***A1***

[3 marks]

(b) let the shaded area be A

METHOD 1

Attempt at finding reflex angle **(M1)**

$\mathrm{A\hat{O}B} = 2\pi - 1.9 \ (= 4.3831...)$

substitution into area formula **(A1)**

$A = \frac{1}{2} \times 5^2 \times 4.3831... \text{ OR } \left(\frac{2\pi - 1.9}{2\pi}\right) \times \pi\left(5^2\right)$

$= 54.7898...$

$= 54.8$ (m^2) **A1**

METHOD 2

let the area of the circle be A_C and the area of the unshaded sector be A_U

$A = A_C - A_U$ **(M1)**

$A = \pi \times 5^2 - \frac{1}{2} \times 5^2 \times 1.9 \ (= 78.5398... - 23.75)$ **(A1)**

$= 54.7898...$

$= 54.8$ (m^2) **A1**

[3 marks]

Total [6 marks]

Question 13

(a) $\vec{AB}=\begin{pmatrix} k-1 \\ -4 \\ -2 \end{pmatrix}, \vec{AC}=\begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$ ***A1A1***

[2 marks]

(b) **METHOD 1**

$k-1=2\times 4$ ***M1***

$k=9$ ***AG***

METHOD 2

in order by y or z-ordinate, the points are $(k, -2, 1), (5, 0, 2)$ $(1, 2, 3)$

$k-5=5-1$ ***M1***

$k=9$ ***AG***

[1 mark]

(c) (i) attempt to set up a vector equation using a point, a parameter and a direction vector ***(M1)***

$\boldsymbol{r}=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}+\lambda\begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$ (or equivalent) ***A1***

Note: "$\boldsymbol{r}=$" or equivalent must be seen for ***A1***.

(ii) **METHOD 1**

point on line L_1 has coordinates $(1+4\lambda, 2-2\lambda, 3-\lambda)$

attempt to use a different parameter for L_2 ***(M1)***

$\frac{x-1}{2}=\frac{y}{3}=1-z=\mu$ or $\boldsymbol{r}=\begin{pmatrix}1\\0\\1\end{pmatrix}+\mu\begin{pmatrix}2\\3\\-1\end{pmatrix}$

point on line L_2 has coordinates $(1+2\mu, 3\mu, 1-\mu)$ ***(A1)***

Note: This ***A1*** may be implied by $\boldsymbol{r}=\begin{pmatrix}1\\0\\1\end{pmatrix}+\mu\begin{pmatrix}2\\3\\-1\end{pmatrix}$.

$1+4\lambda=1+2\mu$

$2-2\lambda=3\mu$

$3-\lambda=1-\mu$

any two of the above equations ***A1***

attempt to solve two simultaneous equations with two parameters ***(M1)***

eg $\lambda=0.25$, $\mu=0.5$ or $\lambda=1.6$, $\mu=-0.4$ or $\lambda=-2$, $\mu=-4$ ***A1***

substitute into third equation or solve a different pair of simultaneous equations ***M1***

obtain contradiction *eg* $3-0.25\neq1-0.5$ or $1+4(1.6)\neq1+2(-0.4)$ or $2-2(-2)\neq3(-4)$ (so the lines do not intersect) ***R1***

Note: Do not award this ***R1*** if it is based on incorrect values.

lines are not parallel ***R1***

so lines are skew ***AG***

METHOD 2

point on line L_1 has coordinates $(1+4\lambda, 2-2\lambda, 3-\lambda)$

attempt to use the equation of L_2 to generate at least two equations in λ ***(M1)***

if the two lines intersect,

$$\frac{(1+4\lambda)-1}{2}=\frac{2-2\lambda}{3}\left(\Rightarrow 2\lambda=\frac{2-2\lambda}{3}\right)$$

$$\frac{(1+4\lambda)-1}{2}=1-(3-\lambda)(\Rightarrow 2\lambda=\lambda-2)$$

$$\frac{2-2\lambda}{3}=1-(3-\lambda)\Rightarrow\left(\frac{2-2\lambda}{3}=\lambda-2\right)$$

any two of the above equations ***A1A1***

attempt to solve at least one equation in λ ***(M1)***

one of $\lambda=\frac{1}{4}$, $\lambda=-2$, $\lambda=\frac{8}{5}$ seen ***A1***

substitute into second equation or solve second equation ***M1***

obtain contradiction *eg* $\lambda=\frac{1}{4}\neq-2$ or $2\left(\frac{1}{4}\right)\neq\frac{1}{4}-2$ (so the lines do not intersect) ***R1***

