Subject – Math AA(Higher Level)
Topic - Number and Algebra
Year - May 2021 – Nov 2024
Paper -3
Questions

Question 1

[Maximum mark: 28]

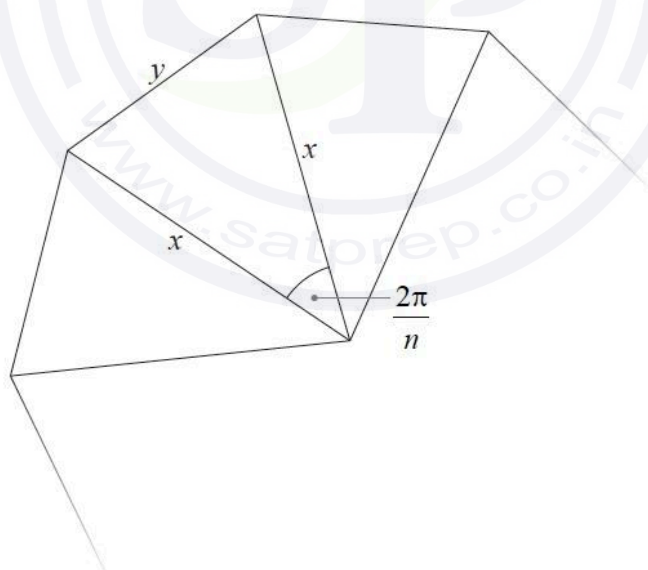
This question asks you to examine various polygons for which the numerical value of the area is the same as the numerical value of the perimeter. For example, a 3 by 6 rectangle has an area of 18 and a perimeter of 18.

For each polygon in this question, let the numerical value of its area be A and let the numerical value of its perimeter be P.

(a) Find the side length, s, where $s > 0$, of a square such that $A = P$. [3]

An n-sided regular polygon can be divided into n congruent isosceles triangles. Let x be the length of each of the two equal sides of one such isosceles triangle and let y be the length of the third side. The included angle between the two equal sides has magnitude $\frac{2\pi}{n}$.

Part of such an n-sided regular polygon is shown in the following diagram.



(b) Write down, in terms of x and n, an expression for the area, A_T, of one of these isosceles triangles. [1]

(c) Show that $y = 2x\sin\frac{\pi}{n}$. [2]

Consider a n-sided regular polygon such that $A = P$.

(d) Use the results from parts (b) and (c) to show that $A = P = 4n\tan\frac{\pi}{n}$. [7]

The Maclaurin series for $\tan x$ is $x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$

(e) (i) Use the Maclaurin series for $\tan x$ to find $\lim_{n\to\infty}\left(4n\tan\frac{\pi}{n}\right)$. [3]

(ii) Interpret your answer to part (e)(i) geometrically. [1]

Consider a right-angled triangle with side lengths a, b and $\sqrt{a^2+b^2}$, where $a \geq b$, such that $A = P$.

(f) Show that $a = \frac{8}{b-4} + 4$. [7]

(g) (i) By using the result of part (f) or otherwise, determine the three side lengths of the only two right-angled triangles for which $a, b, A, P \in \mathbb{Z}$. [3]

(ii) Determine the area and perimeter of these two right-angled triangles. [1]

Question 2

[Maximum mark: 24]

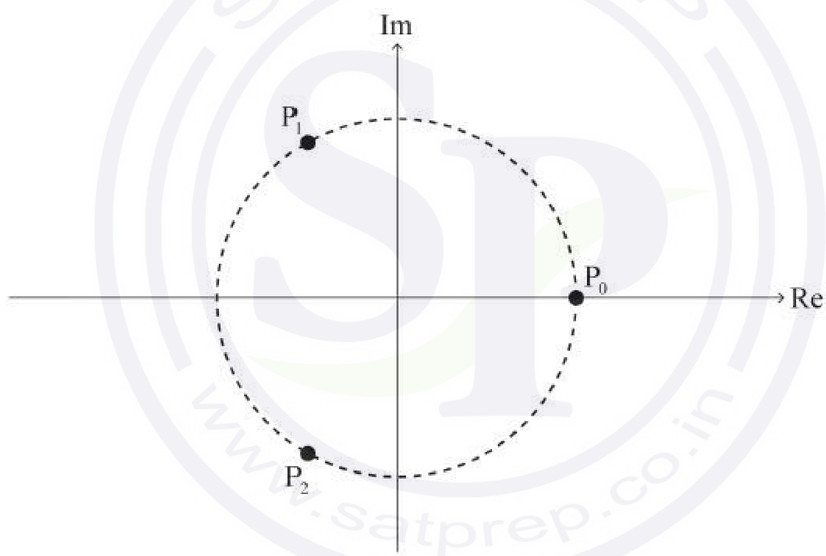
This question asks you to investigate and prove a geometric property involving the roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ for integers n, where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = \mathrm{e}^{\frac{2\pi i}{n}}$. Each root can be represented by a point $P_0, P_1, P_2, \ldots, P_{n-1}$, respectively, on an Argand diagram.

