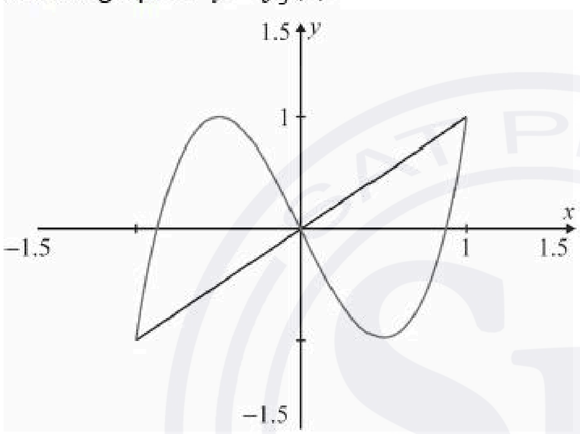
# Subject – Math AA(Higher Level)
# Topic - Calculus
# Year - May 2021 – Nov 2024
# Paper -3
# Answers

## Question 1

(a) correct graph of $y = f_1(x)$ **A1**

correct graph of $y = f_3(x)$ **A1**




***[2 marks]***

(b) (i) graphical or tabular evidence that $n$ has been systematically varied **M1**

*eg* $n = 3$, 1 local maximum point and 1 local minimum point

$n = 5$, 2 local maximum points and 2 local minimum points

$n = 7$, 3 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-1}{2}$ local maximum points **A1**

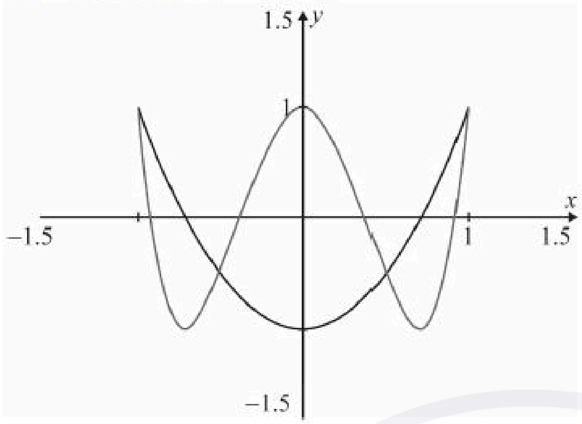
(ii) $\frac{n-1}{2}$ local minimum points **A1**

| **Note:** Allow follow through from an incorrect local maximum formula expression. |
|---|

***[4 marks]***

(c) correct graph of $y = f_2(x)$ **A1**

correct graph of $y = f_4(x)$ **A1**




**[2 marks]**

(d) (i) graphical or tabular evidence that $n$ has been systematically varied **M1**

*eg* $n = 2$, $0$ local maximum point and $1$ local minimum point
$n = 4$, $1$ local maximum points and $2$ local minimum points
$n = 6$, $2$ local maximum points and $3$ local minimum points **(A1)**

$\frac{n-2}{2}$ local maximum points **A1**

(ii) $\frac{n}{2}$ local minimum points **A1**

**[4 marks]**

(e) $f_n(x) = \cos\left(n \arccos(x)\right)$

$$f_n'(x) = \frac{n \sin\left(n \arccos(x)\right)}{\sqrt{1-x^2}}$$ **M1A1**

| **Note:** Award ***M1*** for attempting to use the chain rule. |
|---|

$f_n'(x) = 0 \Rightarrow n \sin\left(n \arccos(x)\right) = 0$ **M1**

$n \arccos(x) = k\pi$ ($k \in \mathbb{Z}^+$) **A1**

leading to

$x = \cos\frac{k\pi}{n}$ ($k \in \mathbb{Z}^+$ and $0 < k < n$) **AG**

**[4 marks]**

(f) $f_2(x) = \cos(2\arccos x)$

$= 2(\cos(\arccos x))^2 - 1$ **M1**

stating that $(\cos(\arccos x)) = x$ **A1**

so $f_2(x) = 2x^2 - 1$ **AG**

***[2 marks]***

(g) $f_{n+1}(x) = \cos((n+1)\arccos x)$

$= \cos(n\arccos x + \arccos x)$ **A1**

use of $\cos(A+B) = \cos A\cos B - \sin A\sin B$ leading to **M1**

$= \cos(n\arccos x)\cos(\arccos x) - \sin(n\arccos x)\sin(\arccos x)$ **AG**

***[2 marks]***

(h) (i) $f_{n-1}(x) = \cos((n-1)\arccos x)$ **A1**

$= \cos(n\arccos x)\cos(\arccos x) + \sin(n\arccos x)\sin(\arccos x)$ **M1**

$f_{n+1}(x) + f_{n-1}(x) = 2\cos(n\arccos x)\cos(\arccos x)$ **A1**

$= 2xf_n(x)$ **AG**

(ii) $f_3(x) = 2xf_2(x) - f_1(x)$ ***(M1)***

$= 2x(2x^2 - 1) - x$

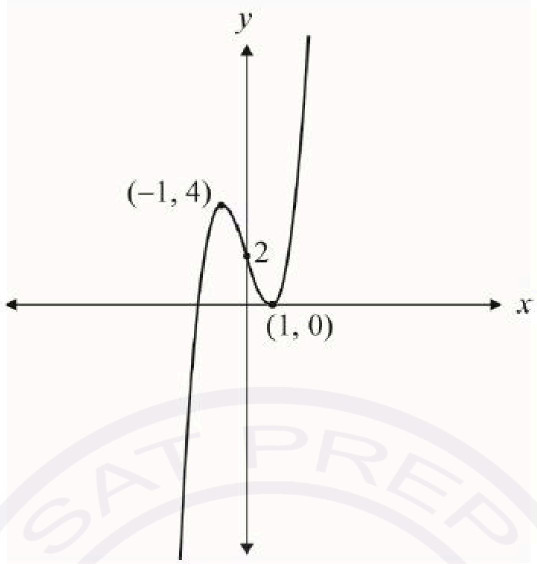
$= 4x^3 - 3x$ **A1**

***[5 marks]***

***Total [25 marks]***

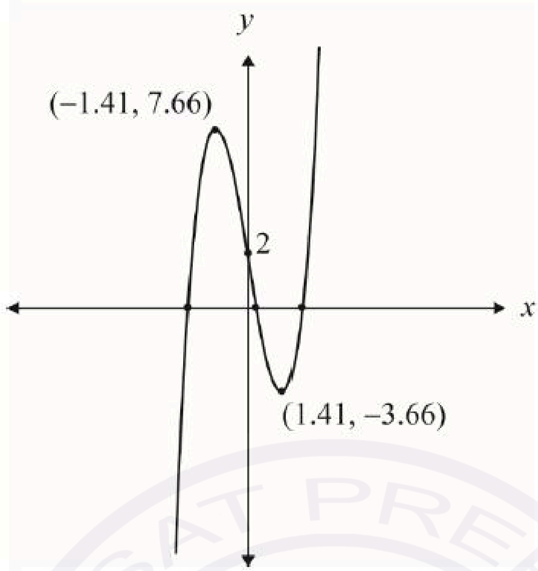
**Question 2**

(a) (i)




| | |
|---|---|
| $c=1$: positive cubic with correct *y*-intercept labelled | ***A1*** |
| local maximum point correctly labelled | ***A1*** |
| local minimum point correctly labelled | ***A1*** |
| | ***[3 marks]*** |

(ii)




$c = 2$: positive cubic with correct *y*-intercept labelled **A1**

local maximum point correctly labelled **A1**

local minimum point correctly labelled **A1**

**Note:** Accept the following exact answers:

Local maximum point coordinates $\left(-\sqrt{2}, 2+4\sqrt{2}\right)$.

Local minimum point coordinates $\left(\sqrt{2}, 2-4\sqrt{2}\right)$.

***[3 marks]***

(b) $f'(x) = 3x^2 - 3c$ **A1**

**Note:** Accept $3x^2 - 3c$ (an expression).

***[1 mark]***

(c) (i) $c = 0$ **A1**

***[1 mark]***

(ii) considers the number of solutions to their $f'(x) = 0$ ***(M1)***

$3x^2 - 3c = 0$

$c > 0$ **A1**

***[2 marks]***

(iii) $c < 0$ **A1**

**Note:** The ***(M1)*** in part (c)(ii) can be awarded for work shown in either (ii) or (iii).

***[1 mark]***

(d) attempts to solve their $f'(x)=0$ for $x$ ***(M1)***

$x=\pm\sqrt{c}$ ***(A1)***

**Note:** Award ***(A1)*** if either $x=-\sqrt{c}$ or $x=\sqrt{c}$ is subsequently considered.
Award the above ***(M1)(A1)*** if this work is seen in part (c).

(i) correctly evaluates $f\left(-\sqrt{c}\right)$ ***A1***

$f\left(-\sqrt{c}\right)=-c^{\frac{3}{2}}+3c^{\frac{3}{2}}+2\ \left(=-c\sqrt{c}+3c\sqrt{c}+2\right)$

the $y$-coordinate of the local maximum point is $2c^{\frac{3}{2}}+2$ ***AG***

***[3 marks]***

(ii) correctly evaluates $f\left(\sqrt{c}\right)$ ***A1***

$f\left(\sqrt{c}\right)=c^{\frac{3}{2}}-3c^{\frac{3}{2}}+2\ \left(=c\sqrt{c}-3c\sqrt{c}+2\right)$

the $y$-coordinate of the local minimum point is $-2c^{\frac{3}{2}}+2$ ***AG***

***[1 mark]***

(e) (i) the graph of $y = f(x)$ will have one $x-$ axis intercept if

**EITHER**

$-2c^{\frac{3}{2}} + 2 > 0$ (or equivalent reasoning) ***R1***

**OR**

the minimum point is above the $x-$ axis ***R1***

**Note:** Award ***R1*** for a rigorous approach that does not (only) refer to sketched graphs.

**THEN**

$0 < c < 1$ ***A1***

**Note:** Condone $c < 1$. The ***A1*** is independent of the ***R1***.

***[2 marks]***

(ii) the graph of $y = f(x)$ will have two $x-$ axis intercepts if

**EITHER**

$-2c^{\frac{3}{2}} + 2 = 0$ (or equivalent reasoning) ***(M1)***

**OR**

evidence from the graph in part(a)(i) ***(M1)***

**THEN**

$c = 1$ ***A1***

***[2 marks]***

(iii) the graph of $y = f(x)$ will have three $x-$ axis intercepts if

**EITHER**

$-2c^{\frac{3}{2}} + 2 < 0$ (or equivalent reasoning) ***(M1)***

**OR**

reasoning from the results in both parts (e)(i) and (e)(ii) ***(M1)***

**THEN**

$c > 1$ ***A1***

***[2 marks]***

(f) case 1:

$c \leq 0$ (independent of the value of $d$) ***A1***

**EITHER**

$g'(x) = 0$ does not have two solutions (has no solutions or 1 solution) ***R1***

**OR**

$\Rightarrow g'(x) \geq 0$ for $x \in$ ~ ***R1***

**OR**

the graph of $y = f(x)$ has no local maximum or local minimum points, hence any vertical translation of this graph ( $y = g(x)$) will also have no local maximum or local minimum points ***R1***

