Subject – Math AA(Higher Level)
Topic - Calculus
Year - May 2021 – Nov 2024
Paper -3
Questions

Question 1

[Maximum mark: 25]

This question asks you to investigate some properties of the sequence of functions of the form $f_n(x) = \cos(n \arccos x)$, $-1 \leq x \leq 1$ and $n \in \mathbb{Z}^+$.

Important: When sketching graphs in this question, you are **not** required to find the coordinates of any axes intercepts or the coordinates of any stationary points unless requested.

(a) On the same set of axes, sketch the graphs of $y = f_1(x)$ and $y = f_3(x)$ for $-1 \leq x \leq 1$. [2]

(b) For odd values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for odd values of n describing, in terms of n, the number of

(i) local maximum points;

(ii) local minimum points. [4]

(c) On a new set of axes, sketch the graphs of $y = f_2(x)$ and $y = f_4(x)$ for $-1 \leq x \leq 1$. [2]

(d) For even values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for even values of n describing, in terms of n, the number of

(i) local maximum points;

(ii) local minimum points. [4]

(e) Solve the equation $f_n'(x) = 0$ and hence show that the stationary points on the graph of $y = f_n(x)$ occur at $x = \cos\frac{k\pi}{n}$ where $k \in \mathbb{Z}^+$ and $0 < k < n$. [4]

The sequence of functions, $f_n(x)$, defined above can be expressed as a sequence of polynomials of degree n.

(f) Use an appropriate trigonometric identity to show that $f_2(x) = 2x^2 - 1$. [2]

Consider $f_{n+1}(x) = \cos\left((n+1)\arccos x\right)$.

(g) Use an appropriate trigonometric identity to show that
$f_{n+1}(x) = \cos(n\arccos x)\cos(\arccos x) - \sin(n\arccos x)\sin(\arccos x)$. [2]

(h) Hence

(i) show that $f_{n+1}(x) + f_{n-1}(x) = 2xf_n(x)$, $n \in \mathbb{Z}^+$;

(ii) express $f_3(x)$ as a cubic polynomial. [5]

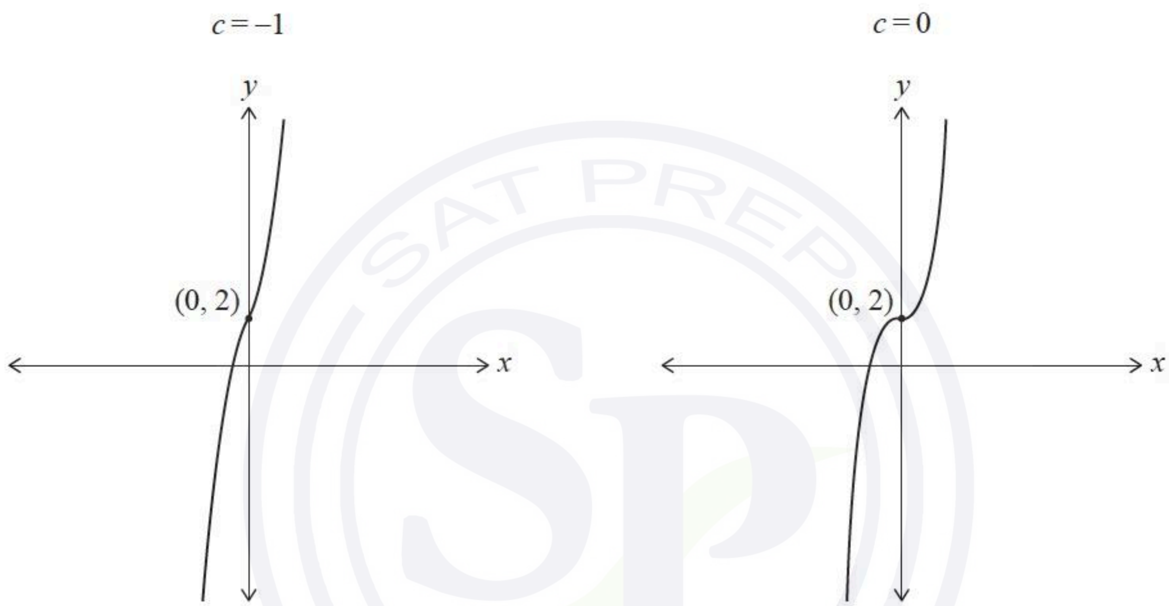
Question 2

[Maximum mark: 27]

This question asks you to explore the behaviour and key features of cubic polynomials of the form $x^3 - 3cx + d$.