For example, the roots of the equation $z^2 = 1$ where $z \in \mathbb{C}$ are 1 and ω. On an Argand diagram, the root 1 can be represented by a point P_0 and the root ω can be represented by a point P_1.

Consider the case where $n = 3$.

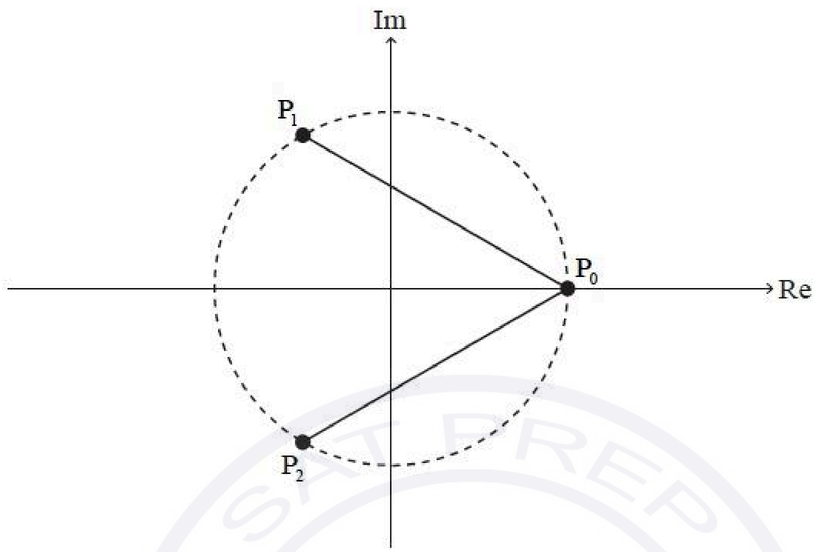
The roots of the equation $z^3 = 1$ where $z \in \mathbb{C}$ are 1, ω and ω^2. On the following Argand diagram, the points P_0, P_1 and P_2 lie on a circle of radius 1 unit with centre $O(0, 0)$.



(a) (i) Show that $(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1$. [2]

(ii) Hence, deduce that $\omega^2 + \omega + 1 = 0$. [2]

Line segments $[P_0P_1]$ and $[P_0P_2]$ are added to the Argand diagram in part (a) and are shown on the following Argand diagram.



P_0P_1 is the length of $[P_0P_1]$ and P_0P_2 is the length of $[P_0P_2]$.

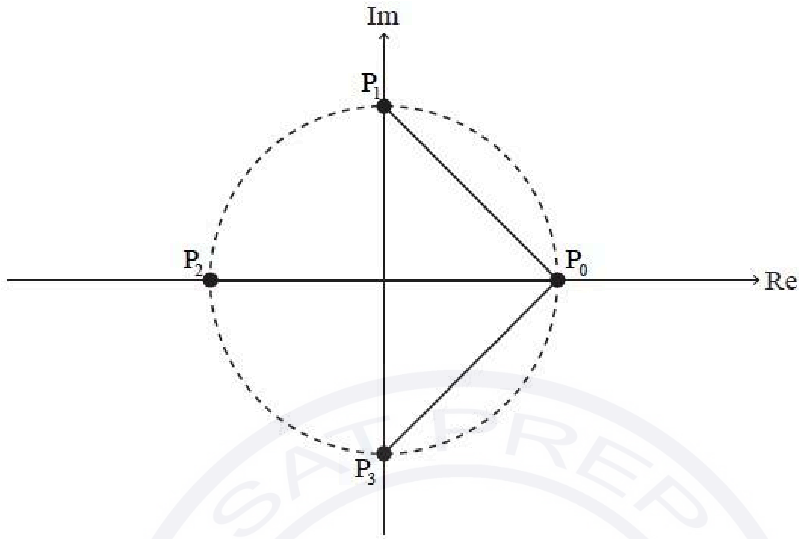
(b) Show that $P_0P_1 \times P_0P_2 = 3$. [3]

Consider the case where $n = 4$.