**THEN**

therefore there is only one $x$- axis intercept ***AG*** case 2

$c > 0$

$\left(-\sqrt{c}, 2c^{\frac{3}{2}} + d\right)$ is a local maximum point and $\left(\sqrt{c}, -2c^{\frac{3}{2}} + d\right)$ is a local minimum point ***(A1)***

**Note:** Award ***(A1)*** for a correct $y$-coordinate seen for either the maximum or the minimum.

considers the positions of the local maximum point and/or the local minimum point ***(M1)***

**EITHER**

considers both points above the $x$-axis or both points below the $x$-axis

**OR**

considers either the local minimum point only above the $x$-axis OR
the local maximum point only below the $x$-axis

**THEN**

$d > 2c^{\frac{3}{2}}$ (both points above the $x$-axis) ***A1***

$d < -2c^{\frac{3}{2}}$ (both points below the $x$-axis) ***A1***

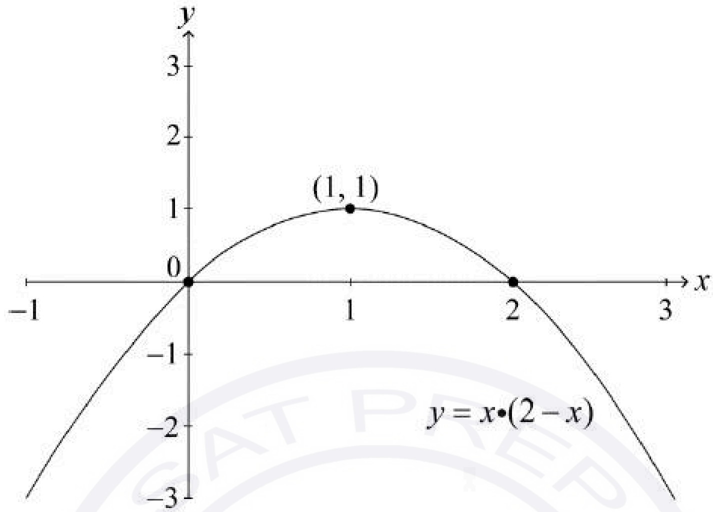
**Note:** Award at most ***(A1)(M1)A0A0*** for case 2 if $c > 0$ is not clearly stated.

***[6 marks]***

***Total [27 marks]***

## Question 3

(a)




inverted parabola extended below the $x$-axis **A1**

$x$-axis intercept values $x = 0, 2$ **A1**

| |
|---|
| **Note:** Accept a graph passing through the origin as an indication of $x = 0$. |

local maximum at $(1,1)$ **A1**

| |
|---|
| **Note:** Coordinates must be stated to gain the final ***A1***.<br>Do not accept decimal approximations. |

***[3 marks]***

(b)

| | Number of local maximum points | Number of local minimum points | Number of points of inflexion with zero gradient |
|---|---|---|---|
| $n=3$ and $n=5$ | 1 | 0 | 2 |
| $n=2$ and $n=4$ | 1 | 2 | 0 |

***A1A1A1A1A1A1***

**Note:** Award ***A1*** for each correct value.

For a table not sufficiently or clearly labelled, assume that their values are in the same order as the table in the question paper and award marks accordingly.

***[6 marks]***

(c) **METHOD 1**

attempts to use the product rule ***(M1)***

$f_n'(x) = -nx^n(a-x)^{n-1} + nx^{n-1}(a-x)^n$ ***A1A1***

| |
|---|
| **Note:** Award ***A1*** for a correct $u\frac{\mathrm{d}v}{\mathrm{d}x}$ and ***A1*** for a correct $v\frac{\mathrm{d}u}{\mathrm{d}x}$. |

**EITHER**

attempts to factorise $f_n'(x)$ (involving at least one of $nx^{n-1}$ or $(a-x)^{n-1}$) ***(M1)***

$= nx^{n-1}(a-x)^{n-1}((a-x)-x)$ ***A1***

**OR**

attempts to express $f_n'(x)$ as the difference of two products with each product containing at least one of $nx^{n-1}$ or $(a-x)^{n-1}$ ***(M1)***

$= (-x)(nx^{n-1})(a-x)^{n-1} + (a-x)(nx^{n-1})(a-x)^{n-1}$ ***A1***

**THEN**

$f_n'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$ ***AG***

| |
|---|
| **Note:** Award the final ***(M1)A1*** for obtaining any of the following forms: |
| $f_n'(x) = nx^n(a-x)^n\left(\frac{a-x-x}{x(a-x)}\right)$; $f_n'(x) = \frac{nx^n(a-x)^n}{x(a-x)}(a-x-x)$; |
| $f_n'(x) = nx^{n-1}\left((a-x)^n - x(a-x)^{n-1}\right)$; |
| $f_n'(x) = (a-x)^{n-1}\left(nx^{n-1}(a-x)^n - nx^n\right)$ |

**METHOD 2**

$f_n(x) = (x(a-x))^n$ ***(M1)***

$= (ax-x^2)^n$ ***A1***

attempts to use the chain rule ***(M1)***

$f_n'(x) = n(a-2x)(ax-x^2)^{n-1}$ ***A1A1***

Note: Award **A1** for $n(a-2x)$ and **A1** for $(ax-x^2)^{n-1}$.

$f_n'(x)=nx^{n-1}(a-2x)(a-x)^{n-1}$ **AG**

***[5 marks]***

(d) $x=0,\ x=\frac{a}{2},\ x=a$ **A2**

Note: Award **A1** for either two correct solutions or for obtaining $x=0,\ x=-a,\ x=-\frac{a}{2}$.
Award **A0** otherwise.

***[2 marks]***

(e) attempts to find an expression for $f_n\left(\frac{a}{2}\right)$ ***(M1)***

$f_n\left(\frac{a}{2}\right)=\left(\frac{a}{2}\right)^n\left(a-\frac{a}{2}\right)^n$

$=\left(\frac{a}{2}\right)^n\left(\frac{a}{2}\right)^n\ \left(=\left(\frac{a}{2}\right)^{2n}\right),\left(=\left(\left(\frac{a}{2}\right)^n\right)^2\right)$ **A1**

**EITHER**

since $a\in\mathbb{R}^+$, $\left(\frac{a}{2}\right)^{2n}>0$ (for $n\in\mathbb{Z}^+, n>1$ and so $f_n\left(\frac{a}{2}\right)>0$) **R1**

Note: Accept any logically equivalent conditions/statements on $a$ and $n$.
Award **R0** if any conditions/statements specified involving $a$, $n$ or both are incorrect.

**OR**

(since $a\in\mathbb{R}^+$), $\frac{a}{2}$ raised to an even power ($2n$) (or equivalent reasoning) is always positive (and so $f_n\left(\frac{a}{2}\right)>0$) **R1**

Note: The condition $a\in\mathbb{R}^+$ is given in the question. Hence some candidates will assume $a\in\mathbb{R}^+$ and not state it. In these instances, award **R1** for a convincing argument.
Accept any logically equivalent conditions/statements on $a$ and $n$.
Award **R0** if any conditions/statements specified involving $a$, $n$ or both are incorrect.

**THEN**

so $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ is always above the horizontal axis **_AG_**

| **Note:** Do not award ***(M1)A0R1***. |
|---|

***[3 marks]***

(f) **METHOD 1**

$f_n'\left(\frac{a}{4}\right) = n\left(\frac{a}{4}\right)^{n-1}\left(a-\frac{a}{2}\right)\left(a-\frac{a}{4}\right)^{n-1}\ \left(= n\left(\frac{a}{4}\right)^{n-1}\left(\frac{a}{2}\right)\left(\frac{3a}{4}\right)^{n-1}\right)$ **_A1_**

**EITHER**

$n\left(\frac{a}{4}\right)^{n-1}\left(\frac{a}{2}\right)\left(\frac{3a}{4}\right)^{n-1} > 0$ as $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$ **_R1_**

**OR**

$n\left(\frac{a}{4}\right)^{n-1}, \left(a-\frac{a}{2}\right)$ and $\left(a-\frac{a}{4}\right)^{n-1}$ are all $> 0$ **_R1_**

**Note:** Do not award ***A0R1***.

Accept equivalent reasoning on correct alternative expressions for $f_n'\left(\frac{a}{4}\right)$ and accept any logically equivalent conditions/statements on $a$ and $n$.

Exceptions to the above are condone $n > 1$ and condone $n > 0$.

An alternative form for $f_n'\left(\frac{a}{4}\right)$ is $(2n)(3)^{n-1}\left(\frac{a}{4}\right)^{2n-1}$.

**THEN**

hence $f_n'\left(\frac{a}{4}\right) > 0$ **_AG_**

***[2 marks]***

**METHOD 2**

$f_n(0) = 0$ and $f_n\left(\frac{a}{2}\right) > 0$ **_A1_** (since $f_n$ is continuous and there are no stationary points between $x=0$ and $x=\frac{a}{2}$)

the gradient (of the curve) must be positive between $x=0$ and $x=\frac{a}{2}$ ***R1***

| **Note:** Do not award ***A0R1***. |
|---|

hence $f_n'\left(\frac{a}{4}\right)>0$ ***AG***

***[2 marks]***

(g) (i) $f_n'(-1)=n(-1)^{n-1}(a+2)(a+1)^{n-1}$

for $n$ even:

$n(-1)^{n-1}(=-n)<0$ (and $(a+2),(a+1)^{n-1}$ are both $>0$) ***R1***

$f_n'(-1)<0$ ***A1***

$f_n'(0)=0$ and $f_n'\left(\frac{a}{4}\right)>0$ (seen anywhere) ***A1***

**Note:** Candidates can give arguments based on the sign of $(-1)^{n-1}$ to obtain the ***R*** mark.

For example, award ***R1*** for the following:

If $n$ is even, then $n-1$ is odd and hence $(-1)^{n-1}<0\ (=-1)$.

Do not award ***R0A1***.

The second ***A1*** is independent of the other two marks.

The ***A*** marks can be awarded for correct descriptions expressed in words.

Candidates can state $(0,0)$ as a point of zero gradient from part (d) or show, state or explain (words or diagram) that $f_n'(0)=0$. The last ***A*** mark can be awarded for a clearly labelled diagram showing changes in the sign of the gradient.

The last ***A1*** can be awarded for use of a specific case (*e.g.* $n=2$).

hence $(0,0)$ is a local minimum point ***AG***

***[3 marks]***

(ii) for $n$ odd:

$n(-1)^{n-1}(=n)>0$, (and $(a+2),(a+1)^{n-1}$ are both $>0$ ) so $f_n'(-1)>0$ **R1**

**Note:** Candidates can give arguments based on the sign of $(-1)^{n-1}$ to obtain the ***R*** mark.

For example, award ***R1*** for the following:

If $n$ is odd, then $n-1$ is even and hence $(-1)^{n-1}>0$ $(=1)$.

$f_n'(0)=0$ and $f_n'\left(\frac{a}{4}\right)>0$ (seen anywhere) **A1**

**Note:** The ***A1*** is independent of the ***R1***.

Candidates can state $(0,0)$ as a point of zero gradient from part (d) or show, state or explain (words or diagram) that $f_n'(0)=0$. The last *A* mark can be awarded for a clearly labelled diagram showing changes in the sign of the gradient.