Consider the function $f(x) = x^3 - 3cx + 2$ for $x \in \mathbb{R}$ and where c is a parameter, $c \in \mathbb{R}$.

The graphs of $y = f(x)$ for $c = -1$ and $c = 0$ are shown in the following diagrams.



(a) On separate axes, sketch the graph of $y = f(x)$ showing the value of the y-intercept and the coordinates of any points with zero gradient, for

(i) $c = 1$; [3]

(ii) $c = 2$. [3]

(b) Write down an expression for $f'(x)$. [1]

(c) Hence, or otherwise, find the set of values of c such that the graph of $y = f(x)$ has

(i) a point of inflexion with zero gradient; [1]

(ii) one local maximum point and one local minimum point; [2]

(iii) no points where the gradient is equal to zero. [1]

(d) Given that the graph of $y = f(x)$ has one local maximum point and one local minimum point, show that

(i) the y-coordinate of the local maximum point is $2c^{\frac{3}{2}} + 2$; [3]

(ii) the y-coordinate of the local minimum point is $-2c^{\frac{3}{2}} + 2$. [1]

(e) Hence, for $c > 0$, find the set of values of c such that the graph of $y = f(x)$ has

(i) exactly one x-axis intercept; [2]

(ii) exactly two x-axis intercepts; [2]

(iii) exactly three x-axis intercepts. [2]

Consider the function $g(x) = x^3 - 3cx + d$ for $x \in \mathbb{R}$ and where c, $d \in \mathbb{R}$.

(f) Find all conditions on c and d such that the graph of $y = g(x)$ has exactly one x-axis intercept, explaining your reasoning. [6]

Question 3

[Maximum mark: 31]

This question asks you to explore the behaviour and some key features of the function $f_n(x) = x^n(a-x)^n$, where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.

In parts (a) and (b), **only** consider the case where $a = 2$.

Consider $f_1(x) = x(2-x)$.

(a) Sketch the graph of $y = f_1(x)$, stating the values of any axes intercepts and the coordinates of any local maximum or minimum points. [3]

Consider $f_n(x) = x^n(2-x)^n$, where $n \in \mathbb{Z}^+$, $n > 1$.

(b) Use your graphic display calculator to explore the graph of $y = f_n(x)$ for

- the odd values $n = 3$ and $n = 5$;
- the even values $n = 2$ and $n = 4$.

Hence, copy and complete the following table. [6]

	Number of local maximum points	Number of local minimum points	Number of points of inflexion with zero gradient
$n = 3$ and $n = 5$			
$n = 2$ and $n = 4$			

Now consider $f_n(x) = x^n(a-x)^n$ where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$, $n > 1$.

(c) Show that $f_n'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$. [5]

(d) State the three solutions to the equation $f_n'(x) = 0$. [2]

(e) Show that the point $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ on the graph of $y = f_n(x)$ is always above the horizontal axis. [3]

(f) Hence, or otherwise, show that $f_n'\left(\frac{a}{4}\right) > 0$, for $n \in \mathbb{Z}^+$. [2]

(g) By using the result from part (f) and considering the sign of $f_n'(-1)$, show that the point $(0, 0)$ on the graph of $y = f_n(x)$ is

(i) a local minimum point for even values of n, where $n > 1$ and $a \in \mathbb{R}^+$; [3]

(ii) a point of inflexion with zero gradient for odd values of n, where $n > 1$ and $a \in \mathbb{R}^+$. [2]

Consider the graph of $y = x^n(a-x)^n - k$, where $n \in \mathbb{Z}^+$, $a \in \mathbb{R}^+$ and $k \in \mathbb{R}$.