The roots of the equation $z^4 = 1$ where $z \in \mathbb{C}$ are 1, ω, ω^2 and ω^3.

(c) By factorizing $z^4 - 1$, or otherwise, deduce that $\omega^3 + \omega^2 + \omega + 1 = 0$. [2]

On the following Argand diagram, the points P_0, P_1, P_2 and P_3 lie on a circle of radius 1 unit with centre $O(0, 0)$. $[P_0P_1]$, $[P_0P_2]$ and $[P_0P_3]$ are line segments.



(d) Show that $P_0P_1 \times P_0P_2 \times P_0P_3 = 4$. [4]

For the case where $n = 5$, the equation $z^5 = 1$ where $z \in \mathbb{C}$ has roots 1, ω, ω^2, ω^3 and ω^4.

It can be shown that $P_0P_1 \times P_0P_2 \times P_0P_3 \times P_0P_4 = 5$.

Now consider the general case for integer values of n, where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are $1, \omega, \omega^2, \ldots, \omega^{n-1}$. On an Argand diagram, these roots can be represented by the points $P_0, P_1, P_2, \ldots, P_{n-1}$ respectively where $[P_0P_1], [P_0P_2], \ldots, [P_0P_{n-1}]$ are line segments. The roots lie on a circle of radius 1 unit with centre $O(0, 0)$.

(e) Suggest a value for $P_0P_1 \times P_0P_2 \times \ldots \times P_0P_{n-1}$. [1]

P_0P_1 can be expressed as $|1 - \omega|$.

(f) (i) Write down expressions for P_0P_2 and P_0P_3 in terms of ω. [2]

(ii) Hence, write down an expression for P_0P_{n-1} in terms of n and ω. [1]

Consider $z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \ldots + z + 1)$ where $z \in \mathbb{C}$.

(g) (i) Express $z^{n-1} + z^{n-2} + \ldots + z + 1$ as a product of linear factors over the set $\mathbb{C}$. [3]

(ii) Hence, using the part (g)(i) and part (f) results, or otherwise, prove your suggested result to part (e). [4]

Question 3

[Maximum marks: 28]

This question asks you to explore cubic polynomials of the form $(x-r)(x^2-2ax+a^2+b^2)$ for $x \in \mathbb{R}$ and corresponding cubic equations with one real root and two complex roots of the form $(z-r)(z^2-2az+a^2+b^2)=0$ for $z \in \mathbb{C}$.

In parts (a), (b) and (c), let $r=1$, $a=4$ and $b=1$.

Consider the equation $(z-1)(z^2-8z+17)=0$ for $z \in \mathbb{C}$.

(a) (i) Given that 1 and $4+\mathrm{i}$ are roots of the equation, write down the third root. [1]

(ii) Verify that the mean of the two complex roots is 4. [1]

Consider the function $f(x)=(x-1)(x^2-8x+17)$ for $x \in \mathbb{R}$.

(b) Show that the line $y=x-1$ is tangent to the curve $y=f(x)$ at the point $\mathrm{A}(4, 3)$. [4]

(c) Sketch the curve $y=f(x)$ and the tangent to the curve at point A, clearly showing where the tangent crosses the x-axis. [2]

Consider the function $g(x)=(x-r)(x^2-2ax+a^2+b^2)$ for $x \in \mathbb{R}$ where $r, a \in \mathbb{R}$ and $b \in \mathbb{R}, b>0$.

(d) (i) Show that $g'(x)=2(x-r)(x-a)+x^2-2ax+a^2+b^2$. [2]

