

Subject - Math AA(Higher Level)
Topic - Number and Algebra
Year - May 2021 - Nov 2024
Paper -3
Answers

Question 1

(a) $A = s^2$ and $P = 4s$ **(A1)**

$A = P \Rightarrow s^2 = 4s$ **(M1)**

$s(s-4) = 0$

$\Rightarrow s = 4 (s > 0)$ **A1**

Note: Award **A1M1A0** if both $s = 4$ and $s = 0$ are stated as final answers.

[3 marks]

(b) $A_T = \frac{1}{2} x^2 \sin \frac{2\pi}{n}$ **A1**

Note: Award **A1** for a correct alternative form expressed in terms of x and n only.

For example, using Pythagoras' theorem, $A_T = x \sin \frac{\pi}{n} \sqrt{x^2 - x^2 \sin^2 \frac{\pi}{n}}$ or

$$A_T = 2 \left(\frac{1}{2} \left(x \sin \frac{\pi}{n} \right) \left(x \cos \frac{\pi}{n} \right) \right) \text{ or } A_T = x^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}.$$

[1 mark]

(c) **METHOD 1**

uses $\sin \theta = \frac{\text{opp}}{\text{hyp}}$ (M1)

$$\frac{\frac{y}{2}}{x} = \sin \frac{\pi}{n} \quad \text{A1}$$

$$y = 2x \sin \frac{\pi}{n} \quad \text{AG}$$

[2 marks]

METHOD 2

uses Pythagoras' theorem $\left(\frac{y}{2}\right)^2 + h^2 = x^2$ and $h = x \cos \frac{\pi}{n}$ (M1)

$$\left(\frac{y}{2}\right)^2 + \left(x \cos \frac{\pi}{n}\right)^2 = x^2 \quad \left(y^2 = 4x^2 \left(1 - \cos^2 \frac{\pi}{n}\right)\right)$$
$$= 4x^2 \sin^2 \frac{\pi}{n} \quad \text{A1}$$

$$y = 2x \sin \frac{\pi}{n} \quad \text{AG}$$

[2 marks]

METHOD 3

uses the cosine rule (M1)

$$y^2 = 2x^2 - 2x^2 \cos \frac{2\pi}{n} \quad \left(= 2x^2 \left(1 - \cos \frac{2\pi}{n}\right)\right)$$
$$= 4x^2 \sin^2 \frac{\pi}{n} \quad \text{A1}$$

$$y = 2x \sin \frac{\pi}{n} \quad \text{AG}$$

[2 marks]

METHOD 4

uses the sine rule

(M1)

$$\frac{y}{\sin \frac{2\pi}{n}} = \frac{x}{\sin \left(\frac{\pi}{2} - \frac{\pi}{n} \right)}$$

$$y \cos \frac{\pi}{n} = 2x \sin \frac{\pi}{n} \cos \frac{\pi}{n}$$

A1

$$y = 2x \sin \frac{\pi}{n}$$

AG**[2 marks]**

(d) $A = P \Rightarrow nA_T = ny$

(M1)

Note: Award M1 for equating correct expressions for A and P .

$$\frac{1}{2}nx^2 \sin \frac{2\pi}{n} = 2nx \sin \frac{\pi}{n} \left(nx^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} = 2nx \sin \frac{\pi}{n} \right)$$

$$\frac{1}{2}x^2 \sin \frac{2\pi}{n} = 2x \sin \frac{\pi}{n} \left(x^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} = 2x \sin \frac{\pi}{n} \right)$$

A1

uses $\sin \frac{2\pi}{n} = 2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}$ (seen anywhere in part (d) or in part (b))

(M1)

$$x^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} = 2x \sin \frac{\pi}{n}$$

attempts to either factorise or divide their expression

(M1)

$$x \sin \frac{\pi}{n} \left(x \cos \frac{\pi}{n} - 2 \right) = 0$$

$$x = \frac{2}{\cos \frac{\pi}{n}}, \left(x \sin \frac{\pi}{n} \neq 0 \right) \text{ (or equivalent)}$$

A1

EITHER

substitutes $x = \frac{2}{\cos \frac{\pi}{n}}$ (or equivalent) into $P = ny$ **(M1)**

$$P = 2n \left(\frac{2}{\cos \frac{\pi}{n}} \right) \left(\sin \frac{\pi}{n} \right) \quad \text{A1}$$

Note: Other approaches are possible. For example, award **A1** for $P = 2nx \cos \frac{\pi}{n} \tan \frac{\pi}{n}$ and **M1** for substituting $x = \frac{2}{\cos \frac{\pi}{n}}$ into P .