The last *A1* can be awarded for use of a specific case (*e.g.* $n=3$).

hence $(0,0)$ is a point of inflexion with zero gradient **AG**

***[2 marks]***

(h) considers the parity of $n$ **(M1)**

| |
|---|
| **Note:** Award ***M1*** for stating at least one specific even value of $n$. |

$n$ must be even (for four solutions) ***A1***

| |
|---|
| **Note:** The above 2 marks are independent of the 3 marks below. |

$0 < k < \left(\frac{a}{2}\right)^{2n}$ ***A1A1A1***

| |
|---|
| **Note:** Award ***A1*** for the correct lower endpoint, ***A1*** for the correct upper endpoint and ***A1*** for strict inequality signs.<br>The third ***A1*** (strict inequality signs) can only be awarded if ***A1A1*** has been awarded.<br>For example, award ***A1A1A0*** for $0 \le k \le \left(\frac{a}{2}\right)^{2n}$. Award ***A1A0A0*** for $k > 0$.<br>Award ***A1A0A0*** for $0 < k < f_n\left(\frac{a}{2}\right)$. |

***[5 marks]***

***Total [31 marks]***

**Question 4**

(a) (i) **METHOD 1**

$\frac{dy}{dt} = y$

$\int \frac{dy}{y} = \int dt$ **(M1)**

$\ln y = t + c$ **OR** $\ln|y| = t + c$ **A1A1**

| |
|---|
| **Note:** Award ***A1*** for $\ln y$ and ***A1*** for $t$ and $c$. |

$y = Ae^t$ ***AG***

**METHOD 2**

rearranging to $\frac{dy}{dt} - y = 0$ AND multiplying by integrating factor $e^{-t}$ ***M1***

$ye^{-t} = A$ ***A1A1***

$y = Ae^t$ ***AG***

***[3 marks]***

(ii) substituting $y = Ae^t$ into differential equation in $x$ ***M1***

$\frac{dx}{dt} = x - Ae^t$

$\frac{dx}{dt} - x = -Ae^t$ ***AG***

***[1 mark]***

(iii) integrating factor (IF) is $e^{\int -1 dt}$ ***(M1)***

$= e^{-t}$ ***(A1)***

$e^{-t}\frac{dx}{dt} - xe^{-t} = -A$

$xe^{-t} = -At + D$ ***(A1)***

$x = (-At + D)e^t$ ***A1***

| |
|---|
| **Note:** The first constant must be $A$, and the second can be any constant for the final ***A1*** to be awarded. Accept a change of constant applied at the end. |

***[4 marks]***

(b) (i) $\frac{d^2y}{dt^2} = -\frac{dx}{dt} + \frac{dy}{dt}$ ***A1***

**EITHER**

$= -x + y + \frac{dy}{dt}$ ***(M1)***

$= \frac{dy}{dt} + \frac{dy}{dt}$ ***A1***

**OR**

$= -x + y + (-x + y)$ ***(M1)***

$= 2(-x + y)$ ***A1***

**THEN**

$= 2\frac{dy}{dt}$ ***AG***

***[3 marks]***

(ii) $\frac{dY}{dt} = 2Y$ ***A1***

$\int \frac{dY}{Y} = \int 2dt$ ***M1***

$\ln|Y| = 2t + c$ OR $\ln Y = 2t + c$ ***A1***

$Y = Be^{2t}$ ***AG***

***[3 marks]***

(iii) $\frac{dy}{dt} = Be^{2t}$

$y = \int Be^{2t}\,dt$ ***M1***

$y = \frac{B}{2}e^{2t} + C$ ***A1***

**Note:** The first constant must be $B$, and the second can be any constant for the final ***A1*** to be awarded. Accept a change of constant applied at the end.

***[2 marks]***

(iv) **METHOD 1**

substituting $\frac{dy}{dt} = Be^{2t}$ and their (iii) into $\frac{dy}{dt} = -x + y$ ***M1(M1)***

$Be^{2t} = -x + \frac{B}{2}e^{2t} + C$ ***A1***

$x = -\frac{B}{2}e^{2t} + C$ ***AG***

**Note:** Follow through from incorrect part (iii) cannot be awarded if it does not lead to the ***AG***.

**METHOD 2**

$\frac{dx}{dt} = x - \frac{B}{2}e^{2t} - C$

$\frac{dx}{dt} - x = -\frac{B}{2}e^{2t} - C$

$\frac{d(xe^{-t})}{dt} = -\frac{B}{2}e^{t} - Ce^{-t}$ ***M1***

$xe^{-t} = \int -\frac{B}{2}e^{t} - Ce^{-t}\, dt$

$xe^{-t} = -\frac{B}{2}e^{t} + Ce^{-t} + D$ ***A1***

$x = -\frac{B}{2}e^{2t} + C + De^{t}$

$\frac{dy}{dt} = -x + y \Rightarrow Be^{2t} = \frac{B}{2}e^{2t} - C - De^{t} + \frac{B}{2}e^{2t} + C \Rightarrow D = 0$ ***M1***

$x = -\frac{B}{2}e^{2t} + C$ ***AG***

***[3 marks]***

(c) (i) $\dfrac{dy}{dt} = -4x + y$

$\dfrac{d^2y}{dt^2} = -4\dfrac{dx}{dt} + \dfrac{dy}{dt}$ seen anywhere **M1**

**METHOD 1**

$\dfrac{d^2y}{dt^2} = -4(x - y) + \dfrac{dy}{dt}$

attempt to eliminate $x$ **M1**

$= -4\left(\dfrac{1}{4}\left(y - \dfrac{dy}{dt}\right) - y\right) + \dfrac{dy}{dt}$

$= 2\dfrac{dy}{dt} + 3y$ **A1**

$\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} - 3y = 0$ **AG**

**METHOD 2**

rewriting LHS in terms of $x$ and $y$ **M1**

$\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} - 3y = (-8x + 5y) - 2(-4x + y) - 3y$ **A1**

$= 0$ **AG**

***[3 marks]***

(ii) $\dfrac{dy}{dt} = F\lambda e^{\lambda t}$, $\dfrac{d^2y}{dt^2} = F\lambda^2 e^{\lambda t}$ ***(A1)***

$F\lambda^2 e^{\lambda t} - 2F\lambda e^{\lambda t} - 3Fe^{\lambda t} = 0$ ***(M1)***

$\lambda^2 - 2\lambda - 3 = 0$ (since $e^{\lambda t} \neq 0$ ) ***A1***

$\lambda_1$ and $\lambda_2$ are 3 and -1 (either order) ***A1***

***[4 marks]***

(iii) **METHOD 1**

$y = Fe^{3t} + Ge^{-t}$

$\frac{dy}{dt} = 3Fe^{3t} - Ge^{-t}$, $\frac{d^2y}{dt^2} = 9Fe^{3t} + Ge^{-t}$ ***(A1)(A1)***

$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 9Fe^{3t} + Ge^{-t} - 2\left(3Fe^{3t} - Ge^{-t}\right) - 3\left(Fe^{3t} + Ge^{-t}\right)$ ***M1***

$= 9Fe^{3t} + Ge^{-t} - 6Fe^{3t} + 2Ge^{-t} - 3Fe^{3t} - 3Ge^{-t}$ ***A1***

$= 0$ ***AG***

**METHOD 2**

$y = Fe^{\lambda_1 t} + Ge^{\lambda_2 t}$

$\frac{dy}{dt} = F\lambda_1 e^{\lambda_1 t} + G\lambda_2 e^{\lambda_2 t}$, $\frac{d^2y}{dt^2} = F\lambda_1^2 e^{\lambda_1 t} + G\lambda_2^2 e^{\lambda_2 t}$ ***(A1)(A1)***

$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = F\lambda_1^2 e^{\lambda_1 t} + G\lambda_2^2 e^{\lambda_2 t} - 2\left(F\lambda_1 e^{\lambda_1 t} + G\lambda_2 e^{\lambda_2 t}\right) - 3\left(Fe^{\lambda_1 t} + Ge^{\lambda_2 t}\right)$ ***M1***

$= Fe^{\lambda_1 t}\left(\lambda^2 - 2\lambda - 3\right) + Ge^{\lambda_2 t}\left(\lambda^2 - 2\lambda - 3\right)$ ***A1***

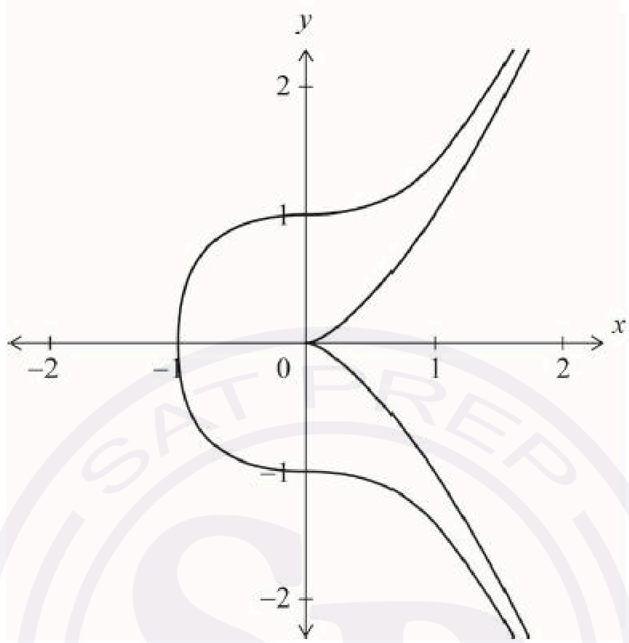
$= 0$ ***AG***

***[4 marks]***

***Total [30 marks]***

**Question 5**

(a) (i)




approximately symmetric about the $x$-axis graph of $y^2 = x^3$ ***A1***

including cusp/sharp point at (0, 0) ***A1***

***[2 marks]***

(ii) approximately symmetric about the x-axis graph of $y^2 = x^3 + 1$ with approximately correct gradient at axes intercepts ***A1***

some indication of position of intersections at $x = -1,\ y = \pm 1$ ***A1***

***[2 marks]***

**Note:** Final ***A1*** can be awarded if intersections are in approximate correct place with respect to the axes shown. Award ***A1A1A1A0*** if graphs 'merge' or 'cross' or are discontinuous at $x$-axis but are otherwise correct. Award ***A1A0A0A0*** if only one correct branch of both curves are seen.

**Note**: If they sketch graphs on separate axes, award a maximum of 2 marks for the 'best' response seen. This is likely to be ***A1A1A0A0.***

(b) (i) $(0, 1)$ and $(0, -1)$ **A1**

**[1 mark]**

(ii) Any **two** from:

$y^2 = x^3$ has a cusp/sharp point, (the other does not)

graphs have different domains

$y^2 = x^3 + 1$ has points of inflexion, (the other does not)

graphs have different $x$-axis intercepts (one goes through the origin, and the other does not)

graphs have different $y$-axis intercepts **A1**

| |
|---|
| **Note:** Follow through from their sketch in part (a)(i). In accordance with marking rules, mark their first two responses and ignore any subsequent. |

**[1 mark]**

(c) Any **two** from:

as $x \to \infty, y \to \pm\infty$

as $x \to \infty$, $y^2 = x^3 + b$ is approximated by $y^2 = x^3$ (or similar)

they have $x$ intercepts at $x = -\sqrt[3]{b}$

they have $y$ intercepts at $y = (\pm)\sqrt{b}$

they all have the same range

$y = 0$ (or $x$-axis) is a line of symmetry

they all have the same line of symmetry $(y = 0)$

they have one $x$-axis intercept

they have two $y$-axis intercepts

they have two points of inflexion

at $x$-axis intercepts, curve is vertical/infinite gradient

there is no cusp/sharp point at $x$-axis intercepts ***A1A1***

**Note:** The last example is the only valid answer for things "not" present. Do not credit an answer of "they are all symmetrical" without some reference to the line of symmetry.