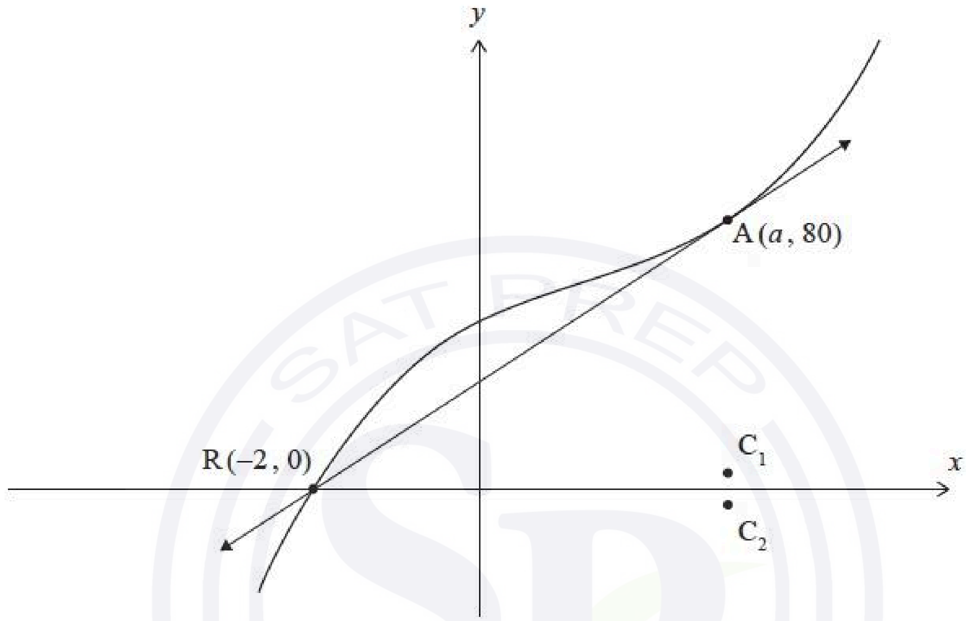
(ii) Hence, or otherwise, prove that the tangent to the curve $y=g(x)$ at the point $\mathrm{A}(a, g(a))$ intersects the x-axis at the point $\mathrm{R}(r, 0)$. [6]

The equation $(z-r)(z^2-2az+a^2+b^2)=0$ for $z \in \mathbb{C}$ has roots r and $a \pm b\mathrm{i}$ where $r, a \in \mathbb{R}$ and $b \in \mathbb{R}, b>0$.

(e) Deduce from part (d)(i) that the complex roots of the equation $(z-r)(z^2-2az+a^2+b^2)=0$ can be expressed as $a \pm \mathrm{i}\sqrt{g'(a)}$. [1]

On the Cartesian plane, the points $C_1\left(a, \sqrt{g'(a)}\right)$ and $C_2\left(a, -\sqrt{g'(a)}\right)$ represent the real and imaginary parts of the complex roots of the equation $(z-r)(z^2-2az+a^2+b^2)=0$.

The following diagram shows a particular curve of the form $y=(x-r)(x^2-2ax+a^2+16)$ and the tangent to the curve at the point $A(a, 80)$. The curve and the tangent both intersect the x-axis at the point $R(-2, 0)$. The points C_1 and C_2 are also shown.



(f) (i) Use this diagram to determine the roots of the corresponding equation of the form $(z-r)(z^2-2az+a^2+16)=0$ for $z \in \mathbb{C}$. [4]

(ii) State the coordinates of C_2. [1]

Consider the curve $y=(x-r)(x^2-2ax+a^2+b^2)$ for $a \neq r$, $b>0$. The points $A(a, g(a))$ and $R(r, 0)$ are as defined in part (d)(ii). The curve has a point of inflexion at point P.

(g) (i) Show that the x-coordinate of P is $\frac{1}{3}(2a+r)$.

You are **not** required to demonstrate a change in concavity. [2]

(ii) Hence describe numerically the horizontal position of point P relative to the horizontal positions of the points R and A. [1]

Consider the special case where $a=r$ and $b>0$.

(h) (i) Sketch the curve $y=(x-r)(x^2-2ax+a^2+b^2)$ for $a=r=1$ and $b=2$. [2]

(ii) For $a=r$ and $b>0$, state in terms of r, the coordinates of points P and A. [1]

Question 4

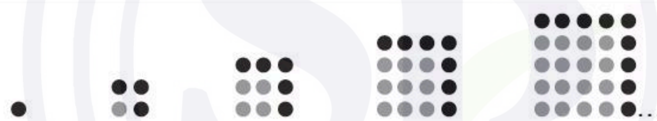
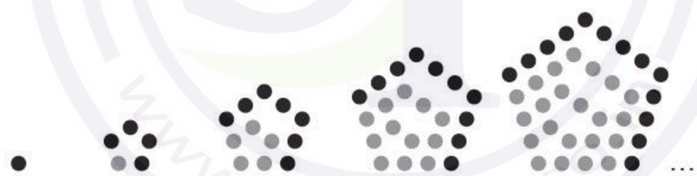
[Maximum marks: 27]

This question asks you to explore some properties of polygonal numbers and to determine and prove interesting results involving these numbers.

A polygonal number is an integer which can be represented as a series of dots arranged in the shape of a regular polygon. Triangular numbers, square numbers and pentagonal numbers are examples of polygonal numbers.

