# Subject – Math AA(Higher Level)
# Topic - Number and Algebra
# Year - May 2021 – Nov 2024
# Paper -2
# Questions

## Question 1

[Maximum mark: 7]

The complex numbers $w$ and $z$ satisfy the equations

$$\frac{w}{z} = 2\mathrm{i}$$
$$z^* - 3w = 5 + 5\mathrm{i}.$$

Find $w$ and $z$ in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{Z}$.

## Question 2

[Maximum mark: 6]

On $1\text{st}$ January $2020$, Laurie invests $\$P$ in an account that pays a nominal annual interest rate of $5.5\%$, compounded **quarterly**.

The amount of money in Laurie's account **at the end of each year** follows a geometric sequence with common ratio, $r$.

(a) Find the value of $r$, giving your answer to four significant figures. [3]

Laurie makes no further deposits to or withdrawals from the account.

(b) Find the year in which the amount of money in Laurie's account will become double the amount she invested. [3]

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## Question 3

[Maximum mark: 7]

Consider the complex numbers $z = 2\left(\cos\frac{\pi}{5} + \mathrm{i}\sin\frac{\pi}{5}\right)$ and $w = 8\left(\cos\frac{2k\pi}{5} - \mathrm{i}\sin\frac{2k\pi}{5}\right)$, where $k \in \mathbb{Z}^+$.

(a) Find the modulus of $zw$. [1]

(b) Find the argument of $zw$ in terms of $k$. [2]

Suppose that $zw \in \mathbb{Z}$.

(c) (i) Find the minimum value of $k$.

(ii) For the value of $k$ found in part (i), find the value of $zw$. [4]

## Question 4

[Maximum mark: 5]

Consider the expansion of $(3 + x^2)^{n+1}$, where $n \in \mathbb{Z}^+$.

Given that the coefficient of $x^4$ is $20412$, find the value of $n$.

## Question 5

[Maximum mark: 5]

Consider $z = \cos\theta + \mathrm{i}\sin\theta$ where $z \in \mathbb{C}$, $z \neq 1$.

Show that $\mathrm{Re}\left(\frac{1+z}{1-z}\right) = 0$.

## Question 6

[Maximum mark: 5]

Eight runners compete in a race where there are no tied finishes. Andrea and Jack are two of the eight competitors in this race.

Find the total number of possible ways in which the eight runners can finish if Jack finishes

(a) in the position immediately after Andrea; [2]

(b) in any position after Andrea. [3]

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## Question 7

[Maximum mark: 5]

An arithmetic sequence has first term $60$ and common difference $-2.5$.

(a) Given that the $k$th term of the sequence is zero, find the value of $k$. [2]

Let $S_n$ denote the sum of the first $n$ terms of the sequence.

(b) Find the maximum value of $S_n$. [3]

## Question 8

[Maximum mark: 8]

(a) Prove the identity $(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$. [2]

The equation $2x^2 - 5x + 1 = 0$ has two real roots, $\alpha$ and $\beta$.

Consider the equation $x^2 + mx + n = 0$, where $m$, $n \in \mathbb{Z}$ and which has roots $\frac{1}{\alpha^3}$ and $\frac{1}{\beta^3}$.

(b) Without solving $2x^2 - 5x + 1 = 0$, determine the values of $m$ and $n$. [6]

## Question 9

[Maximum mark: 9]

The sum of the first $n$ terms of a geometric sequence is given by $S_n = \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r$.

(a) Find the first term of the sequence, $u_1$. [2]

(b) Find $S_\infty$. [3]

(c) Find the least value of $n$ such that $S_\infty - S_n < 0.001$. [4]

## Question 10

[Maximum mark: 7]

Mary, three female friends, and her brother, Peter, attend the theatre. In the theatre there is a row of 10 empty seats. For the first half of the show, they decide to sit next to each other in this row.

(a) Find the number of ways these five people can be seated in this row. [3]

For the second half of the show, they return to the same row of 10 empty seats. The four girls decide to sit at least one seat apart from Peter. The four girls do not have to sit next to each other.