(h) State the conditions on n and k such that the equation $x^n(a-x)^n = k$ has four solutions for x. [5]

Question 4

[Maximum mark: 30]

In this question you will be exploring the strategies required to solve a system of linear differential equations.

Consider the system of linear differential equations of the form:

$$\frac{dx}{dt} = x - y \text{ and } \frac{dy}{dt} = ax + y,$$

where $x, y, t \in \mathbb{R}^+$ and a is a parameter.

First consider the case where $a = 0$.

(a) (i) By solving the differential equation $\frac{dy}{dt} = y$, show that $y = Ae^t$ where A is a constant. [3]

(ii) Show that $\frac{dx}{dt} - x = -Ae^t$. [1]

(iii) Solve the differential equation in part (a)(ii) to find x as a function of t. [4]

Now consider the case where $a = -1$.

(b) (i) By differentiating $\frac{dy}{dt} = -x + y$ with respect to t, show that $\frac{d^2y}{dt^2} = 2\frac{dy}{dt}$. [3]

(ii) By substituting $Y = \frac{dy}{dt}$, show that $Y = Be^{2t}$ where B is a constant. [3]

(iii) Hence find y as a function of t. [2]

(iv) Hence show that $x = -\frac{B}{2}e^{2t} + C$, where C is a constant. [3]

Now consider the case where $a = -4$.

(c) (i) Show that $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$. [3]

From previous cases, we might conjecture that a solution to this differential equation is $y = Fe^{\lambda t}$, $\lambda \in \mathbb{R}$ and F is a constant.

(ii) Find the two values for λ that satisfy $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$. [4]

Let the two values found in part (c)(ii) be λ_1 and λ_2.

(iii) Verify that $y = Fe^{\lambda_1 t} + Ge^{\lambda_2 t}$ is a solution to the differential equation in (c)(i), where G is a constant. [4]

Question 5

[Maximum mark: 28]

This question asks you to explore properties of a family of curves of the type $y^2 = x^3 + ax + b$ for various values of a and b, where $a, b \in \mathbb{N}$.

(a) On the same set of axes, sketch the following curves for $-2 \le x \le 2$ and $-2 \le y \le 2$, clearly indicating any points of intersection with the coordinate axes.

(i) $y^2 = x^3,\ x \ge 0$ [2]

(ii) $y^2 = x^3 + 1,\ x \ge -1$ [2]

(b) (i) Write down the coordinates of the two points of inflexion on the curve $y^2 = x^3 + 1$. [1]

(ii) By considering each curve from part (a), identify two key features that would distinguish one curve from the other. [1]

Now, consider curves of the form $y^2 = x^3 + b$, for $x \ge -\sqrt[3]{b}$, where $b \in \mathbb{Z}^+$.

(c) By varying the value of b, suggest two key features common to these curves. [2]

Next, consider the curve $y^2 = x^3 + x,\ x \ge 0$.

(d) (i) Show that $\frac{dy}{dx} = \pm\frac{3x^2+1}{2\sqrt{x^3+x}}$, for $x > 0$. [3]

(ii) Hence deduce that the curve $y^2 = x^3 + x$ has no local minimum or maximum points. [1]

The curve $y^2 = x^3 + x$ has two points of inflexion. Due to the symmetry of the curve these points have the same x-coordinate.

(e) Find the value of this x-coordinate, giving your answer in the form $x = \sqrt{\frac{p\sqrt{3}+q}{r}}$, where $p, q, r \in \mathbb{Z}$. [7]

$\mathrm{P}(x, y)$ is defined to be a rational point on a curve if x and y are rational numbers.

The tangent to the curve $y^2 = x^3 + ax + b$ at a rational point P intersects the curve at another rational point Q.

Let C be the curve $y^2 = x^3 + 2$, for $x \ge -\sqrt[3]{2}$. The rational point $\mathrm{P}(-1, -1)$ lies on C.