For example, a triangular number is a number that can be arranged in the shape of an equilateral triangle. The first five triangular numbers are $1, 3, 6, 10$ and 15.

The following table illustrates the first five triangular, square and pentagonal numbers respectively. In each case the first polygonal number is one represented by a single dot.

Type of polygonal number	Geometric representation	Values
Triangular numbers		$1, 3, 6, 10, 15, \ldots$
Square numbers		$1, 4, 9, 16, 25, \ldots$
Pentagonal numbers		$1, 5, 12, 22, 35, \ldots$

For an r-sided regular polygon, where $r \in \mathbb{Z}^+$, $r \geq 3$, the nth polygonal number $P_r(n)$ is given by

$$P_r(n) = \frac{(r-2)n^2 - (r-4)n}{2}, \text{ where } n \in \mathbb{Z}^+.$$

Hence, for square numbers, $P_4(n) = \dfrac{(4-2)n^2 - (4-4)n}{2} = n^2$.

(a) (i) For triangular numbers, verify that $P_3(n) = \dfrac{n(n+1)}{2}$. [2]

(ii) The number 351 is a triangular number. Determine which one it is. [2]

(b) (i) Show that $P_3(n) + P_3(n+1) \equiv (n+1)^2$. [2]

(ii) State, in words, what the identity given in part (b)(i) shows for two consecutive triangular numbers. [1]

(iii) For $n = 4$, sketch a diagram clearly showing your answer to part (b)(ii). [1]

(c) Show that $8P_3(n) + 1$ is the square of an odd number for all $n \in \mathbb{Z}^+$. [3]

The nth pentagonal number can be represented by the arithmetic series

$$P_5(n) = 1 + 4 + 7 + \ldots + (3n-2).$$

(d) Hence show that $P_5(n) = \dfrac{n(3n-1)}{2}$ for $n \in \mathbb{Z}^+$. [3]

(e) By using a suitable table of values or otherwise, determine the smallest positive integer, greater than 1, that is both a triangular number and a pentagonal number. [5]

A polygonal number, $P_r(n)$, can be represented by the series

$$\sum_{m=1}^{n} \left(1 + (m-1)(r-2)\right) \text{ where } r \in \mathbb{Z}^+,\ r \geq 3.$$

(f) Use mathematical induction to prove that $P_r(n) = \dfrac{(r-2)n^2 - (r-4)n}{2}$ where $n \in \mathbb{Z}^+$. [8]

Question 5

[Maximum mark: 27]

This question asks you to investigate conditions for the existence of complex roots of polynomial equations of degree 3 and 4.

The cubic equation $x^3 + px^2 + qx + r = 0$, where p, q, $r \in \mathbb{R}$, has roots α, β and γ.

(a) By expanding $(x - \alpha)(x - \beta)(x - \gamma)$ show that:

$$p = -(\alpha + \beta + \gamma)$$

$$q = \alpha\beta + \beta\gamma + \gamma\alpha$$

$$r = -\alpha\beta\gamma.$$ [3]

(b) (i) Show that $p^2 - 2q = \alpha^2 + \beta^2 + \gamma^2$. [3]

(ii) Hence show that $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 = 2p^2 - 6q$. [3]

(c) Given that $p^2 < 3q$, deduce that α, β and γ cannot all be real. [2]

Consider the equation $x^3 - 7x^2 + qx + 1 = 0$, where $q \in \mathbb{R}$.

(d) Using the result from part (c), show that when $q = 17$, this equation has at least one complex root. [2]

Noah believes that if $p^2 \geq 3q$ then α, β and γ are all real.

(e) (i) By varying the value of q in the equation $x^3 - 7x^2 + qx + 1 = 0$, determine the smallest positive integer value of q required to show that Noah is incorrect. [2]

(ii) Explain why the equation will have at least one real root for all values of q. [1]

Now consider polynomial equations of degree 4.

The equation $x^4 + px^3 + qx^2 + rx + s = 0$, where p, q, r, $s \in \mathbb{R}$, has roots α, β, γ and δ.

In a similar way to the cubic equation, it can be shown that:

$$p = -(\alpha + \beta + \gamma + \delta)$$

$$q = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$$

$$r = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$$

$$s = \alpha\beta\gamma\delta.$$

(f) (i) Find an expression for $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ in terms of p and q. [3]

(ii) Hence state a condition in terms of p and q that would imply $x^4 + px^3 + qx^2 + rx + s = 0$ has at least one complex root. [1]

(g) Use your result from part (f)(ii) to show that the equation $x^4 - 2x^3 + 3x^2 - 4x + 5 = 0$ has at least one complex root. [1]

The equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$, has one integer root.