OR

substitutes $x = \frac{2}{\cos \frac{\pi}{n}}$ (or equivalent) into $A = nA_T$ **(M1)**

$$A = \frac{1}{2} n \left(\frac{2}{\cos \frac{\pi}{n}} \right)^2 \left(\sin \frac{2\pi}{n} \right)$$
$$A = \frac{1}{2} n \left(\frac{2}{\cos \frac{\pi}{n}} \right)^2 \left(2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} \right) \quad \text{A1}$$

THEN

$$A = P = 4n \tan \frac{\pi}{n} \quad \text{AG}$$

[7 marks]

(e) (i) attempts to use the Maclaurin series for $\tan x$ with $x = \frac{\pi}{n}$ (M1)

$$\tan \frac{\pi}{n} = \frac{\pi}{n} + \frac{\left(\frac{\pi}{n}\right)^3}{3} + \frac{2\left(\frac{\pi}{n}\right)^5}{15} (+\dots)$$

$$4n \tan \frac{\pi}{n} = 4n \left(\frac{\pi}{n} + \frac{\pi^3}{3n^3} + \frac{2\pi^5}{15n^5} (+\dots) \right) \text{ (or equivalent)} \quad \text{A1}$$

$$= 4 \left(\pi + \frac{\pi^3}{3n^2} + \frac{2\pi^5}{15n^4} + \dots \right)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left(4n \tan \frac{\pi}{n} \right) = 4\pi \quad \text{A1}$$

Note: Award a maximum of **M1A1A0** if $\lim_{n \rightarrow \infty}$ is not stated anywhere.

[3 marks]

(ii) (as $n \rightarrow \infty$, $P \rightarrow 4\pi$ and $A \rightarrow 4\pi$)
the polygon becomes a circle of radius 2

R1

Note: Award **R1** for alternative responses such as:

the polygon becomes a circle of area 4π OR

the polygon becomes a circle of perimeter 4π OR

the polygon becomes a circle with $A = P = 4\pi$.

Award **R0** for polygon becomes a circle.

[1 mark]

(f) $A = \frac{1}{2}ab$ and $P = a + b + \sqrt{a^2 + b^2}$ (A1)(A1)

equates their expressions for A and P M1

$$A = P \Rightarrow a + b + \sqrt{a^2 + b^2} = \frac{1}{2}ab$$

$$\sqrt{a^2 + b^2} = \frac{1}{2}ab - (a + b) \quad \text{M1}$$

Note: Award **M1** for isolating $\sqrt{a^2 + b^2}$ or $\pm 2\sqrt{a^2 + b^2}$. This step may be seen later.

$$a^2 + b^2 = \left(\frac{1}{2}ab - (a + b)\right)^2$$

$$a^2 + b^2 = \frac{1}{4}a^2b^2 - 2\left(\frac{1}{2}ab\right)(a + b) + (a + b)^2 \quad \text{M1}$$

$$\left(= \frac{1}{4}a^2b^2 - a^2b - ab^2 + a^2 + 2ab + b^2 \right)$$

Note: Award **M1** for attempting to expand their RHS of either $a^2 + b^2 = \dots$
or $4(a^2 + b^2) = \dots$

EITHER

$$ab\left(\frac{1}{4}ab - a - b + 2\right) = 0 \quad (ab \neq 0) \quad \text{A1}$$

$$\frac{1}{4}ab - a - b + 2 = 0$$

$$ab - 4a = 4b - 8$$

OR

$$\frac{1}{4}a^2b^2 - a^2b - ab^2 + 2ab = 0$$

$$a\left(\frac{1}{4}b^2 - b\right) + (2b - b^2) = 0 \quad (a(b^2 - 4b) + (8b - 4b^2) = 0) \quad \text{A1}$$

$$a = \frac{4b^2 - 8b}{b^2 - 4b}$$

THEN

$$\Rightarrow a = \frac{4b-8}{b-4}$$

A1

$$a = \frac{4b-16+8}{b-4}$$

$$a = \frac{8}{b-4} + 4$$

AG

Note: Award a maximum of **A1A1M1M1M0A0A0** for attempting to verify.

