Subject – Math AA(Higher Level)
Topic - Functions
Year - May 2021 – Nov 2024
Paper -3
Answers

Question 1

(a) $f'(t)=\frac{e^t-e^{-t}}{2}$ *A1*

$f''(t)=\frac{e^t+e^{-t}}{2}$ *A1*

$=f(t)$ *AG*

[2 marks]

(b) **METHOD 1**

$(f(t))^2+(g(t))^2$

substituting f and g *M1*

$=\frac{(e^t+e^{-t})^2+(e^t-e^{-t})^2}{4}$

$=\frac{(e^t)^2+2+(e^{-t})^2+(e^t)^2-2+(e^{-t})^2}{4}$ *(M1)*

$=\frac{(e^t)^2+(e^{-t})^2}{2}\quad\left(=\frac{e^{2t}+e^{-2t}}{2}\right)$ *A1*

$=f(2t)$ *AG*

METHOD 2

$f(2t)=\frac{e^{2t}+e^{-2t}}{2}$

$=\frac{(e^t)^2+(e^{-t})^2}{2}$ *M1*

$=\frac{(e^t+e^{-t})^2+(e^t-e^{-t})^2}{4}$ *M1A1*

$=(f(t))^2+(g(t))^2$ *AG*

[3 marks]

(c) (i) substituting $e^{iu} = \cos u + i\sin u$ into the expression for f *(M1)*

obtaining $e^{-iu} = \cos u - i\sin u$ *(A1)*

$$f(iu) = \frac{\cos u + i\sin u + \cos u - i\sin u}{2}$$

Note: The ***M1*** can be awarded for the use of sine and cosine being odd and even respectively.

$$= \frac{2\cos u}{2}$$

$= \cos u$ ***A1***

[3 marks]

(ii) $$g(iu) = \frac{\cos u + i\sin u - \cos u + i\sin u}{2}$$

substituting and attempt to simplify *(M1)*

$$= \frac{2i\sin u}{2}$$

$= i\sin u$ ***A1***

[2 marks]

(d) **METHOD 1**

$(f(iu))^2 + (g(iu))^2$

substituting expressions found in part (c) *(M1)*

$= \cos^2 u - \sin^2 u \ (= \cos 2u)$ ***A1***

METHOD 2

$$f(2iu) = \frac{e^{2iu} + e^{-2iu}}{2}$$

$$= \frac{\cos 2u + i\sin 2u + \cos 2u - i\sin 2u}{2}$$ ***M1***

$= \cos 2u$ ***A1***

Note: Accept equivalent final answers that have been simplified removing all imaginary parts eg $2\cos^2 u - 1$ etc

[2 marks]

(e) $\left(f(t)\right)^2-\left(g(t)\right)^2=\frac{(e^t+e^{-t})^2-(e^t-e^{-t})^2}{4}$ ***M1***

$=\frac{(e^{2t}+e^{-2t}+2)-(e^{2t}+e^{-2t}-2)}{4}$ ***A1***

$=\frac{4}{4}=1$ ***A1***

Note: Award ***A1*** for a value of 1 obtained from either LHS or RHS of given expression.

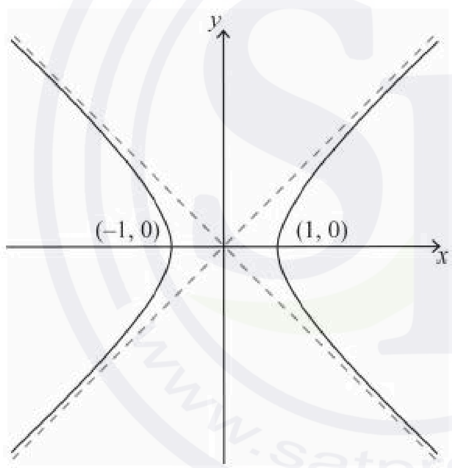
$\left(f(iu)\right)^2-\left(g(iu)\right)^2=\cos^2 u+\sin^2 u$ ***M1***

$=1 \quad \left(\text{hence} \left(f(t)\right)^2-\left(g(t)\right)^2=\left(f(iu)\right)^2-\left(g(iu)\right)^2\right)$ ***AG***

Note: Award full marks for showing that $\left(f(z)\right)^2-\left(g(z)\right)^2=1,\ \forall z\in\mathbb{C}$.

[4 marks]

(f)



A1A1A1A1

Note: Award ***A1*** for correct curves in the upper quadrants, ***A1*** for correct curves in the lower quadrants , ***A1*** for correct x-intercepts of $(-1, 0)$ and $(1, 0)$ (condone $x=-1$ and 1), ***A1*** for $y=x$ and $y=-x$.