(h) (i) State what the result in part (f)(ii) tells us when considering this equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$. [1]

(ii) Write down the integer root of this equation. [1]

(iii) By writing $x^4 - 9x^3 + 24x^2 + 22x - 12$ as a product of one linear and one cubic factor, prove that the equation has at least one complex root. [4]

Question 6

[Maximum mark: 28]

In this question you will investigate series of the form

$$\sum_{i=1}^{n} i^q = 1^q + 2^q + 3^q + \ldots + n^q \text{ **where** } n, q \in \mathbb{Z}^+$$

and use various methods to find polynomials, in terms of n, for such series.

When $q = 1$, the above series is arithmetic.

(a) Show that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$. [1]

Consider the case when $q = 2$.

(b) The following table gives values of n^2 and $\sum_{i=1}^{n} i^2$ for $n = 1, 2, 3$.

n	n^2	$\sum_{i=1}^{n} i^2$
1	1	1
2	4	5
3	9	p

(i) Write down the value of p. [1]

(ii) The sum of the first n square numbers can be expressed as a cubic polynomial with three terms:

$$\sum_{i=1}^{n} i^2 = a_1 n + a_2 n^2 + a_3 n^3 \text{ where } a_1, a_2, a_3 \in \mathbb{Q}^+.$$

Hence, write down a system of three linear equations in a_1, a_2 and a_3. [3]

(iii) Hence, find the values of a_1, a_2 and a_3. [2]

You will now consider a method that can be generalized for all values of q.

Consider the function $f(x) = 1 + x + x^2 + \dots + x^n,\ n \in \mathbb{Z}^+$.

(c) Show that $x f'(x) = x + 2x^2 + 3x^3 + \dots + nx^n$. [1]

Let $f_1(x) = x f'(x)$ and consider the following family of functions:

$$f_2(x) = x f_1{}'(x)$$

$$f_3(x) = x f_2{}'(x)$$

$$f_4(x) = x f_3{}'(x)$$

$$\dots$$

$$f_q(x) = x f_{q-1}{}'(x)$$

(d) (i) Show that $f_2(x) = \sum_{i=1}^{n} i^2 x^i$. [2]

(ii) Prove by mathematical induction that $f_q(x) = \sum_{i=1}^{n} i^q x^i$, $q \in \mathbb{Z}^+$. [6]

(iii) Using sigma notation, write down an expression for $f_q(1)$. [1]

(e) By considering $f(x) = 1 + x + x^2 + \dots + x^n$ as a geometric series, for $x \neq 1$, show that $f(x) = \dfrac{x^{n+1} - 1}{x - 1}$. [2]

(f) For $x \neq 1$, show that $f_1(x) = \dfrac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$. [3]

(g) (i) Show that $\lim_{x \to 1} f_1(x)$ is in indeterminate form. [1]

(ii) Hence, by applying l'Hôpital's rule, show that $\lim_{x \to 1} f_1(x) = \dfrac{1}{2}n(n+1)$. [5]

Question 7

[Maximum mark: 31]

This question asks you to examine linear and quadratic functions constructed in systematic ways using arithmetic sequences.

Consider the function $L(x) = mx + c$ for $x \in \mathbb{R}$ where m, $c \in \mathbb{R}$ and m, $c \neq 0$.

Let $r \in \mathbb{R}$ be the root of $L(x) = 0$.

If m, r and c, in that order, are in arithmetic sequence then $L(x)$ is said to be an AS-linear function.

(a) Show that $L(x) = 2x - 1$ is an AS-linear function. [2]

Consider $L(x) = mx + c$.

(b) (i) Show that $r = -\dfrac{c}{m}$. [1]

(ii) Given that $L(x)$ is an AS-linear function, show that $L(x) = mx - \dfrac{m^2}{m+2}$. [4]

(iii) State any further restrictions on the value of m. [1]

There are only three **integer** sets of values of m, r and c, that form an AS-linear function. One of these is $L(x) = -x - 1$.

(c) Use part (b) to determine the other two AS-linear functions with integer values of m, r and c. [3]

Consider the function $Q(x) = ax^2 + bx + c$ for $x \in \mathbb{R}$ where $a \in \mathbb{R}$, $a \neq 0$ and b, $c \in \mathbb{R}$.