For example, verifying that $A = P = \frac{16}{b-4} + 2b + 4$ gains 4 of the 7 marks.

[7 marks]

- (g) (i) using an appropriate method
eg substituting values for b or using divisibility properties
(5,12,13) and (6,8,10)

(M1)

A1A1

Note: Award **A1A0** for either one set of three correct side lengths or two sets of two correct side lengths.

[3 marks]

- (ii) $A = P = 30$ and $A = P = 24$

A1

Note: Do not award **A1FT**.

[1 mark]

Total [28 marks]

Question 2

(a) (i) **METHOD 1**

attempts to expand $(\omega-1)(\omega^2+\omega+1)$ **(M1)**

$$= \omega^3 + \omega^2 + \omega - \omega^2 - \omega - 1 \quad \text{A1}$$

$$= \omega^3 - 1 \quad \text{AG}$$

[2 marks]

METHOD 2

attempts polynomial division on $\frac{\omega^3-1}{\omega-1}$ **M1**

$$= \omega^2 + \omega + 1 \quad \text{A1}$$

$$\text{so } (\omega-1)(\omega^2+\omega+1) = \omega^3 - 1 \quad \text{AG}$$

[2 marks]

(ii) (since ω is a root of $z^3=1$) $\Rightarrow \omega^3-1=0$ **R1**

and $\omega \neq 1$ **R1**

$$\Rightarrow \omega^2 + \omega + 1 = 0 \quad \text{AG}$$

(b) **METHOD 1**

attempts to find either P_0P_1 or P_0P_2

(M1)

accept any valid method

e.g. $2\sin\frac{\pi}{3}$, $1^2 + 1^2 - 2\cos\frac{2\pi}{3}$, $\frac{1}{\sin\frac{\pi}{6}} = \frac{P_0P_1}{\sin\frac{2\pi}{3}}$ from either $\triangle OP_0P_1$ or $\triangle OP_0P_2$

e.g. use of Pythagoras' theorem

e.g. $\left|1 - e^{i\frac{2\pi}{3}}\right|$, $\left|1 - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right|$ by calculating the distance between 2 points

$$P_0P_1 = \sqrt{3}$$

A1

$$P_0P_2 = \sqrt{3}$$

A1

Note: Award a maximum of **M1A1A0** for any decimal approximation seen in the calculation of either P_0P_1 or P_0P_2 or both.

so $P_0P_1 \times P_0P_2 = 3$

AG

METHOD 2

attempts to find $P_0P_1 \times P_0P_2 = |1 - \omega||1 - \omega^2|$

(M1)

$$P_0P_1 \times P_0P_2 = |\omega^3 - \omega^2 - \omega + 1|$$

A1

$$= |1 - (\omega^2 + \omega + 1) + 2| \text{ and since } \omega^2 + \omega + 1 = 0$$

R1

so $P_0P_1 \times P_0P_2 = 3$

AG

[3 marks]

(c) **METHOD 1**

$$z^4 - 1 = (z-1)(z^3 + z^2 + z + 1)$$

A1

(ω is a root hence) $\omega^4 - 1 = 0$ and $\omega \neq 1$

R1

$$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0$$

AG

Note: Condone the use of ω throughout.