(f) (i) Find the equation of the tangent to C at P. [2]

(ii) Hence, find the coordinates of the rational point Q where this tangent intersects C, expressing each coordinate as a fraction. [2]

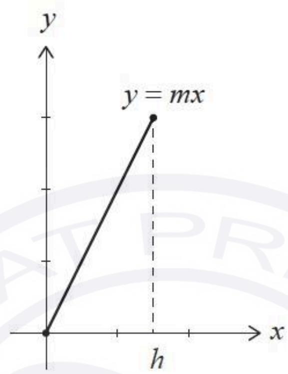
(g) The point $\mathrm{S}(-1, 1)$ also lies on C. The line [QS] intersects C at a further point. Determine the coordinates of this point. [5]

Question 6

[Maximum mark: 27]

In this question you will investigate curved surface areas and use calculus to derive key formulae used in geometry.

Consider the straight line from the origin, $y = mx$, where $0 \le x \le h$ and m, h are positive constants.



When this line is rotated through $360°$ about the x-axis, a cone is formed with a curved surface area A given by:

$$A = 2\pi \int_0^h y\sqrt{1+m^2}\,dx.$$

(a) Given that $m = 2$ and $h = 3$, show that $A = 18\sqrt{5}\,\pi$. [2]

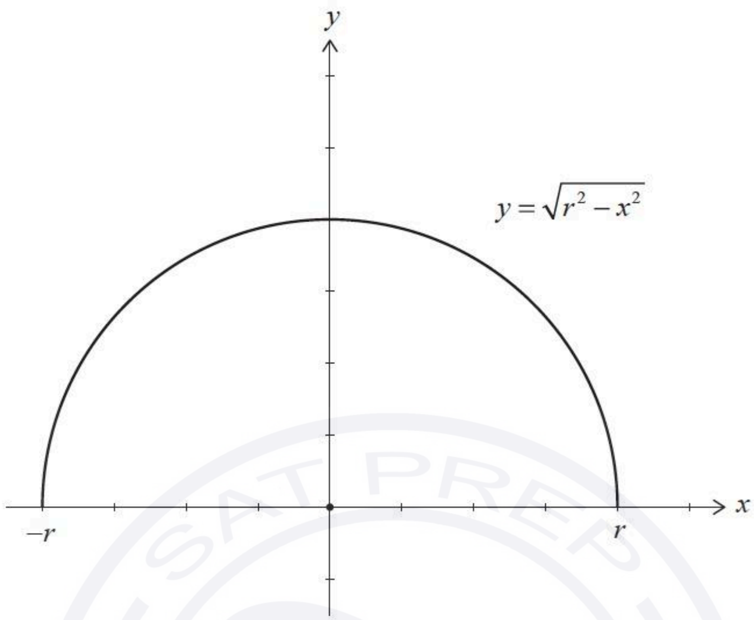
(b) Now consider the general case where a cone is formed by rotating the line $y = mx$ where $0 \le x \le h$ through $360°$ about the x-axis.

(i) Deduce an expression for the radius of this cone r in terms of h and m. [1]

(ii) Deduce an expression for the slant height l in terms of h and m. [2]

(iii) Hence, by using the above integral, show that $A = \pi r l$. [3]

Consider the semi-circle, with radius r, defined by $y = \sqrt{r^2 - x^2}$ where $-r \le x \le r$.



(c) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

A differentiable curve $y = f(x)$ is defined for $x_1 \le x \le x_2$ and $y \ge 0$. When any such curve is rotated through $360°$ about the x-axis, the surface formed has an area A given by:

$$A = 2\pi \int_{x_1}^{x_2} y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x .$$

(d) A sphere is formed by rotating the semi-circle $y = \sqrt{r^2 - x^2}$ where $-r \le x \le r$ through $360°$ about the x-axis. Show by integration that the surface area of this sphere is $4\pi r^2$. [4]

(e) Let $f(x)=\sqrt{r^2-x^2}$ where $-r \le x \le r$.

The graph of $y=f(x)$ is transformed to the graph of $y=f(kx)$, $k>0$. This forms a different curve, called a semi-ellipse.

