

Subject - Math AA(Higher Level)
Topic - Functions
Year - May 2021 - Nov 2024
Paper -3
Questions

Question 1

[Maximum mark: 25]

In this question you will explore some of the properties of special functions f and g and their relationship with the trigonometric functions, sine and cosine.

Functions f and g are defined as $f(z) = \frac{e^z + e^{-z}}{2}$ and $g(z) = \frac{e^z - e^{-z}}{2}$, where $z \in \mathbb{C}$.

Consider t and u , such that $t, u \in \mathbb{R}$.

- (a) Verify that $u = f(t)$ satisfies the differential equation $\frac{d^2u}{dt^2} = u$. [2]
- (b) Show that $(f(t))^2 + (g(t))^2 = f(2t)$. [3]
- (c) Using $e^{iu} = \cos u + i \sin u$, find expressions, in terms of $\sin u$ and $\cos u$, for
- (i) $f(iu)$; [3]
- (ii) $g(iu)$. [2]
- (d) Hence find, and simplify, an expression for $(f(iu))^2 + (g(iu))^2$. [2]
- (e) Show that $(f(t))^2 - (g(t))^2 = (f(iu))^2 - (g(iu))^2$. [4]

The functions $\cos x$ and $\sin x$ are known as circular functions as the general point $(\cos \theta, \sin \theta)$ defines points on the unit circle with equation $x^2 + y^2 = 1$.

The functions $f(x)$ and $g(x)$ are known as hyperbolic functions, as the general point $(f(\theta), g(\theta))$ defines points on a curve known as a hyperbola with equation $x^2 - y^2 = 1$. This hyperbola has two asymptotes.

- (f) Sketch the graph of $x^2 - y^2 = 1$, stating the coordinates of any axis intercepts and the equation of each asymptote. [4]

The hyperbola with equation $x^2 - y^2 = 1$ can be rotated to coincide with the curve defined by $xy = k$, $k \in \mathbb{R}$.

- (g) Find the possible values of k . [5]

Question 2

[Maximum mark: 30]

In this question, you will investigate the maximum product of positive real numbers with a given sum.

Consider the two numbers $x_1, x_2 \in \mathbb{R}^+$, such that $x_1 + x_2 = 12$.

- (a) Find the product of x_1 and x_2 as a function, f , of x_1 only. [2]
- (b) (i) Find the value of x_1 for which the function is maximum. [1]
- (ii) Hence show that the maximum product of x_1 and x_2 is 36. [1]

Consider $M_n(S)$ to be the maximum product of n positive real numbers with a sum of S , where $n \in \mathbb{Z}^+$ and $S \in \mathbb{R}^+$.

For $n = 2$, the maximum product can be expressed as $M_2(S) = \left(\frac{S}{2}\right)^2$.

- (c) Verify that $M_2(S) = \left(\frac{S}{2}\right)^2$ is true for $S = 12$. [1]

Consider n positive real numbers, x_1, x_2, \dots, x_n .

The geometric mean is defined as $(x_1 \times x_2 \times \dots \times x_n)^{\frac{1}{n}}$. It is given that the geometric mean is always less than or equal to the arithmetic mean, so $(x_1 \times x_2 \times \dots \times x_n)^{\frac{1}{n}} \leq \frac{(x_1 + x_2 + \dots + x_n)}{n}$.

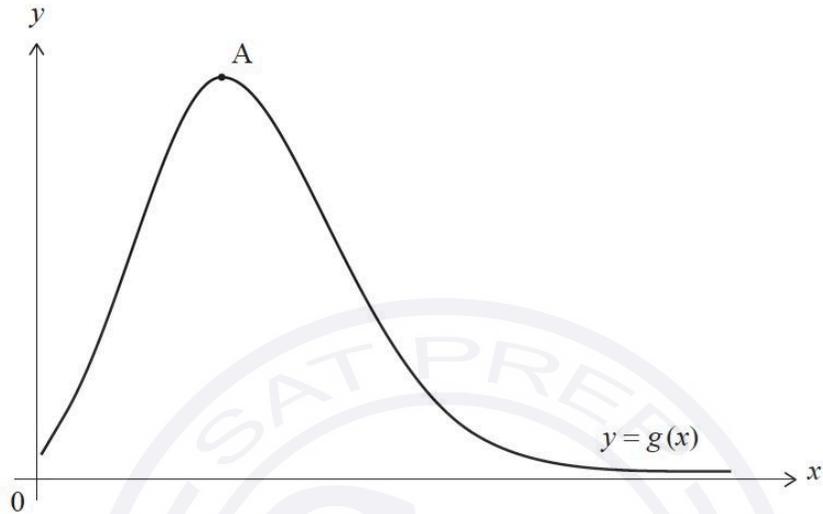
- (d) (i) Show that the geometric mean and arithmetic mean are equal when $x_1 = x_2 = \dots = x_n$. [2]
- (ii) Use this result to prove that $M_n(S) = \left(\frac{S}{n}\right)^n$. [4]
- (e) Hence determine the value of
- (i) $M_3(12)$; [1]
- (ii) $M_4(12)$; [1]
- (iii) $M_5(12)$. [1]

For $n \in \mathbb{Z}^+$, let $P(S)$ denote the maximum value of $M_n(S)$ across all possible values of n .

- (f) Write down the value of $P(12)$ and the value of n at which it occurs. [2]
- (g) Determine the value of $P(20)$ and the value of n at which it occurs. [3]

Consider the function g , defined by $\ln(g(x)) = x \ln\left(\frac{S}{x}\right)$, where $x \in \mathbb{R}^+$.

A sketch of the graph of $y = g(x)$ is shown in the following diagram. Point A is the maximum point on this graph.



- (h) Find, in terms of S , the x -coordinate of point A . [6]
- (i) Verify that $g(x) = M_x(S)$, when $x \in \mathbb{Z}^+$. [2]
- (j) Use your answer to part (h) to find the largest possible product of positive numbers whose sum is 100. Give your answer in the form $a \times 10^k$, where $1 \leq a < 10$ and $k \in \mathbb{Z}^+$. [3]