[2 marks]

METHOD 2

considers the sum of roots of $z^4 - 1 = 0$

(M1)

the sum of roots is zero (there is no z^3 term)

A1

$$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0$$

AG

[2 marks]

METHOD 3

substitutes for ω

(M1)

$$\text{e.g. LHS} = e^{i\frac{3\pi}{2}} + e^{i\pi} + e^{i\frac{\pi}{2}} + 1$$

$$= -i - 1 + i + 1$$

A1

Note: This can be demonstrated geometrically or by using vectors.
Accept Cartesian or modulus-argument (polar) form.

$$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0$$

AG

[2 marks]

METHOD 4

$$\omega^3 + \omega^2 + \omega + 1 = \frac{\omega^4 - 1}{\omega - 1}$$

A1

$$= \frac{0}{\omega - 1} = 0 \text{ as } \omega \neq 1$$

R1

$$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0$$

AG

[2 marks]

(d) **METHOD 1**

$$P_0P_2 = 2$$

A1

attempts to find either P_0P_1 or P_0P_3

(M1)

Note: For example, $P_0P_1 = |1-i|$ and $P_0P_3 = |1+i|$.

Various geometric and trigonometric approaches can be used by candidates.

$$P_0P_1 = \sqrt{2}, P_0P_3 = \sqrt{2}$$

A1A1

Note: Award a maximum of **A1M1A1A0** if labels such as P_0P_1 are not clearly shown.

Award full marks if the lengths are shown on a clearly labelled diagram.

Award a maximum of **A1M1A1A0** for any decimal approximation seen in the calculation of either P_0P_1 or P_0P_3 or both.

$$P_0P_1 \times P_0P_2 \times P_0P_3 = 4$$

AG

[4 marks]

METHOD 2

attempts to find $P_0P_1 \times P_0P_2 \times P_0P_3 = |1-\omega||1-\omega^2||1-\omega^3|$

M1

$$P_0P_1 \times P_0P_2 \times P_0P_3 = |-\omega^6 + \omega^5 + \omega^4 - \omega^2 - \omega + 1|$$

A1

$$= | -(-1) + \omega^5 + 1 - (-1) - \omega + 1 | \text{ since } \omega^6 = \omega^2 = -1 \text{ and } \omega^4 = 1$$

A1

$$= | \omega^5 - \omega + 4 | \text{ and since } \omega^5 = \omega$$

R1

$$\text{so } P_0P_1 \times P_0P_2 \times P_0P_3 = 4$$

AG

[4 marks]

(e) $(P_0P_1 \times P_0P_2 \times \dots \times P_0P_{n-1}) = n$

A1

[1 mark]

(f) (i) $P_0P_2 = |1 - \omega^2|$, $P_0P_3 = |1 - \omega^3|$

A1A1

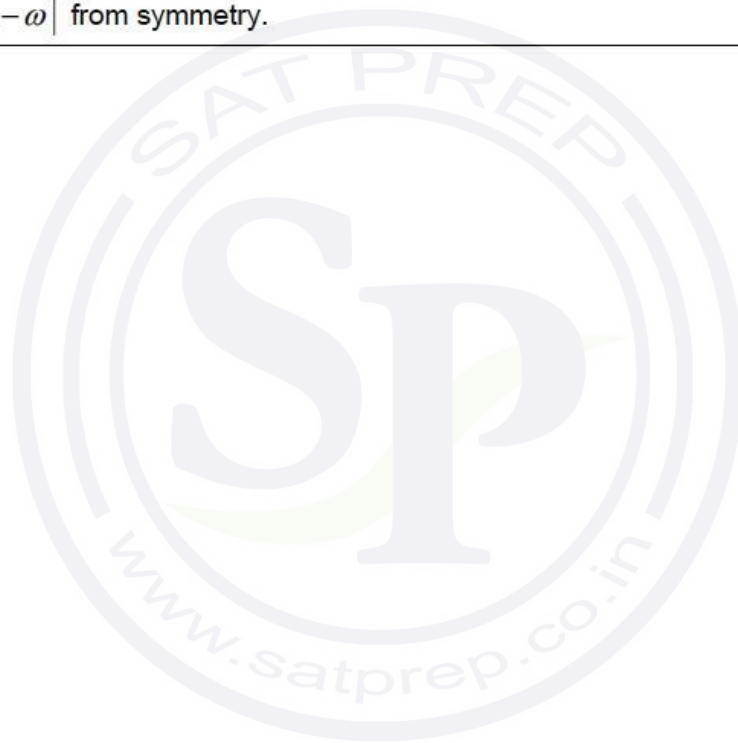
[2 marks]

(ii) $P_0P_{n-1} = |1 - \omega^{n-1}|$

A1

Note: Accept $|1 - \omega|$ from symmetry.