Let r_1, $r_2 \in \mathbb{R}$ be the roots of $Q(x) = 0$.

(d) Write down an expression for

(i) the sum of roots, $r_1 + r_2$, in terms of a and b. [1]

(ii) the product of roots, $r_1 r_2$, in terms of a and c. [1]

If a, r_1, b, r_2 and c, in that order, are in arithmetic sequence, then $Q(x)$ is said to be an AS-quadratic function.

(e) Given that $Q(x)$ is an AS-quadratic function,

(i) write down an expression for $r_2 - r_1$ in terms of a and b; [1]

(ii) use your answers to parts (d)(i) and (e)(i) to show that $r_1 = \dfrac{a^2 - ab - b}{2a}$; [2]

(iii) use the result from part (e)(ii) to show that $b = 0$ or $a = -\dfrac{1}{2}$. [3]

Consider the case where $b = 0$.

(f) Determine the two AS-quadratic functions that satisfy this condition. [5]

Now consider the case where $a = -\dfrac{1}{2}$.

(g) (i) Find an expression for r_1 in terms of b. [2]

(ii) Hence or otherwise, determine the exact values of b and c such that AS-quadratic functions are formed.

Give your answers in the form $\dfrac{-p \pm q\sqrt{s}}{2}$ where p, q, $s \in \mathbb{Z}^+$. [5]

Question 8

[Maximum mark: 24]

This question asks you to explore some properties of the family of curves $y = x^3 + ax^2 + b$ where $x \in \mathbb{R}$ and a, b are real parameters.

Consider the family of curves $y = x^3 + ax^2 + b$ for $x \in \mathbb{R}$, where $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$.

First consider the case where $a = 3$ and $b \in \mathbb{R}$.

(a) By systematically varying the value of b, or otherwise, find the two values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly two x-axis intercepts. [2]

(b) Write down the set of values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly

(i) one x-axis intercept; [1]

(ii) three x-axis intercepts. [1]

Now consider the case where $a = -3$ and $b \in \mathbb{R}$.

(c) Write down the set of values of b such that the curve $y = x^3 - 3x^2 + b$ has exactly

(i) two x-axis intercepts; [1]

(ii) one x-axis intercept; [1]

(iii) three x-axis intercepts. [1]

For the following parts of this question, consider the curve $y = x^3 + ax^2 + b$ for $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$.

(d) Consider the case where the curve has exactly three x-axis intercepts. State whether each point of zero gradient is located above or below the x-axis. [1]

(e) Show that the curve has a point of zero gradient at $\mathrm{P}(0\,,\,b)$ and a point of zero gradient at $\mathrm{Q}\left(-\frac{2}{3}a\,,\,\frac{4}{27}a^3 + b\right)$. [5]

(f) Consider the points P and Q for $a > 0$ and $b > 0$.

(i) Find an expression for $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ and hence determine whether each point is a local maximum or a local minimum. [3]

(ii) Determine whether each point is located above or below the x-axis. [1]

(g) Consider the points P and Q for $a < 0$ and $b > 0$.

(i) State whether P is a local maximum or a local minimum and whether it is above or below the x-axis. [1]

(ii) State the conditions on a and b that determine when Q is below the x-axis. [1]

(h) Prove that if $4a^3 b + 27b^2 < 0$ then the curve, $y = x^3 + ax^2 + b$, has exactly three x-axis intercepts. [5]

Question 9

[Maximum mark: 27]

This question asks you to investigate models for the population of trout in a lake.

Trout is a type of fish. At the start of a year, a lake is estimated to contain 6000 trout.

The owner of the lake estimates that the number of trout will increase by 10% per year.

At the end of each year, the owner proposes to remove 500 trout from the lake to prevent overpopulation.

Therefore, the relationship between T_n, the predicted number of trout at the start of year n, and T_{n+1}, the predicted number of trout at the start of year $n+1$, is given by

$$T_{n+1} = 1.1T_n - 500 \text{ and } T_1 = 6000.$$

For example, the predicted number of trout at the start of the second year is given by

$$T_2 = 1.1T_1 - 500.$$

(a) Use this formula to verify that $T_2 = 6100$. [2]

(b) (i) Verify that $T_3 = 6210$. [1]

(ii) Find T_4. [2]

It is also known that $T_n = 6000(1.1)^{n-1} - \dfrac{500\left((1.1)^{n-1} - 1\right)}{1.1-1}$.