Note: Do not award this ***R1*** if it is based on incorrect values.

lines are not parallel ***R1***

so lines are skew ***AG***

METHOD 3

attempt to use a find Cartesian equation for L_1 ***(M1)***

$\frac{x-1}{4}=\frac{y-2}{-2}=\frac{z-3}{-1}$ ***A1***

attempt to isolate one variable in both equations ***(M1)***

$L_1: z=\frac{1-x}{4}+3=\frac{y-2}{2}+3 \quad L_2: z=\frac{1-x}{2}+1=\frac{-y}{3}+1$ OR

$L_1: y=\frac{1-x}{2}+2=2(z-3)+2 \quad L_2: y=\frac{3(x-1)}{2}=3(1-z)$ OR

$L_1: x=1-2(y-2)=1-4(z-3) \quad L_2: x=\frac{2y}{3}+1=1-2(z-1)$ ***A1***

attempt to solve for each of the other two variables ***(M1)***

e.g. $\frac{1-x}{2}+1=\frac{1-x}{4}+3$ and $\frac{-y}{3}+1=\frac{y-2}{2}+3$

$x=-7$, $y=-1.2$ OR $x=2$, $z=1.4$ OR $y=1.5$, $z=5$ ***A1***

obtain contradiction *eg* $z=5\neq 1.4$ OR $y=1.5\neq -1.2$ OR $x=2\neq -7$

(so the lines do not intersect) ***R1***

Note: Do not award this ***R1*** if it is based on incorrect values.

lines are not parallel ***R1***

so lines are skew ***AG***

[10 marks]

(d) (i) **METHOD 1**

attempt to find cross product of two of $\vec{AB}$, $\vec{AC}$ and $\vec{BC}$ or their opposites ***M1***

eg $\vec{AB} \times \vec{AC} = \begin{pmatrix} 0 \\ k-9 \\ 18-2k \end{pmatrix} \left(= (k-9)\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right)$ ***A1***

attempt to substitute their cross product and a point into the equation of a plane ***(M1)***

$(k-9)y + 2(9-k)z = 2(k-9) + 6(9-k)$

$(k-9)y + 2(9-k)z = 36-4k \ (\Rightarrow y - 2z = -4 \text{ since } k \neq 9)$ ***A1***

METHOD 2

attempt to find vector equation of Π and write x, y and z in parametric form ***M1***

$\left(\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} k-1 \\ -4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix} \Rightarrow \right) x = 1 + \lambda(k-1) + 4\mu,\ y = 2 - 4\lambda - 2\mu,$

$z = 3 - 2\lambda - \mu$ or equivalent ***A1***

attempt to eliminate both parameters to work towards Cartesian form ***M1***

$(k-9)y + 2(9-k)z = 36-4k \ (\Rightarrow y - 2z = -4 \text{ since } k \neq 9)$ ***A1***

(ii) **METHOD 1**

attempt to find the equation of the line through (0, 0, 0) perpendicular to the plane ***(M1)***

EITHER

$$(\boldsymbol{r} =)t\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$ ***(A1)***

attempt to find the point where the line and plane intersect ***(M1)***

$$t + 4t + 4 = 0$$

$$t = -\frac{4}{5}$$ ***(A1)***

OR

$$(\boldsymbol{r} =)t(k-9)\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$ ***(A1)***

attempt to find the point where the line and plane intersect ***(M1)***

$$t(k-9)^2 + 4t(k-9)^2 + 4(k-9) = 0$$

$$t = -\frac{4}{5(k-9)}$$ ***(A1)***

THEN

so the point on the plane closest to the origin is $(0\,,\ -0.8\,,\ 1.6)$ ***A1***

METHOD 2

choose a point on the plane (p, q, r)

$q-2r+4=0$ OR $q(k-9)-2r(k-9)+4(k-9)=0 \Rightarrow q=2r-4$

distance to the origin is $\sqrt{p^2+(2r-4)^2+r^2}$ **(A1)**

since *p* is independent of *r*, distance is minimised when $p=0$ **(R1)**

attempt to find the value of *r* for which their $\sqrt{(2r-4)^2+r^2}$ is minimised **(M1)**

$r=1.6$ **(A1)**

so the point on the plane closest to the origin is $(0, -0.8, 1.6)$ **A1**

METHOD 3

attempt to find a vector from the origin to the closest point on the plane ***(M1)***