(i) Describe this geometric transformation. [2]

(ii) Write down the x-intercepts of the graph $y=f(kx)$ in terms of r and k. [1]

(iii) For $y=f(kx)$, find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x, r and k. [2]

(iv) The semi-ellipse $y=f(kx)$ is rotated $360°$ about the x-axis to form a solid called an ellipsoid.

Find an expression in terms of r and k for the surface area, A, of the ellipsoid.

Give your answer in the form $2\pi\int_{x_1}^{x_2}\sqrt{p(x)}\,\mathrm{d}x$, where $p(x)$ is a polynomial. [4]

(v) Planet Earth can be modelled as an ellipsoid. In this model:

- the ellipsoid has an axis of rotational symmetry running from the North Pole to the South Pole.
- the distance from the North Pole to the South Pole is $12\ 714\,\mathrm{km}$.
- the diameter of the equator is $12\ 756\,\mathrm{km}$.

By choosing suitable values for r and k, find the surface area of Earth in km^2 correct to 4 significant figures. Give your answer in the form $a\times 10^q$ where $1\le a<10$ and $q\in\mathbb{Z}^+$. [4]

Question 7

[Maximum mark: 25]

In this question, you will be investigating the family of functions of the form $f(x) = x^n e^{-x}$.

Consider the family of functions $f_n(x) = x^n e^{-x}$, where $x \geq 0$ and $n \in \mathbb{Z}^+$.

When $n = 1$, the function $f_1(x) = xe^{-x}$, where $x \geq 0$.

(a) Sketch the graph of $y = f_1(x)$, stating the coordinates of the local maximum point. [4]

(b) Show that the area of the region bounded by the graph $y = f_1(x)$, the x-axis and the line $x = b$, where $b > 0$, is given by $\dfrac{e^b - b - 1}{e^b}$. [6]

You may assume that the total area, A_n, of the region between the graph $y = f_n(x)$ and the x-axis can be written as $A_n = \int_0^{\infty} f_n(x)\,dx$ and is given by $\lim_{b \to \infty} \int_0^b f_n(x)\,dx$.

(c) (i) Use l'Hôpital's rule to find $\lim_{b \to \infty} \dfrac{e^b - b - 1}{e^b}$. You may assume that the condition for applying l'Hôpital's rule has been met. [2]

(ii) Hence write down the value of A_1. [1]

You are given that $A_2 = 2$ and $A_3 = 6$.

(d) Use your graphic display calculator, and an appropriate value for the upper limit, to determine the value of

(i) A_4; [2]

(ii) A_5. [1]

(e) Suggest an expression for A_n in terms of n, where $n \in \mathbb{Z}^+$. [1]

(f) Use mathematical induction to prove your conjecture from part (e). You may assume that, for any value of m, $\lim_{x \to \infty} x^m e^{-x} = 0$. [8]

Question 8

[Maximum mark: 24]

This question asks you to examine the number and nature of intersection points of the graph of $y = \log_a x$ where $a \in \mathbb{R}^+$, $a \neq 1$ and the line $y = x$ for particular sets of values of a.

In this question you may either use the change of logarithm base formula $\log_a x = \frac{\ln x}{\ln a}$ or a graphic display calculator "logarithm to any base feature".

The function f is defined by

$$f(x) = \log_a x \text{ where } x \in \mathbb{R}^+ \text{ and } a \in \mathbb{R}^+,\ a \neq 1.$$

(a) Consider the cases $a = 2$ and $a = 10$. On the same set of axes, sketch the following three graphs:

$$y = \log_2 x$$

$$y = \log_{10} x$$

$$y = x.$$

Clearly label each graph with its equation and state the value of any non-zero x-axis intercepts. [4]

In parts (b) and (c), consider the case where $a = \mathrm{e}$. Note that $\ln x \equiv \log_{\mathrm{e}} x$.

