Subject – Math AA(Higher Level)
Topic - Number and Algebra
Year - May 2021 – Nov 2022
Paper -2
Answers

Question 1

substituting $w = 2iz$ into $z^* - 3w = 5 + 5i$ ***M1***

$z^* - 6iz = 5 + 5i$ ***A1***

let $z = x + yi$

comparing real and imaginary parts of $(x - yi) - 6i(x + yi) = 5 + 5i$ ***M1***

to obtain $x + 6y = 5$ and $-6x - y = 5$ ***A1***

attempting to solve for x and y ***M1***

$x = -1$ and $y = 1$ and so $z = -1 + i$ ***A1***

hence $w = -2 - 2i$ ***A1***

Question 2

(a) $\left(1 + \frac{5.5}{4 \times 100}\right)^4$ ***(M1)(A1)***

$= 1.056$ ***A1***

[3 marks]

(b) ***EITHER***

$$2P = P \times \left(1 + \frac{5.5}{100 \times 4}\right)^{4n} \textbf{ OR } 2P = P \times (\text{their } (a))^{m}$$ ***(M1)(A1)***

Note: Award ***(M1)*** for substitution into loan payment formula. Award ***(A1)*** for correct substitution.

OR

$\text{PV} = \pm 1$
$\text{FV} = \mp 2$
$\text{I\%} = 5.5$
$\text{P/Y} = 4$
$\text{C/Y} = 4$
$n = 50.756\ldots$ ***(M1)(A1)***

OR

$\text{PV} = \pm 1$
$\text{FV} = \mp 2$
$\text{I\%} = 100(\text{their } (a) - 1)$
$\text{P/Y} = 1$
$\text{C/Y} = 1$ ***(M1)(A1)***

THEN

$\Rightarrow 12.7$ years
Laurie will have double the amount she invested during 2032 ***A1***

[3 marks]

Total [6 marks]

Question 3

(a) $(|zw| =)16$ ***A1***

[1 mark]

(b) attempt to find $\arg(z) + \arg(w)$ ***(M1)***

$\arg(zw) = \arg(z) + \arg(w)$

$= \frac{\pi}{5} - \frac{2k\pi}{5} \left(= \frac{(1-2k)\pi}{5}\right)$ ***A1***

[2 marks]

(c) (i) $zw \in \mathbb{Z} \Rightarrow \arg(zw)$ is a multiple of π *(M1)*

$\Rightarrow 1-2k$ is a multiple of 5 *(M1)*

$k=3$ ***A1***

(ii) $zw = 16(\cos(-\pi)+\mathrm{i}\sin(-\pi))$

-16 ***A1***

[4 marks]

Total [7 marks]

Question 4

METHOD 1

product of a binomial coefficient, a power of 3 (and a power of x^2) seen ***(M1)***
evidence of correct term chosen ***(A1)***

$$^{n+1}C_2 \times 3^{n+1-2} \times \left(x^2\right)^2 \quad \left(= \frac{n(n+1)}{2} \times 3^{n-1} \times x^4\right) \text{ OR } n-r=1$$

equating their coefficient to 20412 or their term to $20412x^4$ ***(M1)***

EITHER

$$^{n+1}C_2 \times 3^{n-1} = 20412$$ ***(A1)***

OR

$$^{r+2}C_r \times 3^r = 20412 \Rightarrow r = 6$$ ***(A1)***

THEN

$n = 7$ ***A1***

METHOD 2

$$3^{n+1}\left(1+\frac{x^2}{3}\right)^{n+1}$$

product of a binomial coefficient and a power of $\frac{x^2}{3}$ OR $\frac{1}{3}$ seen ***(M1)***
evidence of correct term chosen ***(A1)***

$$3^{n+1} \times {}^{n+1}C_2 \times \left(\frac{x^2}{3}\right)^2 \quad \left(= 3^{n-1}\frac{n(n+1)}{2}x^4\right)$$

equating their coefficient to 20412 or their term to $20412x^4$ ***(M1)***

$$3^{n-1} \times \frac{n(n+1)}{2} = 20412$$ ***(A1)***

$n = 7$ ***A1***

Total [5 marks]