(b) Find the number of ways these five people can now be seated in this row. [4]

## Question 11

[Maximum mark: 4]

Consider the equation $kx^2 - (k+3)x + 2k + 9 = 0$, where $k \in \mathbb{R}$.

(a) Write down an expression for the product of the roots, in terms of $k$. [1]

(b) Hence or otherwise, determine the values of $k$ such that the equation has one positive and one negative real root. [3]

## Question 12

[Maximum mark: 4]

Consider the set of six-digit positive integers that can be formed from the digits $0$, $1$, $2$, $3$, $4$, $5$, $6$, $7$, $8$ and $9$.

Find the total number of six-digit positive integers that can be formed such that

(a) the digits are distinct; [2]

(b) the digits are distinct and are in increasing order. [2]

## Question 13

[Maximum mark: 6]

Consider the expansion of $\frac{(ax+1)^9}{21x^2}$, where $a \neq 0$. The coefficient of the term in $x^4$ is $\frac{8}{7}a^5$.

Find the value of $a$.

## Question 14

[Maximum mark: 5]

A geometric sequence has a first term of $50$ and a fourth term of $86.4$.

The sum of the first $n$ terms of the sequence is $S_n$.

Find the smallest value of $n$ such that $S_n > 33\,500$.

## Question 15

[Maximum mark: 6]

Prove by contradiction that $p^2 - 8q - 11 \neq 0$, for any $p, q \in \mathbb{Z}$.

## Question 16

[Maximum mark: 7]

Consider $z = \cos\frac{11\pi}{18} + \mathrm{i}\sin\frac{11\pi}{18}$.

(a) Find the smallest value of $n$ that satisfies $z^n = -\mathrm{i}$, where $n \in \mathbb{Z}^+$. [4]

(b) Hence or otherwise, describe a single geometric transformation applied to $z$ on the Argand diagram that results in $z^{10}$. [3]

## Question 17

[Maximum mark: 7]

The coefficient of $x^6$ in the expansion of $(ax^3 + b)^8$ is $448$.

The coefficient of $x^6$ in the expansion of $(ax^3 + b)^{10}$ is $2880$.

Find the value of $a$ and the value of $b$, where $a, b > 0$.

## Question 18

[Maximum mark: 5]

Let $S$ be the set of $30$ positive integers $\{1, 2, 3, ..., 28, 29, 30\}$.

Raghu randomly selects three positive integers from $S$ without replacement. He then adds them together and determines whether the sum is divisible by $3$.

Determine the total number of selections Raghu can make to obtain a sum that is divisible by $3$.

You may assume that order is not important, for example, $\{1, 2, 3\}, \{1, 3, 2\}, \{2, 3, 1\}, \{2, 1, 3\}, \{3, 1, 2\}, \{3, 2, 1\}$ are all considered to be the same selection. [5]

## Question 19

[Maximum mark: 7]

The expansion of $(x + h)^8$, where $h \in \mathbb{Q}^+$, can be written as $x^8 + ax^7 + bx^6 + cx^5 + dx^4 + ... + h^8$, where $a, b, c, d, ... \in \mathbb{R}$.

Given that the coefficients, $a$, $b$ and $d$, are the first three terms of a geometric sequence, find the value of $h$.

## Question 20

[Maximum mark: 6]

A junior baseball team consists of six boys and three girls.

The team members are to be placed in a line to have their photograph taken.

(a) In how many ways can the team members be placed if

(i) there are no restrictions;

(ii) the girls must be placed next to each other. [3]

(b) Five members of the team are selected to attend a baseball summer camp. Find the number of possible selections that contain at least two girls. [3]

## Question 21

[Maximum mark: 10]

Let $z = 1 + \cos 2\theta + \mathrm{i}\sin 2\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

(a) Show that

(i) $\arg z = \theta$;

(ii) $|z| = 2\cos\theta$. [7]

(b) Hence or otherwise, find the value of $\theta$ such that $\arg(z^2) = |z^3|$. [3]

## Question 22

[Maximum mark: 6]

The loudness of a sound, $L$, measured in decibels, is related to its intensity, $I$ units, by $L = 10\log_{10}(I \times 10^{12})$.