**Note:** Do not allow same/ similar shape or equivalent.

***Note:*** In accordance with marking rules, mark their first two responses and ignore any subsequent.

***[2 marks]***

(d) (i) **METHOD 1**

attempt to differentiate implicitly ***M1***

$2y\frac{dy}{dx} = 3x^2 + 1$ ***A1***

$\frac{dy}{dx} = \frac{3x^2+1}{2y}$ OR $(\pm)2\sqrt{x^3+x}\frac{dy}{dx} = 3x^2 + 1$ ***A1***

$\frac{dy}{dx} = \pm\frac{3x^2+1}{2\sqrt{x^3+x}}$ ***AG***

**METHOD 2**

attempt to use chain rule $y = (\pm)\sqrt{x^3+x}$ ***M1***

$\frac{dy}{dx} = (\pm)\frac{1}{2}(x^3+x)^{-\frac{1}{2}}(3x^2+1)$ ***A1A1***

**Note:** Award ***A1*** for $(\pm)\frac{1}{2}(x^3+x)^{-\frac{1}{2}}$, ***A1*** for $(3x^2+1)$.

$\frac{dy}{dx} = \pm\frac{3x^2+1}{2\sqrt{x^3+x}}$ ***AG***

***[3 marks]***

(ii) **EITHER**

local minima/maxima occur when $\frac{dy}{dx} = 0$

$1 + 3x^2 = 0$ has no (real) solutions (or equivalent) ***R1***

**OR**

$\left(x^2 \geq 0 \Rightarrow\right)$ $3x^2 + 1 > 0$, so $\frac{dy}{dx} \neq 0$ ***R1***

**THEN**

so, no local minima/maxima exist ***AG***

***[1 mark]***

(e) **EITHER**

attempt to use quotient rule to find $\frac{d^2y}{dx^2}$ ***M1***

$$\frac{d^2y}{dx^2} = (\pm)\frac{12x\sqrt{x+x^3} - (1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)}{4(x+x^3)}$$ ***A1A1***

| |
|---|
| **Note:** Award ***A1*** for correct $12x\sqrt{x+x^3}$ and correct denominator, ***A1*** for correct $-(1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)$. |
| **Note:** Future ***A*** marks may be awarded if the denominator is missing or incorrect. |

stating or using $\frac{d^2y}{dx^2} = 0$ (may be seen anywhere) ***(M1)***

$$12x\sqrt{x+x^3} = (1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)$$

**OR**

attempt to use product rule to find $\frac{d^2y}{dx^2}$ ***M1***

$$\frac{d^2y}{dx^2} = \frac{1}{2}(3x^2+1)\left(-\frac{1}{2}\right)(3x^2+1)(x^3+x)^{-\frac{3}{2}} + 3x(x^3+x)^{-\frac{1}{2}}$$ ***A1A1***

| |
|---|
| **Note:** Award ***A1*** for correct first term, ***A1*** for correct second term. |

setting $\frac{d^2y}{dx^2} = 0$ ***(M1)***

**OR**

attempts implicit differentiation on $2y\dfrac{dy}{dx} = 3x^2 + 1$ **M1**

$2\left(\dfrac{dy}{dx}\right)^2 + 2y\dfrac{d^2y}{dx^2} = 6x$ **A1**

recognizes that $\dfrac{d^2y}{dx^2} = 0$ ***(M1)***

$\dfrac{dy}{dx} = \pm\sqrt{3x}$

$(\pm)\dfrac{3x^2+1}{2\sqrt{x^3+x}} = (\pm)\sqrt{3x}$ ***(A1)***

**THEN**

$12x(x+x^3) = (1+3x^2)^2$

$12x^2 + 12x^4 = 9x^4 + 6x^2 + 1$

$3x^4 + 6x^2 - 1 = 0$ **A1**

attempt to use quadratic formula or equivalent ***(M1)***

$x^2 = \dfrac{-6 \pm \sqrt{48}}{6}$

$(x > 0 \Rightarrow) x = \sqrt{\dfrac{2\sqrt{3}-3}{3}}$ $(p = 2,\ q = -3,\ r = 3)$ **A1**

| |
|---|
| **ote:** Accept any integer multiple of $p$, $q$ and $r$ (e.g. $4, -6$ and $6$). |

***[7 marks]***

(f) (i) attempt to find tangent line through $(-1,-1)$ ***(M1)***

$y+1=-\frac{3}{2}(x+1)$ OR $y=-1.5x-2.5$ ***A1***

***[2 marks]***

(ii) attempt to solve simultaneously with $y^2=x^3+2$ ***(M1)***

| | |
|---|---|
| **Note:** | The ***M1*** mark can be awarded for an unsupported correct answer in an incorrect format (e.g. (4.25, -8.875)). |

obtain $\left(\frac{17}{4},-\frac{71}{8}\right)$ ***A1***

***[2 marks]***

(g) attempt to find equation of $[\mathrm{QS}]$ **(M1)**

$$\frac{y-1}{x+1}=-\frac{79}{42}(=-1.88095\ldots)$$ **(A1)**

solve simultaneously with $y^2=x^3+2$ **(M1)**

$$x=0.28798\ldots\left(=\frac{127}{441}\right)$$ **A1**

$$y=-1.4226\ldots\left(=\frac{13175}{9261}\right)$$ **A1**

$(0.288,-1.42)$

**OR**

attempt to find vector equation of $[\mathrm{QS}]$ **(M1)**

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}+\lambda\begin{pmatrix}\frac{21}{4}\\-\frac{79}{8}\end{pmatrix}$$ **(A1)**

$$x=-1+\frac{21}{4}\lambda$$

$$y=1-\frac{79}{8}\lambda$$

attempt to solve $\left(1-\frac{79}{8}\lambda\right)^2=\left(-1+\frac{21}{4}\lambda\right)^3+2$ **(M1)**

$\lambda=0.2453\ldots$

$$x=0.28798\ldots\left(=\frac{127}{441}\right)$$ **A1**

$$y=-1.4226\ldots\left(=\frac{13175}{9261}\right)$$ **A1**

$(0.288,-1.42)$

***[5 marks]***

***[Total 28 marks]***

## Question 6

(a) **EITHER**

$A = 2\pi\int_0^3 2x\sqrt{1+2^2}\ \mathrm{d}x \ \left(= 4\sqrt{5}\pi\int_0^3 x\,\mathrm{d}x\right)$ ***(A1)***

$= 2\pi\sqrt{5}\ \left[x^2\right]_0^3 \ \left(= 2\pi\sqrt{5}\left(3^2 - 0^2\right)\right)$ ***A1***

**OR**

$A = 2\pi m\sqrt{1+m^2}\ \left[\frac{x^2}{2}\right]_0^h \ \left(= 2\pi m\sqrt{1+m^2}\left(\frac{h^2}{2}\right)\right)$ ***(A1)***

$= 2\pi(2)\sqrt{5}\ \left[\frac{x^2}{2}\right]_0^3 \ \left(= 2\pi(2)\sqrt{5}\left(\frac{3^2}{2}\right)\right)$ ***A1***

**THEN**

$= 18\sqrt{5}\,\pi$ ***AG***

***[2 marks]***

(b) (i) $r = mh$ ***A1***

***[1 mark]***

(ii) $l = \sqrt{h^2 + r^2}$ ***(M1)***

$l = \sqrt{h^2 + h^2m^2}\ \left(= h\sqrt{1+m^2}\right)$ ***A1***

***[2 marks]***

(iii) $A = 2\pi\int_0^h mx\sqrt{1+m^2}\ \mathrm{d}x$ **(A1)**

$= 2\pi m\sqrt{1+m^2}\left[\frac{1}{2}x^2\right]_0^h$ **(M1)**

**Note:** Award **(M1)** for $(c)\pi m\sqrt{1+m^2}\left[\frac{1}{2}x^2\right]_0^h$.

At least one of the above two lines needs to be seen.

$= \pi h^2 m\sqrt{1+m^2}\ \left(= \pi hm \times \sqrt{\left(h^2+h^2m^2\right)}\right)$ **A1**

$= \pi rl$ **AG**

**Note:** Award as above if $\frac{l}{h} = \sqrt{1+m^2}$ is used, for example.

***[3 marks]***

(c) **METHOD 1**

attempts to use the chain rule **(M1)**

**Note:** Award **(M1)** for $\frac{\mathrm{d}y}{\mathrm{d}x} = (c)x\left(r^2-x^2\right)^{-\frac{1}{2}}$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(r^2-x^2\right)^{-\frac{1}{2}}\ (-2x)\ \left(= -x\left(r^2-x^2\right)^{-\frac{1}{2}}\right)$ **A1**

**METHOD 2**

attempts implicit differentiation on $y^2 = r^2 - x^2$ (or equivalent) **(M1)**

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ **A1**

***[2 marks]***

(d) **EITHER**

attempts to substitute $y=\sqrt{r^2-x^2}$ and their $\frac{dy}{dx}$ into $A$ ***(M1)***

$$A=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+\left(-x\left(r^2-x^2\right)^{-\frac{1}{2}}\right)^2}\,dx$$

$$=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$$ ***A1***

**OR**

attempts to substitute $y$ and their $\frac{dy}{dx}$ in terms of $x$ and $y$ into $A$ ***(M1)***

$$A=2\pi\int_{-r}^{r}y\sqrt{1+\left(-\frac{x}{y}\right)^2}\,dx$$

$$=2\pi\int_{-r}^{r}y\sqrt{1+\frac{x^2}{y^2}}\,dx\ \left(=2\pi\int_{-r}^{r}\sqrt{x^2+y^2}\,dx\right)$$ ***A1***

**THEN**

attempts to perform valid algebraic simplification to form a definite integral in terms of $r$ only ***M1***

$$=2\pi\int_{-r}^{r}r\,dx$$

$$=2\pi r\left[x\right]_{-r}^{r}\ \left(=2\pi r\left(r-(-r)\right)\right)$$ ***A1***

$$=4\pi r^2$$ ***AG***

**Note:** Award marks as above for $A=4\pi\int_{0}^{r}\sqrt{r^2-x^2}\sqrt{1+\left(-x\left(r^2-x^2\right)^{-\frac{1}{2}}\right)^2}\,dx$.

***[4 marks]***

continued…

(e) (i) **EITHER**

horizontal stretch **_A1_**

factor $\frac{1}{k}$ **_A1_**

**OR**

horizontal compression **_A1_**

factor $k$ (invariant line $y$- axis) **_A1_**

> **Note:** Award ***A1A1*** as above for correct alternative descriptions.
> For example, dilation by a factor of $\frac{1}{k}$ from the $y$- axis.