(b) Use calculus to find the minimum value of the expression $x - \ln x$, justifying that this value is a minimum. [5]

(c) Hence deduce that $x > \ln x$. [1]

(d) There exist values of a for which the graph of $y = \log_a x$ and the line $y = x$ do have intersection points. The following table gives three intervals for the value of a.

Interval	Number of intersection points
$0 < a < 1$	p
$1 < a < 1.4$	q
$1.5 < a < 2$	r

By investigating the graph of $y = \log_a x$ for different values of a, write down the values of p, q and r. [4]

In parts (e) and (f), consider $a \in \mathbb{R}^+$, $a \neq 1$.

For $1.4 \leq a \leq 1.5$, a value of a exists such that the line $y = x$ is a tangent to the graph of $y = \log_a x$ at a point P.

(e) Find the exact coordinates of P and the exact value of a. [8]

(f) Write down the exact set of values for a such that the graphs of $y = \log_a x$ and $y = x$ have

(i) two intersection points; [1]

(ii) no intersection points. [1]

Question 8

[Maximum mark: 31]

This question begins by asking you to examine families of curves that intersect every member of another family of curves at right-angles. You will then examine a family of curves that intersects every member of another family of curves at an acute angle, α.

(a) Consider a family of straight lines, L, with equation $y = mx$, where m is a parameter. Each member of L intersects every member of a family of curves, C, at right-angles.

Note: In parts (i), (ii) and (iii), you are not required to consider the case where $x = 0$.

(i) Write down an expression for the gradient of L in terms of x and y. [1]

(ii) Hence show that the gradient of C is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$. [1]

(iii) By solving the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$, show that the family of curves, C, has equation $x^2 + y^2 = k$ where k is a parameter. [2]

A family of curves has equation $y^2 = 4a^2 - 4ax$ where a is a positive real parameter.

A second family of curves has equation $y^2 = 4b^2 + 4bx$ where b is a positive real parameter.

(b) Consider the case where $a = 2$ and $b = 1$. On the same set of axes, sketch the curves $y^2 = 16 - 8x$ and $y^2 = 4 + 4x$. On your sketch, clearly label each curve and any x-intercepts.

Note: You are not required to find the coordinates of any points of intersection of the two curves. [3]

(c) By solving $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ simultaneously, show that these curves intersect at the points $\mathrm{M}\left(a-b, 2\sqrt{ab}\right)$ and $\mathrm{N}\left(a-b, -2\sqrt{ab}\right)$. [6]

(d) At point M, show that the curves $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ intersect at right-angles. [5]

Consider two families of curves, F and G.

The gradient of F is denoted by $f(x, y)$.

The gradient of G is denoted by $g(x, y)$.

Each member of F intersects every member of G at an acute angle, α.

It can be shown that

$$g(x, y) = \frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha}.$$

In part (e), consider the specific case where $f(x, y) = -\frac{x}{y}$, for $x \neq 0$, $y \neq 0$ and $\alpha = \frac{\pi}{4}$.

(e) (i) Show that $g(x, y) = \frac{y - x}{y + x}$. [2]

(ii) Hence, by solving the homogeneous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y - x}{y + x}$, find a general equation that represents this family of curves, G. Give your answer in the form $h(x, y) = d$ where d is a parameter. [9]

(f) By considering $\lim_{\alpha \to \frac{\pi}{2}} \tan\alpha$, show that, for all finite $f(x, y)$,

$$\lim_{\alpha \to \frac{\pi}{2}} g(x, y) = -\frac{1}{f(x, y)}.$$ [2]

Question 9

[Maximum mark: 24]

If two functions $f(x)$ and $g(x)$ are differentiable, then their product is differentiable and the two functions satisfy the product rule: $(f(x)g(x))' = f(x)g'(x) + g(x)f'(x)$.

In this question, you will meet examples of pairs of differentiable functions, $f(x)$ and $g(x)$, that also satisfy $(f(x)g(x))' = f'(x)g'(x)$.

In part (a), consider $f(x) = \frac{1}{(2-x)^2}$, where $x \in \mathbb{R}$, $x \neq 2$, and $g(x) = x^2$, where $x \in \mathbb{R}$.