Question 5

$$\frac{1+z}{1-z}=\frac{1+\cos\theta+\mathrm{i}\sin\theta}{1-\cos\theta-\mathrm{i}\sin\theta}$$

attempt to use the complex conjugate of their denominator **_M1_**

$$=\frac{(1+\cos\theta+\mathrm{i}\sin\theta)(1-\cos\theta+\mathrm{i}\sin\theta)}{(1-\cos\theta-\mathrm{i}\sin\theta)(1-\cos\theta+\mathrm{i}\sin\theta)}$$ **_A1_**

$$\mathrm{Re}\left(\frac{1+z}{1-z}\right)=\frac{1-\cos^2\theta-\sin^2\theta}{(1-\cos\theta)^2+\sin^2\theta}\ \left(=\frac{1-\cos^2\theta-\sin^2\theta}{2-2\cos\theta}\right)$$ **_M1A1_**

ote: Award ***M1*** for expanding the numerator and ***A1*** for a correct numerator. Condone either an incorrect denominator or the absence of a denominator.

using $\cos^2\theta+\sin^2\theta=1$ to simplify the numerator **_(M1)_**

$$\mathrm{Re}\left(\frac{1+z}{1-z}\right)=0$$ **_AG_**

[5 marks]

Question 6

(a) Jack and Andrea finish in that order (as a unit) so we are considering the arrangement of 7 objects **(*M1*)**

$7! \; (=5040)$ ways ***A1***

[2 marks]

(b) **METHOD 1**

the number of ways that Andrea finishes in front of Jack is equal to the number of ways that Jack finishes in front of Andrea **(*M1*)**

total number of ways is $8!$ ***(A1)***

$\frac{8!}{2} \; (=20160)$ ways ***A1***

[3 marks]

METHOD 2

the other six runners can finish in $6! \; (=720)$ ways ***(A1)***

when Andrea finishes first, Jack can finish in 7 different positions

when Andrea finishes second, Jack can finish in 6 different positions etc

$7+6+5+4+3+2+1 \; (=28)$ ways ***(A1)***

hence there are $(7+6+5+4+3+2+1)\times 6!$ ways

$28\times 6! \; (=20160)$ ways ***A1***

[3 marks]
Total [5 marks]

Question 7

(a) attempt to use $u_1 + (n-1)d = 0$ *(M1)*

$60 - 2.5(k-1) = 0$

$k = 25$ ***A1***

[2 marks]

(b) **METHOD 1**

attempting to express S_n in terms of n *(M1)*

use of a graph or a table to attempt to find the maximum sum *(M1)*

$= 750$ ***A1***

METHOD 2

EITHER

recognizing maximum occurs at $n = 25$ *(M1)*

$S_{25} = \frac{25}{2}(60+0),\ S_{25} = \frac{25}{2}(2\times 60 + 24\times -2.5)$ *(A1)*

OR

attempting to calculate S_{24} *(M1)*

$S_{24} = \frac{24}{2}(2\times 60 + 23\times -2.5)$ *(A1)*

THEN

$= 750$ ***A1***

[3 marks]

Total [5 marks]

Question 8

(a) **METHOD 1**

$(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$

attempts to expand $(p+q)^3$ **M1**

$p^3 + 3p^2q + 3pq^2 + q^3$

$(p+q)^3 - 3pq(p+q) \equiv p^3 + 3p^2q + 3pq^2 + q^3 - 3pq(p+q)$

$\equiv p^3 + 3p^2q + 3pq^2 + q^3 - 3p^2q - 3pq^2$ **A1**

$\equiv p^3 + q^3$ **AG**

Note: Condone the use of equals signs throughout.

METHOD 2

$(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$

attempts to factorise $(p+q)^3 - 3pq(p+q)$ **M1**

$\equiv (p+q)\left((p+q)^2 - 3pq\right) \quad \left(\equiv (p+q)\left(p^2 - pq + q^2\right)\right)$

$\equiv p^3 - p^2q + pq^2 + p^2q - pq^2 + q^3$ **A1**

$\equiv p^3 + q^3$ **AG**

Note: Condone the use of equals signs throughout.

METHOD 3

$p^3 + q^3 \equiv (p+q)^3 - 3pq(p+q)$

attempts to factorise $p^3 + q^3$ **M1**

$\equiv (p+q)\left(p^2 - pq + q^2\right)$

$\equiv (p+q)\left((p+q)^2 - 3pq\right)$ **A1**

$\equiv (p+q)^3 - 3pq(p+q)$ **AG**

Note: Condone the use of the equals sign throughout.