***[2 marks]***

(ii) $\pm\frac{r}{k}$ **_A1_**

> **Note:** Award ***A0*** for $\frac{r}{k}$ only and ***A0*** for $-\frac{r}{k}$ only.

***[1 mark]***

(iii) **METHOD 1**

attempts to use the chain rule ***(M1)***

$$\frac{dy}{dx}=\frac{1}{2}\left(r^2-(kx)^2\right)^{-\frac{1}{2}}\times\left(-k^2 2x\right)\ \left(=-k^2x\left(r^2-(kx)^2\right)^{-\frac{1}{2}}\ \left(=\frac{-k^2x}{\sqrt{r^2-k^2x^2}}\right)\right)$$ **_A1_**

**METHOD 2**

attempts implicit differentiation on $y^2=r^2-k^2x^2$ (or equivalent) ***(M1)***

$$\frac{dy}{dx}=-\frac{k^2x}{y}$$

$$\frac{dy}{dx}=\frac{-k^2x}{\sqrt{r^2-k^2x^2}}$$ **_A1_**

***[2 marks]***

(iv) **EITHER**

$$A = 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} \sqrt{r^2 - k^2x^2} \sqrt{1 + \frac{k^4x^2}{r^2 - k^2x^2}}\, dx$$ ***(A1)***

**Note:** Award ***(A1)*** for the correct substitution of $y$ and $\frac{dy}{dx}$.

attempts to simplify to find $p(x)$, eg. forming a common denominator of $r^2 - k^2x^2$ and then cancelling $r^2 - k^2x^2$ ***(M1)***

**OR**

$$A = 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} y\sqrt{1 + \left(-\frac{k^2x}{y}\right)^2}\, dx \quad \left( = 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} y\sqrt{1 + \frac{k^4x^2}{y^2}}\, dx \right)$$ ***(A1)***

**Note:** Award ***(A1)*** for the correct substitution of $\frac{dy}{dx}$.

attempts to simplify to find $p(x)$, eg. forming a common denominator of $y^2$ and cancelling $y^2$ ***(M1)***

**THEN**

$$= 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} \sqrt{r^2 - k^2x^2 + k^4x^2}\, dx \quad \left( = 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} \sqrt{r^2 + \left(k^4 - k^2\right)x^2}\, dx \right)$$ ***A1A1***

**Note:** Award ***A1*** for correct limits (seen anywhere) and ***A1*** for $p(x)$ correct.
The above ***A1*** for correct limits is independent of the ***(M1)***.

***[4 marks]***

(v) $r = 6378\ (\text{km})$ **(A1)**

$k = 1.00330...\left(= \frac{6378}{6357} = \frac{2126}{2119}\right)$ **(A1)**

attempts to form a definite integral for surface area **(M1)**

$$= 2\pi \int_{-6357}^{6357} \sqrt{6378^2 - \left(\frac{6378}{6357}\right)^2 x^2 + \left(\frac{6378}{6357}\right)^4 x^2}\ \mathrm{d}x$$

$= 510064226.3...$

$= 5.101 \times 10^8\ (\text{km}^2)$ **A1**

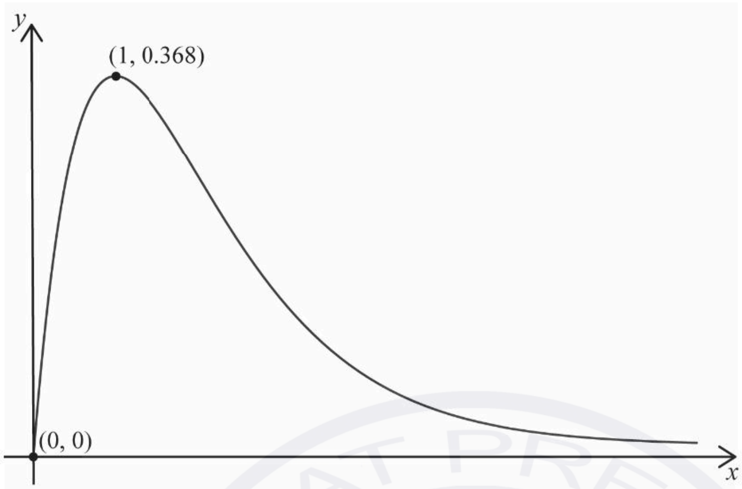
**Note:** Award ***A0A1FT(M1)A0*** for using $r = 6357\ (\text{km})$ and $k = 0.996707...$ leading to an answer of $5.089 \times 10^8\ (\text{km}^2)$.

***[4 marks]***

***Total [27 marks]***

**Question 7**

(a)




***A1*** for $(1, 0.368)$ or $\left(1, \frac{1}{e}\right)$ labelled at local maximum (accept correct coordinates written away from the graph)

***A1*** for graph clearly starting at, or passing through,the origin

***A1*** for correct domain

***A1*** for correct shape i.e.: single maximum, and asymptotic behaviour (equation not required) (or point of inflexion)

***[4 marks]***

(b) $\int_0^b x\,\mathrm{e}^{-x}\,\mathrm{d}x$ ***(A1)***

| **Note:** Award ***(A1)*** for correct integrand and limits (which can be seen later in the question) |
|---|

Use of integration by parts

$= \left[-x\,\mathrm{e}^{-x}\right]_0^b + \int_0^b \mathrm{e}^{-x}\,\mathrm{d}x$ ***A1A1***

| **Note:** Award ***A1*** for each part (including the correct sign with each) |
|---|

$= \left[-x\,\mathrm{e}^{-x}\right]_0^b - \left[\mathrm{e}^{-x}\right]_0^b$ ***A1***

| **Note:** Award ***A1*** for correct second term. Condone absence of limits to this point |
|---|

attempt to substitute limits ***M1***

$= -b\,\mathrm{e}^{-b} - \mathrm{e}^{-b} + 1$ ***A1***

$= \dfrac{\mathrm{e}^b - b - 1}{\mathrm{e}^b}$ ***AG***

***[6 marks]***

(c) (i) $\lim_{b\to\infty} \dfrac{\mathrm{e}^b - b - 1}{\mathrm{e}^b} = \lim_{b\to\infty} \dfrac{\mathrm{e}^b - 1}{\mathrm{e}^b}$ ***A1***

| **Note:** Award ***A1*** for correct quotient. Condone absence of limit. |
|---|

$\left( = \lim_{b\to\infty} \dfrac{\mathrm{e}^b}{\mathrm{e}^b} \right) = 1$ ***A1***

***[2 marks]***

(ii) $\left( \int_0^\infty x\,\mathrm{e}^{-x}\,\mathrm{d}x = \right) 1$ ***A1***

***[1 mark]***

(d) (i) correct integral *(M1)*

| **Note:** Award ***M1*** for correct integrand with limits from 0 to a larger number. |
| --- |

24 ***A1***

***[2 marks]***

(ii) 120 ***A1***

| **Note:** The ***M1*** can be awarded if either part (d)(i) or part (d)(ii) is correct. |
| --- |

***[1 mark]***

(e) $A_n = n!$ ***A1***

***[1 mark]***

(f)

| |
|---|
| **Note:** Accept starting at $n = 0$, throughout this proof. |

$n = 1$

$A_1 = 1$ and $1! = 1$ ***M1A1***

| |
|---|
| **Note:** Award ***M1*** for considering the case where $n = 1$, and ***A1*** if it is clear that both $A_1 = 1$ and $1! = 1$ have been considered. |

so true for $n = 1$

assume true for $n = k$, ( $A_k = \int_0^\infty x^k e^{-x}\, dx = k!$) ***M1***

| |
|---|
| **Note:** Award ***M0*** for statements such as "let $n = k$". <br> **Note:** Subsequent marks after this ***M1*** are independent of this mark and can be awarded. |

when $n = k + 1$

attempt to integrate by parts ***M1***

| |
|---|
| **Note:** To obtain the ***M1***, a minimum of an expression +/- an integral must be seen. |

$\int_0^\infty x^{k+1} e^{-x}\, dx = \left[-x^{k+1} e^{-x}\right]_0^\infty + (k+1)\int_0^\infty x^k e^{-x}\, dx$ ***A1***

$(k+1)\int_0^\infty x^k e^{-x}\, dx$ simplified to $(k+1)k!$ seen ***A1***

$= 0 + (k+1)k!$

| |
|---|
| **Note:** Condone omission of the zero. |

$= (k+1)!$ ***A1***

Hence if true for $n = k$ then also true for $n = k + 1$. As true for $n = 1$ so true for $n \in \mathbb{Z}^+$. ***R1***

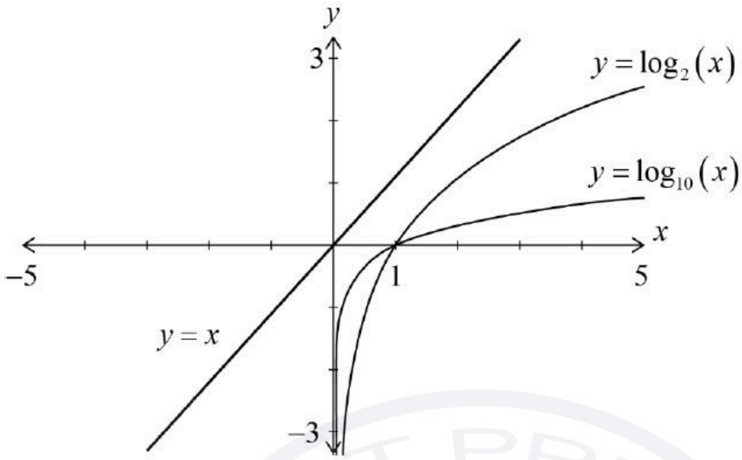
| |
|---|
| **Note:** Award the final ***R1*** mark provided at least four of the previous marks are gained. |

***[8 marks]***

***Total [25 marks]***

**Question 8**

(a)




clearly labelled graphs of $y = \log_2 x$ and $y = \log_{10} x$ with correct domain, asymptotic behaviour and concavity evident **A1**

correct relative positions of the two log graphs both above and below the $x$- axis **A1**

$(1,0)$ indicated (coordinates not required) **A1**

correct graph of $y = x$ **A1**

***[4 marks]***

(b) $\frac{d}{dx}(x-\ln x)$

$=1-\frac{1}{x}$ ***A1***

attempts to solve their $\frac{dy}{dx}=0$ for $x$ ***(M1)***

$1-\frac{1}{x}=0 \Rightarrow x=1$

(when $x=1$,) $x-\ln x=1$ ***A1***

**EITHER**

$\frac{d}{dx}\left(1-\frac{1}{x}\right)$

$=\frac{1}{x^2}$ ***A1***

$\frac{1}{x^2}>0$ (when $x=1$) ***R1***

hence $x-\ln x$ has a minimum value of $1$

**Note:** Award ***R1*** for either '$1>0$' or a graph of $y=\frac{1}{x^2}>0$ or 'the graph of $y=x-\ln x$ is concave-up'. Do not award ***R1*** if the second derivative is incorrect.