EITHER

$$(\boldsymbol{r} =)t\begin{pmatrix}0\\1\\-2\end{pmatrix}$$ ***(A1)***

distance to the origin $=\left(\frac{4}{\sqrt{1^2+(-2)^2}}=\frac{4}{\sqrt{5}}\right)=\frac{4\sqrt{5}}{5}$ ***(A1)***

$t=\pm\frac{4}{5}$

check in equation of plane $y-2z=-4$ to get $t=-\frac{4}{5}$ ***(R1)***

OR

$$(\boldsymbol{r} =)t(k-9)\begin{pmatrix}0\\1\\-2\end{pmatrix}$$ ***(A1)***

distance to the origin $=\left(\frac{4}{\sqrt{1^2+(-2)^2}}=\frac{4}{\sqrt{5}}\right)=\frac{4\sqrt{5}}{5}$ ***(A1)***

$t=\pm\frac{4}{5(k-9)}$

check in equation of plane $y-2z=-4$ to get $t=-\frac{4}{5(k-9)}$ ***(R1)***

THEN

so the point on the plane closest to the origin is $(0\,,-0.8\,,\,1.6)$ ***A1***

[9 marks]

Total [22 marks]

Question 14

(a) (i) attempt to use the cosine rule **(M1)**

$$\text{AC} = \sqrt{2^2 + 4^2 - 2(2)(4)\cos\alpha}\left(= \sqrt{20 - 16\cos\alpha} = 2\sqrt{5 - 4\cos\alpha}\right)$$ **A1**

(ii) $$\text{AC} = \sqrt{6^2 + 8^2 - 2(6)(8)\cos\beta}\left(= \sqrt{100 - 96\cos\beta} = 2\sqrt{25 - 24\cos\beta}\right)$$ **A1**

(iii) $5 - 4\cos\alpha = 25 - 24\cos\beta$

$\alpha = \arccos(6\cos\beta - 5)$ **A1**

[4 marks]

(b) attempt to find the sum of two triangle areas using $A = \frac{1}{2}ab\sin C$ **(M1)**

Note: Do not award this ***M1*** if the triangle is assumed to be right angled.

$\text{Area} = \frac{1}{2}(8)\sin\alpha + \frac{1}{2}(48)\sin\beta$ **(A1)**

attempt to express the area in terms of one variable only **(M1)**

$= 4\sqrt{1-(6\cos\beta - 5)^2} + 24\sin\beta$ or $4\sin(\arccos(6\cos\beta - 5)) + 24\sin\beta$ OR

$4\sin\alpha + 24\sqrt{1-\left(\frac{5+\cos\alpha}{6}\right)^2}$ or $4\sin\alpha + 24\sin\left(\arccos\left(\frac{5+\cos\alpha}{6}\right)\right)$

Max area = 19.5959…

= 19.6 **A1**

[4 marks]

Total [8 marks]

Question 15

(a) **METHOD 1**

attempt to use scalar product or formula for angle between two vectors ***(M1)***

$\boldsymbol{u}.\boldsymbol{v} = \cos\frac{1}{n} + \sin\frac{1}{n}$ (seen anywhere) ***(A1)***

$$\cos\theta = \frac{\cos\frac{1}{n} + \sin\frac{1}{n}}{\sqrt{2}\sqrt{\left(\cos^2\frac{1}{n} + \sin^2\frac{1}{n}\right)}} \left(= \frac{\cos\frac{1}{n} + \sin\frac{1}{n}}{\sqrt{2}} \right)$$ ***A1***

METHOD 2

attempt to use an Argand diagram showing two complex numbers in the first quadrant with the angle between them marked as θ ***(M1)***

$\arg(\boldsymbol{u}) = \frac{\pi}{4}$ (accept 45° or $\arctan(1)$) **and** $\arg(\boldsymbol{v}) = \frac{1}{n}$ ***(A1)***

$\cos\theta = \cos\left|\frac{\pi}{4} - \frac{1}{n}\right|$ ***A1***

(b) use of $\frac{1}{n} \to 0$ as $n \to \infty$ ***(M1)***

EITHER

$(\cos\theta \to)\frac{1}{\sqrt{2}}$ ***(A1)***

OR

$(v \to)\boldsymbol{i}$ ***(A1)***

THEN

the limit is $\frac{\pi}{4}$ ***A1***

Note: Accept $45°$. Do not accept rounded values such as 0.785.

[3 marks]

Total [6 marks]