Question 1

[Maximum mark: 25]

In this question you will explore some of the properties of special functions f and g and their relationship with the trigonometric functions, sine and cosine.

Functions f and g are defined as $f(z) = \dfrac{e^z + e^{-z}}{2}$ and $g(z) = \dfrac{e^z - e^{-z}}{2}$, where $z \in \mathbb{C}$.

Consider t and u, such that $t, u \in \mathbb{R}$.

(a) Verify that $u = f(t)$ satisfies the differential equation $\dfrac{d^2 u}{dt^2} = u$. [2]

(b) Show that $(f(t))^2 + (g(t))^2 = f(2t)$. [3]

(c) Using $e^{iu} = \cos u + i\sin u$, find expressions, in terms of $\sin u$ and $\cos u$, for

    (i) $f(iu)$; [3]

    (ii) $g(iu)$. [2]

(d) Hence find, and simplify, an expression for $(f(iu))^2 + (g(iu))^2$. [2]

(e) Show that $(f(t))^2 - (g(t))^2 = (f(iu))^2 - (g(iu))^2$. [4]

The functions $\cos x$ and $\sin x$ are known as circular functions as the general point $(\cos\theta, \sin\theta)$ defines points on the unit circle with equation $x^2 + y^2 = 1$.

The functions $f(x)$ and $g(x)$ are known as hyperbolic functions, as the general point $(f(\theta), g(\theta))$ defines points on a curve known as a hyperbola with equation $x^2 - y^2 = 1$. This hyperbola has two asymptotes.

(f) Sketch the graph of $x^2 - y^2 = 1$, stating the coordinates of any axis intercepts and the equation of each asymptote. [4]

The hyperbola with equation $x^2 - y^2 = 1$ can be rotated to coincide with the curve defined by $xy = k$, $k \in \mathbb{R}$.

(g) Find the possible values of k. [5]