**OR**

for $(0<)x<1$, $1-\frac{1}{x}<0$ ***R1***

for $x>1$, $1-\frac{1}{x}>0$ ***R1***

hence $x-\ln x$ has a minimum value of $1$

**Note:** Award ***R1R1*** for either a clearly labelled sign diagram/table (accept correct numerical values) or the graph of $y=1-\frac{1}{x}$ with sign change in gradient indicated.

**Note:** Award a maximum of ***A0(M1)A1A0R1*** or ***A0(M1)A1R0R1*** if no symbolic derivatives are seen.

***[5 marks]***

(c)

**EITHER**

$x-\ln x\geq 1$ $\left(x\in\mathbb{R}^{+}\right)$ ***R1***

**OR**

$x-\ln x>0$ $\left(x\in\mathbb{R}^{+}\right)$ ***R1***

**THEN**

so $x>\ln x$ ***AG***

***[1 mark]***

(d)

| Interval | Number of intersection points |
|---|---|
| $0 < a < 1$ | $p = 1$ |
| $1 < a < 1.4$ | $q = 2$ |
| $1.5 < a < 2$ | $r = 0$ |

***A1A2A1***

**Note:** Award ***A1*** for $p = 1$, ***A2*** for $q = 2$ and ***A1*** for $r = 0$.

***[4 marks]***

(e) **METHOD 1**

**EITHER**

$y = \log_a x$

$\frac{dy}{dx} = \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $\frac{1}{x \ln a} = 1$ for $x$ ***(M1)***

**OR**

$y = x - \log_a x$

$\frac{dy}{dx} = 1 - \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $1 - \frac{1}{x \ln a} = 0$ for $x$ ***(M1)***

**THEN**

$x = \frac{1}{\ln a}$ OR $x \ln a = 1$ OR $\ln a = \frac{1}{x}$ OR $\ln a^x = 1$ OR $\frac{1}{a^x \ln a} = 1$ ***A1***

at $x = \frac{1}{\ln a}$, $\log_a x = x$

attempts to solve $\frac{\ln x}{\ln a} = \frac{1}{\ln a}$ OR $\ln x = 1$ OR $\left(e^{\frac{1}{x}}\right)^x = x$ for $x$ ***(M1)***

$x = e$

coordinates of P are $(e, e)$ (accept $x = e$, $y = e$) ***A1A1***

attempts to solve $\frac{1}{\ln a} = e$ OR $\log_a e = e$ for $a$ analytically ***(M1)***

$\ln a = \frac{1}{e}$ OR $a^e = e$

$a = e^{\frac{1}{e}}$ ***A1***

**METHOD 2**

**EITHER**

$y = \log_a x$

$\frac{dy}{dx} = \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $\frac{1}{x \ln a} = 1$ for $x$ ***(M1)***

**OR**

$y = x - \log_a x$

$\frac{dy}{dx} = 1 - \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $1 - \frac{1}{x \ln a} = 0$ for $x$ ***(M1)***

**THEN**

$x = \frac{1}{\ln a}$ OR $x \ln a = 1$ OR $\ln a = \frac{1}{x}$ OR $\ln a^x = 1$ OR $\frac{1}{a^x \ln a} = 1$ ***A1***

at $x = \frac{1}{\ln a}$, $\log_a x = x$

attempts to solve $\log_a\left(\frac{1}{\ln a}\right) = \frac{1}{\ln a}$ for $a$ ***(M1)***

**EITHER**

$$\frac{\ln\left(\frac{1}{\ln a}\right)}{\ln a} = \frac{1}{\ln a} \Rightarrow \ln\left(\frac{1}{\ln a}\right) = 1$$

**OR**

for example, writes $a^{\log_a\left(\frac{1}{\ln a}\right)} = a^{\frac{1}{\ln a}}$ and then attempts to apply appropriate index/log laws to both sides: $\ln a = \frac{\log_a a}{\log_a e}$ and so $\frac{1}{\ln a} = \log_a e$

$$a^{\frac{1}{\ln a}} = a^{\log_a e} = e$$

**THEN**

attempts to solve $\frac{1}{\ln a} = \mathrm{e}$ OR $\log_a \mathrm{e} = \mathrm{e}$ for $a$ analytically ***(M1)***

$\ln a = \frac{1}{\mathrm{e}}$ OR $a^{\mathrm{e}} = \mathrm{e}$

$a = \mathrm{e}^{\frac{1}{\mathrm{e}}}$ ***A1***

$x = \frac{1}{\ln \mathrm{e}^{\frac{1}{\mathrm{e}}}} = \frac{1}{\frac{1}{\mathrm{e}}}$

coordinates of P are $(\mathrm{e}, \mathrm{e})$ (accept $x = \mathrm{e}$, $y = \mathrm{e}$) ***A1A1***

**METHOD 3**

$y = \log_a x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln a}$ ***(A1)***

(equation of the tangent at $(x_1, y_1)$ is) $y = \frac{1}{x_1 \ln a}(x - x_1) + \frac{\ln x_1}{\ln a}$ (or equivalent) ***A***

compares this equation with $y = x$ and attempts to form at least one of the following ***M***

$\frac{1}{x_1 \ln a} = 1$ OR $\frac{\ln x_1 - 1}{\ln a} = 0$

attempts to solve $\frac{1}{x_1 \ln a} = 1$ OR $\frac{\ln x_1 - 1}{\ln a} = 0$ for $x_1$ ***(M1)***

$x_1 = \mathrm{e}$

coordinates of P are $(\mathrm{e}, \mathrm{e})$ (accept $x = \mathrm{e}$, $y = \mathrm{e}$) ***A1A1***

attempts to solve $\frac{1}{\mathrm{e} \ln a} = 1$ (or equivalent) for $a$ analytically ***(M1)***

$\ln a = \frac{1}{\mathrm{e}}$ **OR** $a^{\mathrm{e}} = \mathrm{e}$

$a = \mathrm{e}^{\frac{1}{\mathrm{e}}}$ ***A1***

***[8 marks]***

(f) (i) $1 < a < e^{\frac{1}{e}}$ ***A1***

**Note:** Award ***A0*** for $a < e^{\frac{1}{e}}$.

***[1 mark]***

(ii) $a > e^{\frac{1}{e}}$ ***A1***

**Note:** Only award ***FT*** for $1.4 < a < 1.5$. If the value of $a$ is not exact, e.g. $1.44$, award at most ***A0A1*** in part (f) for a consistent approximate endpoint value.
If a value of $a$ is not found in part (e), award at most ***A0A1*** in part (f) for a consistent approximate endpoint value provided that $1.4 < a < 1.5$.

***[1 mark]***

***Total [24 marks]***

**Question 8**

(a) (i) $\frac{y}{x}$ ***A1***

***[1 mark]***

(ii) $\frac{dy}{dx} = -\frac{1}{\left(\frac{y}{x}\right)} \left(= -\frac{1}{m}\right)$ ***A1***

**Note:** Award ***A1*** for responses such as 'the gradient is the negative (opposite) reciprocal of $\frac{y}{x}$ or $\frac{y}{x} \times m = -1$ (or equivalent).
Award ***A1*** for $\frac{y}{x} \times \left(-\frac{x}{y}\right) = -1$.
Do not award ***FT*** from part (a) (i).

so $\frac{dy}{dx} = -\frac{x}{y}$ ***AG***

***[1 mark]***

(iii) attempts to separate variables $x$ and $y$ ***(M1)***

$$\int y \, dy = -\int x \, dx$$

**Note:** Award ***(M1)*** for $y \, dy = -x \, dx$.

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$ ***A1***

**Note:** Award ***A1*** for $\frac{y^2}{2} + c_1 = -\frac{x^2}{2} + c_2$.

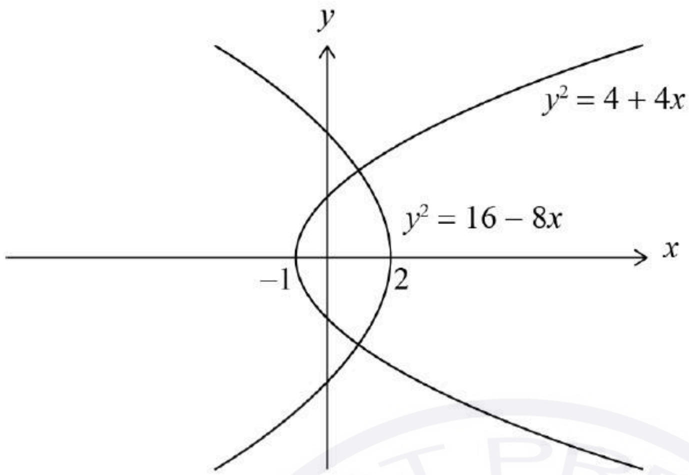
Award ***A0*** for $\frac{y^2}{2} = -\frac{x^2}{2}$.

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

$\Rightarrow x^2 + y^2 = k$ (where $k = 2c$) ***AG***

***[2 marks]***

(b)




two parabolic shaped curves with approximately correct shape/position
(e.g. two intersection points, in first and fourth quadrant) ***A1A1***

$x$-intercepts $-1$ and $2$ ***A1***

***[3 marks]***

(c) at intersection, $4a^2-4ax=4b^2+4bx$

$4a^2-4b^2=4ax+4bx$ ($a^2-b^2=ax+bx,\ 4a^2-4b^2-4ax-4bx=0$) ***A1***

attempts to factorize either the LHS or the RHS of the first two equations above (or equivalent) OR attempts to partially factorize the LHS side of $a^2-b^2-ax-bx=0$ (or equivalent) ***(M1)***

$(a+b)(a-b)=(a+b)x$

**Note:** Accept alternative forms such as $4(a+b)(a-b)=4(a+b)x$ or $(a+b)((a-b)-x)=0$.

recognition that $a+b>0$ (or equivalent, eg. $a>0, b>0$) (allows division by $a+b$) ***R1***

**Note:** Subsequent marks are not dependent on this ***R1***.

$x=a-b$ ***A1***

**Note:** As $x=a-b$ is an ***AG***, only award the above ***A1*** if $a^2-b^2=(a+b)(a-b)$ has been used.

substitutes $x=a-b$ into either $y^2=4a^2-4ax$ or $y^2=4b^2+4bx$ and attempts to simplify ***(M1)***

$y^2=4a^2-4a(a-b)$ OR $y^2=4b^2+4b(a-b)$

$y^2=4a^2-4a^2+4ab \Rightarrow y=\pm 2\sqrt{ab}$ ***A1***

so $\mathrm{M}\left(a-b,\ 2\sqrt{ab}\right)$ and $\mathrm{N}\left(a-b,\ -2\sqrt{ab}\right)$ ***AG***

***[6 marks]***

(d) **METHOD 1**

attempts implicit differentiation on either curve ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4a}{2y}$ (or equivalent) and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4b}{2y}$ (or equivalent) ***A1***

substitutes $y = 2\sqrt{ab}$ into either $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4a}{2y}$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4b}{2y}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{\frac{a}{b}}$ $\left(= -\frac{a}{\sqrt{ab}}\right)$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{b}{a}}$ $\left(= \frac{b}{\sqrt{ab}}\right)$ (or equivalent) ***A1***

**EITHER**

$-\sqrt{\frac{a}{b}} \times \sqrt{\frac{b}{a}} = -1$ OR $-\frac{a}{\sqrt{ab}} \times \frac{b}{\sqrt{ab}} = -1$ (or equivalent) ***A1***