[4 marks]

(g) attempt to rotate by $45°$ in either direction ***(M1)***

> **Note:** Evidence of an attempt to relate to a sketch of $xy = k$ would be sufficient for this ***(M1).***

attempting to rotate a particular point, *eg* $(1, 0)$ ***(M1)***

$(1, 0)$ rotates to $\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$ (or similar) ***(A1)***

hence $k = \pm\frac{1}{2}$ ***A1A1***

[5 marks]
Total [25 marks]

Question 2

(a) $x_2 = 12 - x_1$ ***(M1)***

$f(x) = x_1(12 - x_1)$ ***A1***

[2 marks]

(b) (i) $(x_1 =)\ 6$ ***A1***

Note: Award the ***A1*** if 6 seen in part (ii).

[1 mark]

(ii) $f(6)$ **OR** 6^2 **OR** graph with maximum at $(6, 36)$ ***M1***

$= 36$ ***AG***

[1 mark]

(c) $M_2(12) = \left(\frac{12}{2}\right)^2 = 36$ which is the maximum product (from (b)(ii)) ***A1***

Note: Both the 36 **AND** a link to part (b), which may be simply seeing the word "maximum" must be seen to award the ***A1***.

[1 mark]

(d) (i) let all x_i be labelled as x (or x_1 or x_n etc.) ***(M1)***

Note: Do not accept use of a specific number for x or n.

$(x^n)^{\frac{1}{n}} = x$ and $\frac{nx}{n} = x$ ***A1***

[2 marks]

(ii) $x_1 + x_2 + \ldots + x_n = S$ ***(A1)***

$(x_1 \times x_2 \times \ldots \times x_n)^{\frac{1}{n}} \le \frac{S}{n}$ ***M1***

Note: Award ***M1*** for use of the inequality, which may be seen as an equality.

$x_1 \times x_2 \times \ldots \times x_n \le \left(\frac{S}{n}\right)^n$ (as both sides are positive) ***M1***

LHS and RHS are equal when all values of x_i are equal (to $\frac{S}{n}$) ***R1***

$M_n(S) = \left(\frac{S}{n}\right)^n$ ***AG***

[4 marks]

(e) (i) $M_3(12) = 4^3 = 64$ ***A1***

[1 mark]

(ii) $M_4(12) = 3^4 = 81$ ***A1***

[1 mark]

(iii) $M_5(12) = 2.4^5 = 79.6 \ (79.6262\ldots)$ ***A1***

[1 mark]

(f) considering $M_n(12)$ for higher values of n

$P(12)=81$ **A1**

$n=4$
A1

Note: Award ***A0A0*** for $P(12)=82.6$ and $n=4.41$.

[2 marks]

(g) Consideration of graph or table of $\left(\frac{20}{n}\right)^n$ including values either side of 7 ***(M1)***

Maximum occurs when $n=7$ **A1**

$P(20)=\left(\frac{20}{7}\right)^7=1550\ (1554.260...)$ **A1**

Note: Award ***(M1)A0A1*** for $n=7.36$ and $P(20)=1570$.

[3 marks]

(h) **EITHER**

$\ln(g(x)) = x(\ln(S) - \ln x)$ ***M1***

attempt to use implicit differentiation and product rule ***M1M1***

$\frac{g'(x)}{g(x)} = \ln S - \ln x - x\frac{1}{x}$ ***A1***

OR

attempt to use implicit differentiation, product rule and chain rule ***M1M1M1***

$\frac{g'(x)}{g(x)} = \ln\frac{S}{x} + \left(x\frac{x}{S} \times \frac{-S}{x^2}\right)$ ***A1***

OR

attempt to make equation explicit to $g(x) = \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}$ ***M1***

attempt to use product rule and chain rule ***M1M1***

$g'(x) = \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}\left[x \times \frac{x}{S} \times \left(-Sx^{-2}\right) + \ln\left(\frac{S}{x}\right)\right]$

$= \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}\left[\ln\left(\frac{S}{x}\right) - 1\right]$ ***A1***

THEN

$g'(x) = \left(\ln\frac{S}{x} - 1\right)g(x)$

$g(x) \neq 0$

$g'(x) = 0 \Rightarrow \ln\frac{S}{x} - 1 = 0$ ***M1***

$x = \frac{S}{\mathrm{e}}$ $(0.368S, 0.36789...S)$ ***A1***

[6 marks]

(i) $\ln(g(x)) = x\ln\left(\frac{S}{x}\right) \Rightarrow \ln(g(x)) = \ln\left(\frac{S}{x}\right)^x$ ***M1***

$g(x) = \left(\frac{S}{x}\right)^x$ ***A1***

$= M_x(S)$ for $x \in \mathbb{Z}^+$ ***AG***

[2 marks]

(j) $\frac{100}{\mathrm{e}} = 36.8$ ***M1***

$\left(\frac{100}{36}\right)^{36} = 9.3996... \times 10^{15}$ **AND** $\left(\frac{100}{37}\right)^{37} = 9.47406... \times 10^{15}$ ***R1***

largest possible product is 9.47×10^{15} $(9.47406... \times 10^{15})$ ***A1***

Note: Award ***A1*** independently of the ***R1*** (but not independently of the ***M1***).

[3 marks]

Total [30 marks]