(a) (i) Find an expression for $f'(x)$. [2]

(ii) Show that $f'(x)g'(x) = \frac{4x}{(2-x)^3}$. [2]

(iii) Show that $f(x)g'(x) + g(x)f'(x) = \frac{4x}{(2-x)^3}$. [4]

In parts (b) and (c), consider two non-constant functions, $f(x)$ and $g(x)$, where $f(x) > 0$ and $g(x) \neq g'(x)$.

(b) By rearranging the equation $f(x)g'(x) + g(x)f'(x) = f'(x)g'(x)$, show that $\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$. [2]

(c) Hence, by integrating both sides of $\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$, show that $f(x) = Ae^{\left(\int \frac{g'(x)}{g'(x)-g(x)} dx\right)}$, where A is an arbitrary positive constant. [2]

The result from part (c) can be used to find pairs of functions, $f(x)$ and $g(x)$, which satisfy **both** of the following:

$$(f(x)g(x))' = f(x)g'(x) + g(x)f'(x) \text{ and } (f(x)g(x))' = f'(x)g'(x).$$

In parts (d) and (e), use the result in part (c) with $A = 1$.

(d) Consider $g(x) = xe^x$.

Find $f(x)$ such that $f(x)$ and $g(x)$ satisfy the above two equations. [5]

(e) Consider $g(x) = \sin x + \cos x$.

Find $f(x)$ such that $f(x)$ and $g(x)$ satisfy the above two equations over the domain $0 < x < \pi$.

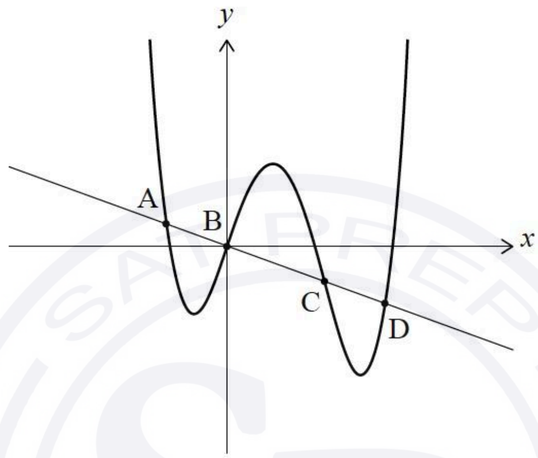
Give your answer in the form $f(x) = \sqrt{e^x h(x)}$, where $h(x)$ is a function to be determined. [7]

Question 10

[Maximum mark: 28]

This question investigates a ratio of lengths found from the line passing through the points of inflexion of a quartic curve of the form $y = x^4 - mx^3 + nx$.

The curve $y = x^4 - 3x^3 + 3x$ has points of inflexion at B and C. The line passing through B and C intersects the curve again at points A and D. This is shown in the following graph.



(a) Find $\frac{d^2y}{dx^2}$. [3]

(b) Find the coordinates of B and C. [4]

(c) Show that the equation of the line through B and C is $y = -0.375x$. [2]

(d) Find, correct to three decimal places, the x-coordinate of D. [2]

Now consider the general curve $y = x^4 - mx^3 + nx$, where m, $n \in \mathbb{R}$ and $m > 0$.

(e) Find the x-coordinates of the two points of inflexion in terms of m. [3]

Let these points of inflexion be B and C. The line passing through B and C intersects the curve again at points A and D. Let x_A be the x-coordinate of point A, and similarly for x_B, x_C and x_D. It is given that $x_A < x_B < x_C < x_D$.

(f) (i) Write down the coordinates of B. [1]

(ii) Find, in terms of m and n, the coordinates of C. [2]

(g) Show that the equation of the line through B and C is $y=\left(-\frac{m^3}{8}+n\right)x$. [2]

(h) Show that $x_A=\frac{m}{4}-\frac{m}{4}\sqrt{5}$. [7]

(i) Hence, find the exact value of $\frac{x_B-x_A}{x_C-x_B}$. [2]