(c) (i) Show that $T_n = 1000(1.1)^{n-1} + 5000$. [2]

(ii) Hence, or otherwise, find T_6. Give your answer to the nearest whole number. [2]

After deciding that the trout population would increase too quickly, the lake owner proposes instead to remove 750 trout at the end of each year.

The relationship between D_n, the predicted number of trout at the start of year n, and D_{n+1}, the predicted number of trout at the start of year $n+1$, is now given by

$$D_{n+1} = 1.1D_n - 750 \text{ and } D_1 = 6000.$$

It is also known that $D_n = -1500(1.1)^{n-1} + 7500$.

(d) (i) Show that $D_{n+1} - D_n = -150(1.1)^{n-1}$. [3]

(ii) Use the result in part (d)(i) to deduce that the predicted number of trout at the start of any year will be greater than the predicted number at the start of the next year. [1]

(e) Determine the first year during which there will be no trout remaining in the lake. [4]

The lake owner now considers a more general approach where d trout are removed at the end of each year.

Let C_n denote the predicted number of trout in the lake at the start of the nth year where

$$C_n = 6000(1.1)^{n-1} - 10d((1.1)^{n-1} - 1).$$

(f) Find the value of d such that the predicted number of trout at the start of each year is constant. [3]

To model predicted numbers of trout, the lake owner has been using sequences generated by

$$u_{n+1} = ru_n - d, \text{ where } d, r \in \mathbb{R}^+ \text{ and } r \neq 1.$$

(g) Use mathematical induction to prove that $u_n = u_1 r^{n-1} - \frac{d\left(r^{n-1} - 1\right)}{r-1}$, for $n \in \mathbb{Z}^+$. [7]

Question 10

[Maximum mark: 28]

A polynomial is said to be palindromic if the sequence of its coefficients remains the same in reverse. This question asks you to investigate some properties and solutions of palindromic polynomial equations.

In parts (a) and (b), consider quadratic equations of the form $ax^2 + bx + a = 0$, where $a \neq 0$.

The sequence of coefficients, $\{a, b, a\}$, remains the same in reverse.

The following table shows three palindromic quadratic equations and their sequence of coefficients.

Palindromic quadratic equation	**Sequence of coefficients**
$2x^2 - 5x + 2 = 0$	$\{2, -5, 2\}$
$x^2 + 4x + 1 = 0$	$\{1, 4, 1\}$
$x^2 + 1 = 0$	$\{1, 0, 1\}$

The quadratic equation $2x^2 - 5x + 2 = 0$ has roots 2 and $\frac{1}{2}$.

These roots form a "reciprocal pair", since one root is the reciprocal of the other.

(a) (i) Determine the roots of $x^2 + 4x + 1 = 0$.

Give these roots in the form $s \pm \sqrt{t}$, where $s \in \mathbb{Z}$ and $t \in \mathbb{Z}^+$. [3]

(ii) Hence, or otherwise, show that these roots form a reciprocal pair. [2]

(b) Show that the complex roots of $x^2 + 1 = 0$ form a reciprocal pair. [2]

Let $p(x) = ax^2 + bx + a$, where $a \neq 0$.

(c) Verify that $p(x) = x^2 p\left(\frac{1}{x}\right)$ where $x \neq 0$. [2]

In parts (d) and (e), you may assume the result that a polynomial, $p(x)$, of degree n is palindromic if and only if $p(x) = x^n p\left(\frac{1}{x}\right)$.

(d) Use $p(x) = x^n p\left(\frac{1}{x}\right)$ to show that if $\alpha \neq 0$ is a root of $p(x) = 0$, then $\frac{1}{\alpha}$ is also a root. [2]

Let $f(x) = p(x)q(x)$, where p and q are palindromic polynomials of degree n and m respectively.

(e) Show that f is a palindromic polynomial. [4]

Consider the palindromic polynomial $f(x) = x^4 + 2x^3 - x^2 + 2x + 1$.

This polynomial can be expressed in the form

$$f(x) = (x^2 + ux + 1)(x^2 + vx + 1) \text{, where } u, v \in \mathbb{Z} \text{ and } u < v.$$

(f) By forming and solving an appropriate system of equations in u and v, determine the value of u and the value of v. [6]

(g) Hence, find all the exact complex and purely real roots of $x^4 + 2x^3 - x^2 + 2x + 1 = 0$. [3]

Consider the palindromic polynomial equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_2x^2 + a_1x + 1 = 0 \text{, where } n \text{ is odd.}$$

(h) Show that -1 is always a root of this equation. [4]