**OR**

eg. the negative (opposite) reciprocal of $-\sqrt{\frac{a}{b}}$ is $\sqrt{\frac{b}{a}}$ (or equivalent) ***A1***

**OR**

the product of the two gradients is $-1$ ***A1***

**THEN**

so at point $\mathrm{M}$, the curves intersect at right angles ***AG***

**METHOD 2**

attempts chain rule differentiation on either $y=\sqrt{4a^2-4ax}$ or $y=\sqrt{4b^2+4bx}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2a}{\sqrt{4a^2-4ax}}$ (or equivalent) and $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2b}{\sqrt{4b^2+4bx}}$ (or equivalent) ***A1***

substitutes $x=a-b$ into either $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2a}{\sqrt{4a^2-4ax}}$ or $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2b}{\sqrt{4b^2+4bx}}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\sqrt{\frac{a}{b}}$ $\left(=-\frac{a}{\sqrt{ab}}\right)$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{\frac{b}{a}}$ $\left(=\frac{b}{\sqrt{ab}}\right)$ (or equivalent) ***A1***

**EITHER**

$-\sqrt{\frac{a}{b}}\times\sqrt{\frac{b}{a}}=-1$ OR $-\frac{a}{\sqrt{ab}}\times\frac{b}{\sqrt{ab}}=-1$ (or equivalent) ***A1***

**OR**

eg. the negative reciprocal of $-\sqrt{\frac{a}{b}}$ is $\sqrt{\frac{b}{a}}$ (or equivalent) ***A1***

**OR**

the product of the two gradients is $-1$ ***A1***

**THEN**

so at point $\mathrm{M}$, the curves intersect at right angles ***AG***

***[5 marks]***

(e) (i) $$g(x, y) = \frac{-\frac{x}{y} + \tan\frac{\pi}{4}}{1 - \left(-\frac{x}{y}\right)\tan\frac{\pi}{4}}$$ ***(A1)***

$$g(x, y) = \frac{-\frac{x}{y} + 1}{1 + \frac{x}{y}} \left( = \frac{\frac{-x+y}{y}}{\frac{y+x}{y}} \right)$$ ***A1***

so $g(x, y) = \dfrac{y-x}{y+x}$ ***AG***

***[2 marks]***

(ii) let $y = vx$ ***(M1)***

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$ ***(A1)***

$$\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \right)\frac{vx - x}{vx + x} \left(= \frac{v-1}{v+1}\right)$$ ***(A1)***

attempts to express $x\frac{\mathrm{d}v}{\mathrm{d}x}$ as a single fraction in $v$ ***(M1)***

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{v^2+1}{v+1}$$ (or equivalent) ***(A1)***

attempts to separate variables $x$ and $v$ ***(M1)***

$$\int \frac{v+1}{v^2+1}\,\mathrm{d}v = -\int \frac{1}{x}\,\mathrm{d}x$$ (or equivalent)

$$\frac{1}{2}\ln\left(v^2+1\right) + \arctan v = -\ln|x| (+d)$$ (or equivalent) ***A1A1***

$$\frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right) + \arctan\frac{y}{x} + \ln|x| = d$$ (or equivalent) ***A1***

***[9 marks]***

(f) **METHOD 1**

$$g(x,\ y)=\frac{\dfrac{1}{\tan\alpha}f\left(x,y\right)+1}{\dfrac{1}{\tan\alpha}-f\left(x,y\right)}$$ ***M1***

**EITHER**

as $\alpha\to\frac{\pi}{2}$, $\frac{1}{\tan\alpha}\to 0$, (hence $g(x,\ y)\to -\frac{1}{f(x,\ y)}$) ***R1***

**OR**

as $\alpha\to\frac{\pi}{2}$, $\tan\alpha\to\infty$ and so $g(x,\ y)\to\frac{0\times f\left(x,y\right)+1}{0-f\left(x,y\right)}$ ***R1***

**THEN**

$$\lim_{\alpha\to\frac{\pi}{2}} g(x,\ y)=-\frac{1}{f(x,\ y)}$$ ***AG***

| |
|---|
| **Note:** The ***R1*** is dependent on the ***M1***. |

**METHOD 2**

uses either $\tan\alpha=\frac{\sin\alpha}{\cos\alpha}$ or $\frac{1}{\tan\alpha}=\frac{\cos\alpha}{\sin\alpha}$ to form $g(x,\ y)=\frac{\cos\alpha f\left(x,y\right)+\sin\alpha}{\cos\alpha-\sin\alpha f\left(x,y\right)}$ ***M1***

as $\alpha\to\frac{\pi}{2}$, $\cos\alpha\to 0$ and $\sin\alpha\to 1$ and so $g(x,\ y)\to\frac{0\times f\left(x,y\right)+1}{0-f\left(x,y\right)}$ ***R1***

$$\lim_{\alpha\to\frac{\pi}{2}} g(x,\ y)=-\frac{1}{f(x,\ y)}$$ ***AG***

| |
|---|
| **Note:** The ***R1*** is dependent on the ***M1***. |

***[2 marks]***

***Total [31 marks]***

## Question 9

(a) (i) attempts chain rule differentiation to find $f'(x)$ ***(M1)***

$$f'(x) = \frac{2}{(2-x)^3} \left(= (-1)(-2)(2-x)^{-3}\right)$$ ***A1***

**Note:** Award ***(M1)*** for attempting chain rule differentiation on $\left(4-4x+x^2\right)^{-1}$ or attempting quotient rule differentiation on $\frac{1}{\left(4-4x+x^2\right)} \left(= \frac{1}{(2-x)^2}\right)$.

Award ***A1*** for $f'(x) = \frac{2}{(2-x)^3} \left(= (-1)(-2)(2-x)^{-3}\right)$.

***[2 marks]***

(ii) $g'(x) = 2x$ ***(A1)***

$$f'(x)g'(x) = \left(2(2-x)^{-3}\right)(2x) \left(= \frac{2(2x)}{(2-x)^3}\right)$$ (or equivalent) ***A1***

$$= \frac{4x}{(2-x)^3}$$ ***AG***

***[2 marks]***

(iii)

| |
|---|
| **Note:** Award a maximum of ***(M1)A1(M1)A0FT*** from parts (a) (i) and (ii). |

substitutes $f(x), g(x)$ and their $g'(x), f'(x)$ into the given expression ***(M1)***

**EITHER**

$f(x)g'(x)+g(x)f'(x)=2x(2-x)^{-2}+2x^2(2-x)^{-3}$ ***A1***

| |
|---|
| **Note:** Award ***A1*** if $f(x)g'(x)=2x(2-x)^{-2}$ and $g(x)f'(x)=2x^2(2-x)^{-3}$ are stated separately. |

attempts to factorise their expression ***(M1)***

$=2x(2-x)^{-3}((2-x)+x)$ ***A1***

**OR**

$f(x)g'(x)+g(x)f'(x)=\frac{2x}{(2-x)^2}+\frac{2x^2}{(2-x)^3}$ ***A1***

| |
|---|
| **Note:** Award ***A1*** if $f(x)g'(x)=\frac{2x}{(2-x)^2}$ and $g(x)f'(x)=\frac{2x^2}{(2-x)^3}$ are stated separately. |

attempts to form an expression with a common denominator ***(M1)***

| |
|---|
| **Note:** Award ***(M1)*** for $(2-x)^2(2-x)^3$ as a common denominator. |

$=\frac{2x(2-x)}{(2-x)^3}+\frac{2x^2}{(2-x)^3}\ \left(=\frac{4x-2x^2+2x^2}{(2-x)^3}\right)$ ***A1***

**THEN**

$=\frac{4x}{(2-x)^3}$

***AG***

| |
|---|
| **Note:** Award marks as appropriate for attempting to find the derivative of $f(x)g(x)=\frac{x^2}{(2-x)^2}$ (or equivalent). |

***[4 marks]***

(b) **METHOD 1**

$$f'(x)g'(x) - g(x)f'(x) = f(x)g'(x)$$ ***(A1)***

$$\left(f'(x)g'(x) - g(x)f'(x) - f(x)g'(x) = 0\right)$$

$$f'(x)\left(g'(x) - g(x)\right) = f(x)g'(x) \quad \left(f'(x)\left(g(x) - g'(x)\right) = -f(x)g'(x)\right)$$ ***A1***

$$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$$ ***AG***

| |
|---|
| **Note:** Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a). |

**METHOD 2**

$$g'(x) = \frac{f'(x)g'(x)}{f(x)} - \frac{g(x)f'(x)}{f(x)}$$ ***(A1)***

$$g'(x) = \frac{f'(x)}{f(x)}\left(g'(x) - g(x)\right)$$ ***A1***

$$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$$ ***AG***

| |
|---|
| **Note:** Candidates may not show the steps exactly as shown above.<br>Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a). |

**METHOD 3**

$g'(x) = \dfrac{f(x)g'(x)}{f'(x)} + g(x)$ ***(A1)***

$\dfrac{f(x)}{f'(x)} = \dfrac{g'(x) - g(x)}{g'(x)}$ ***A1***

$\dfrac{f'(x)}{f(x)} = \dfrac{g'(x)}{g'(x) - g(x)}$ ***AG***

**Note:** Candidates may not show the steps exactly as shown above.
Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a).

**METHOD 4**

$\dfrac{f(x)}{f'(x)} + \dfrac{g(x)}{g'(x)} = 1$ ***(A1)***

$\dfrac{f(x)}{f'(x)} = \dfrac{g'(x) - g(x)}{g'(x)}$ ***A1***

$\dfrac{f'(x)}{f(x)} = \dfrac{g'(x)}{g'(x) - g(x)}$ ***AG***

**Note:** Candidates may not show the steps exactly as shown above.
Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a).

***[2 marks]***

(c) **METHOD 1**

**Note:** Condone the absence of '$\mathrm{d}x$' and the modulus sign throughout. Only award the second ***A*** mark if the constant of integration has been dealt with correctly.

**EITHER**

$$\ln f(x) = \int \frac{g'(x)}{g'(x)-g(x)}\,\mathrm{d}x\ (+C)$$ ***A1***

**Note:** Condone the absence of '$+C$' when awarding the first ***A*** mark.

$$f(x) = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}\mathrm{e}^{C} \quad \left(f(x) = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x+C\right)}\right)$$ ***A1***

**Note:** Award ***A1*** for $f(x) = A\mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}$, where $A = \mathrm{e}^{C}$.

**OR**

$$\ln f(x)(+C) = \int \frac{g'(x)}{g'(x)-g(x)}\,\mathrm{d}x$$ ***A1***

**Note:** Condone the absence of '$+C$' when awarding the first ***A*** mark.

$$f(x) = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}\mathrm{e}^{-C} \quad \left(f(x)\mathrm{e}^{C} = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}, f(x) = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x-C\right)}\right)$$ ***A1***

**Note:** Award ***A1*** for $f(x) = A\mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}$, where $A = \mathrm{e}^{-C}$.