[1 mark]



(g) (i) $z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \dots + z + 1)$

considers the equation $z^{n-1} + z^{n-2} + \dots + z + 1 = 0$

(M1)

the roots are $\omega, \omega^2, \dots, \omega^{n-1}$

(A1)

so $(z - \omega)(z - \omega^2) \dots (z - \omega^{n-1})$

A1

[3 marks]

(ii) **METHOD 1**

substitutes $z = 1$ into $(z - \omega)(z - \omega^2) \dots (z - \omega^{n-1}) \equiv z^{n-1} + z^{n-2} + \dots + z + 1$

M1

$$(1 - \omega)(1 - \omega^2) \dots (1 - \omega^{n-1}) = n$$

(A1)

takes modulus of both sides

M1

$$|(1 - \omega)(1 - \omega^2) \dots (1 - \omega^{n-1})| = |n|$$

$$|1 - \omega| |1 - \omega^2| \dots |1 - \omega^{n-1}| = n$$

A1

$$\text{so } P_0 P_1 \times P_0 P_2 \times \dots \times P_0 P_{n-1} = n$$

AG

Note: Award a maximum of **M1A1FTM1A0** from part (e).

[4 marks]

METHOD 2

$(1-\omega), (1-\omega^2), \dots, (1-\omega^{n-1})$ are the roots of $(1-v)^{n-1} + (1-v)^{n-2} + \dots + (1-v) + 1 = 0$

M1

coefficient of v^{n-1} is $(-1)^{n-1}$ and the coefficient of 1 is n

A1

product of the roots is $\frac{(-1)^{n-1} n}{(-1)^{n-1}} = n$

A1

$|1-\omega||1-\omega^2|\dots|1-\omega^{n-1}| = n$

A1

so $P_0 P_1 \times P_0 P_2 \times \dots \times P_0 P_{n-1} = n$

AG

[4 marks]

Total[24 marks]



Question 3

(a) (i) $4-i$

A1
[1 mark]

(ii) $\text{mean} = \frac{1}{2}(4+i+4-i)$
 $= 4$

A1
AG
[1 mark]

(b) **METHOD 1**

attempts product rule differentiation

(M1)

Note: Award **(M1)** for attempting to express $f(x)$ as $f(x) = x^3 - 9x^2 + 25x - 17$

$$f'(x) = (x-1)(2x-8) + x^2 - 8x + 17 \quad (f'(x) = 3x^2 - 18x + 25)$$

A1

$$f'(4) = 1$$

A1

Note: Where $f'(x)$ is correct, award **A1** for solving $f'(x) = 1$ and obtaining $x = 4$.

EITHER

$$y-3 = 1(x-4)$$

A1

OR

$$y = x + c$$

$$3 = 4 + c \Rightarrow c = -1$$

A1

OR

states the gradient of $y = x - 1$ is also 1 and verifies that $(4, 3)$ lies on the line $y = x - 1$

A1

THEN

so $y = x - 1$ is the tangent to the curve at $A(4, 3)$

AG

METHOD 2

sets $f(x) = x - 1$ to form $x - 1 = (x - 1)(x^2 - 8x + 17)$ (M1)