Consider two sounds, $S_1$ and $S_2$.

$S_1$ has an intensity of $10^{-6}$ units and a loudness of $60$ decibels.

$S_2$ has an intensity that is twice that of $S_1$.

(a) State the intensity of $S_2$. [1]

(b) Determine the loudness of $S_2$. [2]

The maximum loudness of thunder in a thunderstorm was measured to be $115$ decibels.

(c) Find the corresponding intensity, $I$, of the thunder. [3]

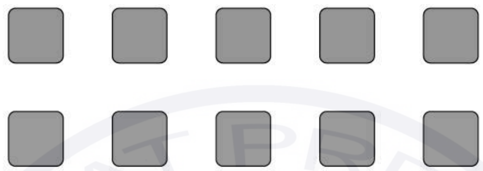
## Question 23

[Maximum mark: 7]

A group of 10 children includes one pair of brothers, Alvin and Bobby, and one pair of sisters, Catalina and Daniela.

The children are to be seated at 10 desks which are arranged in two rows of five as shown in the following diagram.

front



Alvin and Bobby must be seated next to each other in the same row.

(a) Find the total number of ways the children can be seated. [3]

After an argument, Catalina and Daniela must not be seated next to each other. Alvin and Bobby must still be seated next to each other.

(b) Find the total number of ways the children can be seated. [4]

## Question 24

[Maximum mark: 7]

Darren buys a car for $\$35\,000$. The value of the car decreases by $15\%$ in the first year.

(a) Find the value of the car at the end of the first year. [2]

After the first year, the value of the car decreases by $11\%$ in each subsequent year.

(b) Find the value of Darren's car 10 years after he buys it, giving your answer to the nearest dollar. [2]

When Darren has owned the car for $n$ complete years, the value of the car is less than $10\%$ of its original value.

(c) Find the least value of $n$. [3]

## Question 25

[Maximum mark: 7]

The sum of the first $n$ terms of a geometric sequence is given by $S_n = \sum_{r=0}^{n-1} 5(\log_2 c)^r$.

(a) Given that $S_n$ converges, find the range of possible values of $c$. [3]

(b) In the case where $c = 1.5$, find the least value of $n$ such that $|S_\infty - S_n| < 0.1$. [4]

## Question 26

[Maximum mark: 6]

(a) Given that $w \in \mathbb{C}$, prove that $ww^* = |w|^2$. [2]

Two complex numbers $z$ and $w$ satisfy the following equations:

$$5w^* = (1 - 2\mathrm{i})z^2$$

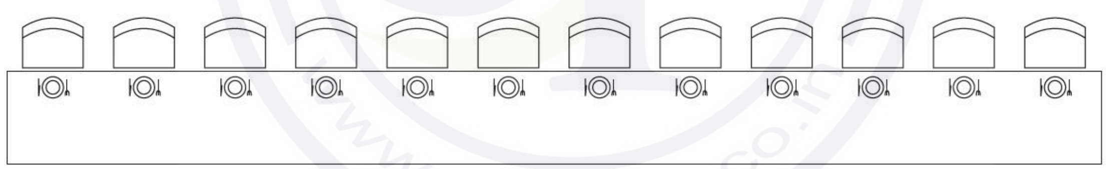
$$zw = 10 + 10\mathrm{i}.$$

(b) Given that $|w| = 2\sqrt{5}$, find $z$ expressing your answer in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{Z}$. [4]

## Question 27

[Maximum mark: 6]

A self-service sushi restaurant has a row of $12$ available seats, as shown in the following diagram.



Anvi, Vanya and Parita decide to go to the restaurant for lunch.

(a) Find the number of possible ways that they can be seated in this row, if they decide **not** to sit together as a group of $3$. [3]

The next day, Anvi, Vanya and Parita are joined by $3$ additional people in the same restaurant, and they sit in the same row of $12$ available seats. Anvi, Vanya and Parita now decide to sit next to each other as a group of $3$.

(b) Find the number of possible ways that these 6 people can be seated. [3]

## Question 28

[Maximum mark: 4]

Find the coefficient of $x^6$ in the expansion of $(2x - 5)^9$.

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