**THEN**

$$f(x) = A\mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}$$ ***AG***

**METHOD 2**

**Note:** Condone the absence of '$\mathrm{d}x$' throughout.

$$f'(x)-\frac{g'(x)}{g'(x)-g(x)}f(x)=0$$

Integrating factor: $\mathrm{e}^{\left(-\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}$ ***A1***

**Note:** Award ***A1*** for $f(x)\mathrm{e}^{\left(-\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}=C$.

$$\frac{d}{dx}\left[f(x)\mathrm{e}^{\left(-\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}\right]=0\Rightarrow f(x)\mathrm{e}^{\left(-\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}=A$$ ***A1***

**Note:** Award ***A1*** for $f(x)\mathrm{e}^{\left(-\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}=C$.

$$f(x)=A\mathrm{e}^{\left(\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}$$ ***AG***

***[2 marks]***

(d) $g'(x) = xe^x + e^x$ (seen anywhere) ***(A1)***

attempts to find an expression for $\dfrac{g'(x)}{g'(x) - g(x)}$ ***(M1)***

$= \dfrac{xe^x + e^x}{e^x} \left( = \dfrac{e^x(x+1)}{e^x} \right)$

$= x + 1$ (as $e^x \neq 0$) ***(A1)***

attempts to integrate their $\dfrac{g'(x)}{g'(x) - g(x)}$ ***(M1)***

$\int (x+1)\,dx = \dfrac{1}{2}x^2 + x(+C)$

$f(x) = e^{\left(\frac{1}{2}x^2 + x\right)}$ ***A1***

**Note:** Award ***A0*** for $f(x) = e^{\left(\frac{1}{2}x^2 + x + C\right)}$ (or equivalent expressed with an arbitrary constant).

***[5 marks]***

(e) $g'(x) = \cos x - \sin x$ (seen anywhere) ***(A1)***

attempts to find an expression for $\dfrac{g'(x)}{g'(x) - g(x)}$ ***(M1)***

$$= \frac{\cos x - \sin x}{\cos x - \sin x - \sin x - \cos x}\left(= \frac{\sin x - \cos x}{2\sin x}\right)$$

$= \dfrac{1}{2} - \dfrac{1}{2}\cot x$ (as $\sin x \neq 0$) OR $= \dfrac{1}{2} - \dfrac{1}{2}\dfrac{\cos x}{\sin x}$ (as $\sin x \neq 0$) ***A1***

$$f(x) = \mathrm{e}^{\int\left(\frac{1}{2} - \frac{1}{2}\cot x\right)\mathrm{d}x}$$

attempts to find the indefinite integral of $(\pm k)\cot x$ OR $(\pm k)\dfrac{\cos x}{\sin x}$ ***(M1)***

**Note:** As $|\sin x| = \sin x$ for $0 < x < \pi$, condone the presence or omission of the modulus sign throughout a candidate's solution.
Condone the presence of an arbitrary constant except when awarding the final ***A*** mark.

$\int\left(\dfrac{1}{2} - \dfrac{1}{2}\cot x\right)\mathrm{d}x = \dfrac{x}{2} - \dfrac{1}{2}\ln|\sin x|(+C)\ \left(= \dfrac{1}{2}\left(x - \ln|\sin x|(+C)\right)\right)$ ***A1***

$$f(x) = \mathrm{e}^{\frac{x}{2}}\mathrm{e}^{-\frac{1}{2}\ln|\sin x|}\left(\mathrm{e}^{C}\right)$$

$= \mathrm{e}^{\frac{x}{2}}\mathrm{e}^{\ln\sqrt{\frac{1}{\sin x}}}\left(\mathrm{e}^{C}\right)\ \left(= \mathrm{e}^{\frac{x}{2}}\mathrm{e}^{\frac{1}{2}\ln\left(\frac{1}{\sin x}\right)}\left(\mathrm{e}^{C}\right), = \sqrt{\mathrm{e}^{x - \ln(\sin x)}}\left(\mathrm{e}^{C}\right)\right)$ ***A1***

$$= \mathrm{e}^{\frac{x}{2}}\sqrt{\frac{1}{\sin x}}$$

$= \sqrt{\mathrm{e}^{x}\operatorname{cosec} x}\ \left(= \sqrt{\dfrac{\mathrm{e}^{x}}{\sin x}}\right)$ (where $h(x) = \dfrac{1}{\sin x}$) ***A1***

***[7 marks]***

***Total [24 marks]***

## Question 10

2. (a) $\frac{dy}{dx} = 4x^3 - 9x^2 + 3$ **(M1)(A1)**

**Note:** Award ***M1*** for at least two correct terms.

$\frac{d^2y}{dx^2} = 12x^2 - 18x$ ***A1***

***[3 marks]***

(b) valid attempt to find $x$-coordinates (e.g. solving $12x^2 - 18x = 0$ or graphing $\frac{dy}{dx} = 4x^3 - 9x^2 + 3$ **(M1)**

$x = 0, 1.5 \left(\frac{3}{2}\right)$ **(A1)**

point B is $(0, 0)$ ***A1***

point C is $(1.5, -0.5625)$ $\left((1.5, -0.563), \left(\frac{3}{2}, -\frac{9}{16}\right)\right)$ ***A1***

**Note:** Award ***M0A0A1A0*** for an unsupported answer of point B is $(0, 0)$.

***[4 marks]***

(c) $y$-intercept $= 0$ (as line passes through $(0, 0)$) ***R1***

**Note:** Award ***R1*** for correctly substituting point B or point C to show that $y = 0$.

Award ***R1*** for $y = -0.375x + 0$.

gradient $= \frac{-0.5625}{1.5}$ $\left(= \frac{-\frac{9}{16}}{\frac{3}{2}}\right)$ ***A1***

**Note:** Award ***A0FT*** if their answer to (b) doesn't lead to the given answer, but condone $\frac{-0.563}{1.5}$.

$= -0.375$

so equation is $y = -0.375x$ ***AG***

**Note:** Award at most ***A1R0*** for working backwards to verify points B and C lie on the given line.

***[2 marks]***

(d) $x^4 - 3x^3 + 3x = -0.375x$ **(M1)**

$2.427$ ***A1***

**Note:** Correct answer must be given to three decimal places for the ***A1*** to be awarded.

Award ***(M1)A1*** for $(2.427, -0.910)$.

Award ***(M1)A0*** for an unsupported answer of $2.43$.

***[2 marks]***

(e) $\frac{dy}{dx} = 4x^3 - 3mx^2 + n$ (***A1***)

$\frac{d^2y}{dx^2} = 12x^2 - 6mx$ (***A1***)

$6x(2x - m) = 0$

$x = 0, \frac{m}{2}$ ***A1***

**Note:** Accept $0m$ in place of $0$.

***[3 marks]***

(f) (i) point B is $(0, 0)$ ***A1***

***[1 mark]***

(ii) $y_c = \left(\frac{m}{2}\right)^4 - m\left(\frac{m}{2}\right)^3 + n\left(\frac{m}{2}\right)$ $\left(= \frac{m^4}{16} - \frac{m^4}{8} + \frac{nm}{2}\right)$ ***(A1)***

$= -\frac{m^4}{16} + \frac{nm}{2}$ OR $\frac{8nm - m^4}{16}$ OR $\frac{m}{2}\left(n - \frac{m^3}{8}\right)$ OR equivalent simplification ***A1***

Point C is $\left(\frac{m}{2}, -\frac{m^4}{16} + \frac{nm}{2}\right)$

**Note:** Award the second ***A1*** if the simplified y-coordinate is seen here or in part (g).

***[2 marks]***

(g) attempt to divide their $y_C$ by their $x_C$ OR substitute into $\frac{y_2 - y_1}{x_2 - x_1}$ ***(M1)***

e.g. $\left(-\frac{m^4}{16}+\frac{nm}{2}\right)\div\frac{m}{2}$

$\left(-\frac{m^4}{16}+\frac{nm}{2}\right)\times\frac{2}{m}$ **OR** equivalent manipulation leading to given answer ***A1***

**Note:** Award ***A0 FT*** if their answer does not lead to the ***AG***.

$= -\frac{m^3}{8}+n$ and $y$-intercept $= 0$,

so equation is $y=\left(-\frac{m^3}{8}+n\right)x$ ***AG***

**Note:** Award at most ***M1A0*** for working backwards to verify points B and C lie on the given line.

***[2 marks]***

(h) **METHOD 1**

$x^4 - mx^3 + nx = -\frac{m^3}{8}x + nx$

attempt to rearrange this equation to equal zero ***(M1)***

$x^4 - mx^3 + \frac{m^3}{8}x = 0$

recognizing their $(x - x_C)$ is a factor of this expression ***(M1)***

$x\left(x-\frac{m}{2}\right)\left(x^2-\frac{m}{2}x-\frac{m^2}{4}\right)$ $(=0)$ OR equivalent ***A1M1A1***

**Note:** Award ***A1*** for $x\left(x-\frac{m}{2}\right)$, ***M1A1*** for $\left(x^2-\frac{m}{2}x-\frac{m^2}{4}\right)$.

If a candidate divides by $x$ without justification, do not award the first ***A1***, but all subsequent marks can still be awarded.

use of quadratic formula to find roots, $x_A$ (and $x_D$), of the quadratic **M1**

$$x_A = \frac{\frac{m}{2} - \sqrt{\frac{m^2}{4} + m^2}}{2}$$ OR equivalent **A1**

**Note:** Condone a $\pm$ in place of the minus sign, provided given answer is restated.

$$x_A = \frac{m}{4} - \frac{m}{4}\sqrt{5}$$ **AG**

**Note:** Award ***(M1)(M0)A0M0A0M0A0*** for attempting to verify the given answer satisfies

$$x^4 - mx^3 + nx = -\frac{m^3}{8}x + nx.$$

**METHOD 2**

$$x^4 - mx^3 + nx = -\frac{m^3}{8}x + nx$$

attempt to rearrange this equation to equal zero ***(M1)***

$$x^4 - mx^3 + \frac{m^3}{8}x = 0$$

$$x\left(x^3 - mx^2 + \frac{m^3}{8}\right) = 0$$

Recognise that their $x_C$ is a root of this equation ***(M1)***

Attempt to find sum and product of roots ***(M1)***

$$x_A + x_D + \frac{m}{2} = m \quad \text{AND} \quad x_A \times x_D \times \frac{m}{2} = -\frac{m^3}{8}$$ **A1**

$$x_A \times \left(\frac{m}{2} - x_A\right) \times \frac{m}{2} = -\frac{m^3}{8}$$

$$x_A{}^2 - \frac{m}{2}x_A - \frac{m^2}{4} = 0$$ **A1**

use of quadratic formula to find roots, $x_A$ (and $x_D$), of the quadratic **M1**

$$x_A = \frac{\frac{m}{2} - \sqrt{\frac{m^2}{4} + m^2}}{2}$$ OR equivalent ***A1***

**Note:** Condone a ± in place of the minus sign, provided given answer is restated.

$$x_A = \frac{m}{4} - \frac{m}{4}\sqrt{5}$$ ***AG***

***[7 marks]***

(i) $$\frac{0 - \left(\frac{m}{4} - \frac{m}{4}\sqrt{5}\right)}{\frac{m}{2} - 0}$$ ***(A1)***

$$= \frac{\sqrt{5} - 1}{2}$$ ***A1***

**Note:** Answer must be exact.

***[2 marks]***

***Total [28 marks]***