EITHER

$(x - 1)(x^2 - 8x + 16) = 0$ ($x^3 - 9x^2 + 24x - 16 = 0$) A1

attempts to solve a correct cubic equation (M1)

$$(x - 1)(x - 4)^2 = 0 \Rightarrow x = 1, 4$$

OR

recognises that $x \neq 1$ and forms $x^2 - 8x + 17 = 1$ ($x^2 - 8x + 16 = 0$) A1

attempts to solve a correct quadratic equation (M1)

$$(x - 4)^2 = 0 \Rightarrow x = 4$$

THEN

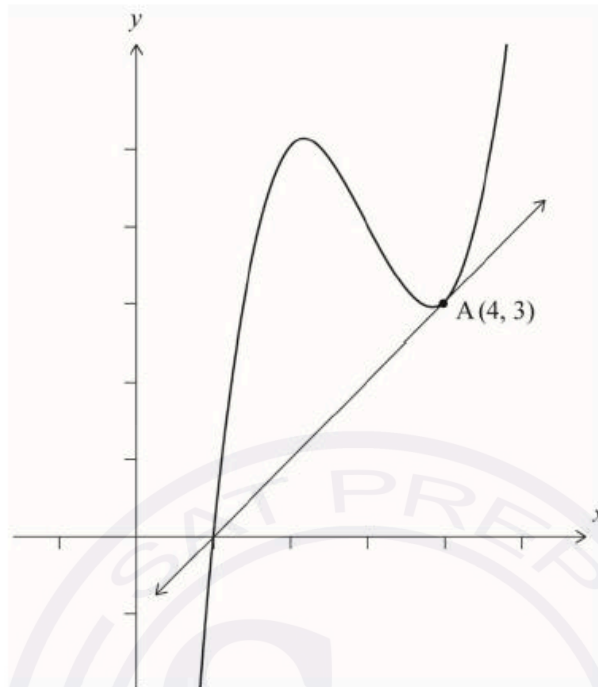
$x = 4$ is a double root R1

so $y = x - 1$ is the tangent to the curve at $A(4, 3)$ AG

Note: Candidates using this method are not required to verify that $y = 3$.

[4 marks]

(c)



a positive cubic with an x -intercept ($x=1$), and a local maximum and local minimum in the first quadrant both positioned to the left of A

A1

Note: As the local minimum and point A are very close to each other, condone graphs that seem to show these points coinciding.
For the point of tangency, accept labels such as A, (4,3) or the point labelled from both axes. Coordinates are not required.

a correct sketch of the tangent passing through A and crossing the x -axis at the same point ($x=1$) as the curve

A1

Note: Award **A1A0** if both graphs cross the x -axis at distinctly different points.

[2 marks]

(d) (i) EITHER

$$g'(x) = (x-r)(2x-2a) + x^2 - 2ax + a^2 + b^2$$

(M1)A1

OR

$$g(x) = x^3 - (2a+r)x^2 + (a^2 + b^2 + 2ar)x - (a^2 + b^2)r$$

attempts to find $g'(x)$

M1

$$g'(x) = 3x^2 - 2(2a+r)x + a^2 + b^2 + 2ar$$

$$= 2x^2 - 2(a+r)x + 2ar + x^2 - 2ax + a^2 + b^2$$

A1

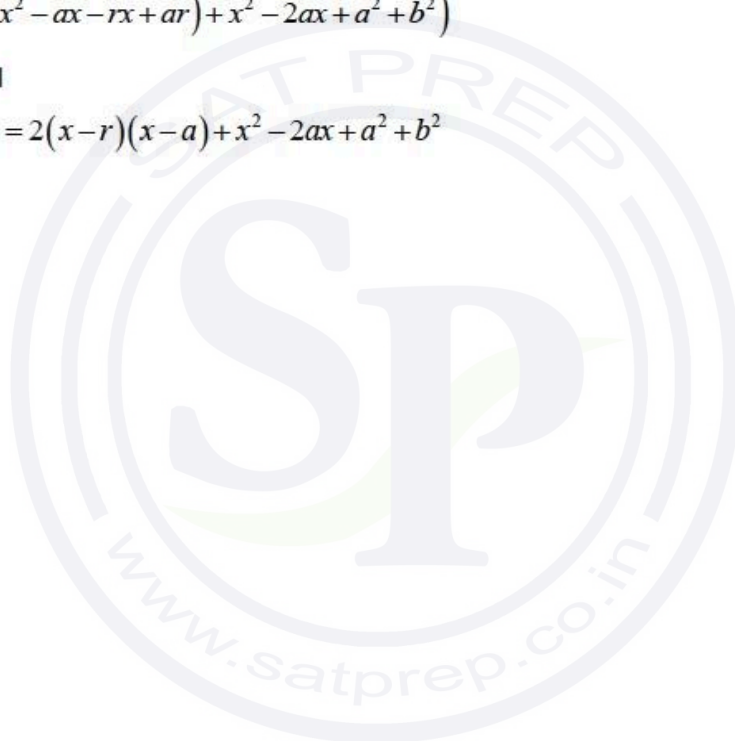
$$= 2(x^2 - ax - rx + ar) + x^2 - 2ax + a^2 + b^2$$