[2 marks]

(b)

Note: Award a maximum of ***A1M0A0A1M0A0*** for $m=-95$ and $n=8$ found by using $\alpha, \beta = \frac{5 \pm \sqrt{17}}{4}$ $(\alpha, \beta = 0.219..., 2.28...)$.

Condone, as appropriate, solutions that state but clearly do not use the values of α and β.

Special case: Award a maximum of ***A1M1A0A1M0A0*** for $m=-95$ and $n=8$ obtained by solving simultaneously for α and β from product of roots and sum of roots equations.

product of roots of $x^2 - \frac{5}{2}x + \frac{1}{2} = 0$

$\alpha\beta = \frac{1}{2}$ (seen anywhere) ***A1***

considers $\left(\frac{1}{\alpha^3}\right)\left(\frac{1}{\beta^3}\right)$ by stating $\frac{1}{(\alpha\beta)^3} (= n)$ ***M1***

Note: Award ***M1*** for attempting to substitute their value of $\alpha\beta$ into $\frac{1}{(\alpha\beta)^3}$.

$$\frac{1}{(\alpha\beta)^3} = \frac{1}{\left(\frac{1}{2}\right)^3}$$

$n = 8$ ***A1***

sum of roots of $x^2 - \frac{5}{2}x + \frac{1}{2} = 0$

$\alpha + \beta = \frac{5}{2}$ (seen anywhere) ***A1***

considers $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$ by stating $\frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}$ $\left(\left(\frac{\alpha+\beta}{\alpha\beta}\right)^3 - \frac{3(\alpha+\beta)}{(\alpha\beta)^2}\right) (= -m)$ ***M1***

Note: Award ***M1*** for attempting to substitute their values of $\alpha+\beta$ and $\alpha\beta$ into their expression. Award ***M0*** for use of $(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$ only.

$$= \frac{\left(\frac{5}{2}\right)^3 - \left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{\frac{1}{8}} \quad (=125-30=95)$$

$m = -95$ ***A1***

$(x^2 - 95x + 8 = 0)$

[6 marks]
Total [8 marks]

Question 9

(a) $u_1 = S_1 = \frac{2}{3} \times \frac{7}{8}$ **(M1)**

$= \frac{14}{24}\left(= \frac{7}{12} = 0.583333\ldots\right)$ **A1**

[2 marks]

(b) $r = \frac{7}{8}(= 0.875)$ **(A1)**

substituting their values for u_1 and r into $S_\infty = \frac{u_1}{1-r}$ **(M1)**

$= \frac{14}{3}(= 4.66666\ldots)$ **A1**

[3 marks]

(c) attempt to substitute their values into the inequality or formula for S_n **(M1)**

$$\frac{14}{3} - \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r < 0.001 \text{ OR } S_n = \frac{\frac{7}{12}\left(1-\left(\frac{7}{8}\right)^n\right)}{\left(1-\frac{7}{8}\right)}$$

attempt to solve their inequality using a table, graph or logarithms
(must be exponential) **(M1)**

Note: Award ***(M0)*** if the candidate attempts to solve $S_\infty - u_n < 0.001$.

correct critical value or at least one correct crossover value **(A1)**

$63.2675\ldots$ OR $S_\infty - S_{63} = 0.001036...$ OR $S_\infty - S_{64} = 0.000906...$

OR $S_\infty - S_{63} - 0.001 = 0.0000363683...$ OR $S_\infty - S_{64} - 0.001 = -0.0000931777...$

least value is $n = 64$ **A1**

[4 marks]

Total [9 marks]

Question 10

(a) $6 \times 5!$ **(A1)(A1)**

$= 720$ (accept $6!$) **A1**

[3 marks]

(b) **METHOD 1**

(Peter apart from girls, in an end seat) ${}^{8}P_{4}\,(=1680)$ OR

(Peter apart from girls, not in end seat) ${}^{7}P_{4}\,(=840)$ **(A1)**

case 1: Peter at either end

$2 \times {}^{8}P_{4}\,(=3360)$ OR $2 \times {}^{8}C_{4} \times 4!\,(=3360)$ **(A1)**

case 2: Peter not at the end

$8 \times {}^{7}P_{4}\,(=6720)$ OR $8 \times {}^{7}C_{4} \times 4!\,(=6720)$ **(A1)**

Total number of ways $= 3360 + 6720$

$= 10080$ **A1**

METHOD 2

(Peter next to girl, in an end seat) $4 \times {}^{8}P_{3}\,(=1344)$ OR

(Peter next to one girl, not in end seat) $2 \times 4 \times {}^{7}P_{3}\,(=1680)$ OR

(Peter next to two girls, not in end seat) $4 \times 3 \times {}^{7}P_{2}\,(=504)$ **(A1)**

case 1: Peter at either end

$2 \times 4 \times {}^{8}P_{3}\,(=2688)$ **(A1)**

case 2: Peter not at the end

$8\left(2 \times 4 \times {}^{7}P_{3} + 4 \times 3 \times {}^{7}P_{2}\right)(=17472)$ **(A1)**

Total number of ways $= {}^{10}P_{5} - (2688 + 17472)$

$= 10080$ **A1**

[4 marks]

Total [7 marks]

Question 11

(a) product of roots $= \dfrac{2k+9}{k}$ ***A1***

[1 mark]

(b) recognition that the product of the roots will be negative ***(M1)***

$$\frac{2k+9}{k} < 0$$

critical values $k = 0, -\dfrac{9}{2}$ seen ***(A1)***

$$-\frac{9}{2} < k < 0$$ ***A1***

[3 marks]

Total [4 marks]

Question 12

(a) $9\times9\times8\times7\times6\times5 \left(=9\times{}^{9}P_5\right)$ **(M1)**

$=136080 \left(=9\times\frac{9!}{4!}\right)$ **A1**

Note: Award ***M1A0*** for $10\times9\times8\times7\times6\times5 \left(={}^{10}P_6=151200 =\frac{10!}{4!}\right)$.

Note: Award ***M1A0*** for ${}^{9}P_6=60480$

[2 marks]

(b) **METHOD 1**

EITHER

every unordered subset of 6 digits from the set of 9 non-zero digits can be arranged in exactly one way into a 6-digit number with the digits in increasing order. ***A1***

OR

${}^{9}C_6(\times1)$ ***A1***

THEN

$=84$ ***A1***

METHOD 2

EITHER

removes 3 digits from the set of 9 non-zero digits and these 6 remaining digits can be arranged in exactly one way into a 6-digit number with the digits in increasing order. ***A1***

OR

${}^{9}C_3(\times1)$ ***A1***

THEN

$=84$ ***A1***

[2 marks]

Total [4 marks]

Question 13

> **Note:** Do not award any marks if there is clear evidence of adding instead of multiplying, for example ${}^9C_r + (ax)^{9-r} + (1)^r$.

valid approach for expansion (must be the product of a binomial coefficient with $n=9$ and a power of ax) ***(M1)***

${}^9C_r(ax)^{9-r}(1)^r$ OR ${}^9C_{9-r}(ax)^r(1)^{9-r}$ OR ${}^9C_0(ax)^0(1)^9 + {}^9C_1(ax)^1(1)^8 + \ldots$

recognizing that the term in x^6 is needed ***(M1)***

$\frac{\text{Term in } x^6}{21x^2} = kx^4$ OR $r=6$ OR $r=3$ OR $9-r=6$

correct term or coefficient in binomial expansion (seen anywhere) ***(A1)***

${}^9C_6(ax)^6(1)^3$ OR ${}^9C_3\,a^6x^6$ OR $84\left(a^6x^6\right)(1)$ OR $84a^6$

EITHER

correct term in x^4 or coefficient (may be seen in equation) ***(A1)***

$\frac{{}^9C_6}{21}a^6x^4$ OR $4a^6x^4$ OR $4a^6$

Set their term in x^4 or coefficient of x^4 equal to $\frac{8}{7}a^5x^4$ or $\frac{8}{7}a^5$ (do not accept other powers of x) ***(M1)***

$\frac{{}^9C_3}{21}a^6x^4 = \frac{8}{7}a^5x^4$ OR $4a^6 = \frac{8}{7}a^5$

Question 14

$86.4 = 50r^3$ **(A1)**

$r = 1.2\left(= \sqrt[3]{\dfrac{86.4}{50}}\right)$ seen anywhere **(A1)**

$\dfrac{50\left(1.2^n - 1\right)}{0.2} > 33500$ OR $250\left(1.2^n - 1\right) = 33500$ **(A1)**

attempt to solve their geometric S_n inequality or equation **(M1)**

sketch OR $n > 26.9045$, $n = 26.9$ OR $S_{26} = 28368.8$ OR $S_{27} = 34092.6$ OR algebraic manipulation involving logarithms

$n = 27$ (accept $n \geq 27$) **A1**

Total [5 marks]