THEN

$$g'(x) = 2(x-r)(x-a) + x^2 - 2ax + a^2 + b^2$$

AG

[2 marks]



(ii) **METHOD 1**

$$g(a) = b^2(a-r) \quad \text{(A1)}$$

$$g'(a) = b^2 \quad \text{(A1)}$$

attempts to substitute their $g(a)$ and $g'(a)$ into $y - g(a) = g'(a)(x - a)$ **M1**

$$y - b^2(a-r) = b^2(x-a)$$

EITHER

$$y = b^2(x-r) \quad (y = b^2x - b^2r) \quad \text{A1}$$

sets $y=0$ so $b^2(x-r) = 0$ **M1**

$$b > 0 \Rightarrow x=r \quad \text{OR} \quad b \neq 0 \Rightarrow x=r \quad \text{R1}$$

OR

sets $y=0$ so $-b^2(a-r) = b^2(x-a)$ **M1**

$$b > 0 \quad \text{OR} \quad b \neq 0 \Rightarrow -(a-r) = x-a \quad \text{R1}$$

$$x=r \quad \text{A1}$$

THEN

so the tangent intersects the x -axis at the point $R(r, 0)$ **AG**

METHOD 2

$$g'(a) = b^2 \quad \text{(A1)}$$

$$g(a) = b^2(a - r) \quad \text{(A1)}$$

attempts to substitute their $g(a)$ and $g'(a)$ into $y = g'(a)x + c$ and attempts to find c

M1

$$c = -b^2r$$

EITHER

$$y = b^2(x - r) \quad (y = b^2x - b^2r) \quad \text{A1}$$

$$\text{sets } y = 0 \text{ so } b^2(x - r) = 0 \quad \text{M1}$$

$$b > 0 \Rightarrow x = r \text{ OR } b \neq 0 \Rightarrow x = r \quad \text{R1}$$

OR

$$\text{sets } y = 0 \text{ so } b^2(x - r) = 0 \quad \text{M1}$$

$$b > 0 \text{ OR } b \neq 0 \Rightarrow x - r = 0 \quad \text{R1}$$

$$x = r \quad \text{A1}$$

METHOD 3

$$g'(a) = b^2 \quad \text{(A1)}$$

the line through $R(r, 0)$ parallel to the tangent at A has equation

$$y = b^2(x - r) \quad \text{A1}$$

$$\text{sets } g(x) = b^2(x - r) \text{ to form } b^2(x - r) = (x - r)(x^2 - 2ax + a^2 + b^2) \quad \text{M1}$$

$$b^2 = x^2 - 2ax + a^2 + b^2, \quad (x \neq r) \quad \text{A1}$$

$$(x - a)^2 = 0 \quad \text{A1}$$

since there is a double root ($x = a$), this parallel line through

$R(r, 0)$ is the required tangent at A **R1**

[6 marks]

(e) EITHER

$$g'(a) = b^2 \Rightarrow b = \sqrt{g'(a)} \text{ (since } b > 0)$$

R1

Note: Accept $b = \pm\sqrt{g'(a)}$.

OR

$$(a \pm bi) = a \pm i\sqrt{b^2} \text{ and } g'(a) = b^2$$

R1

THEN

hence the complex roots can be expressed as $a \pm i\sqrt{g'(a)}$

AG

[1 mark]

(f) (i) $b = 4$ (seen anywhere)

A1

EITHER

attempts to find the gradient of the tangent in terms of a and equates to 16 (M1)

OR

substitutes $r = -2, x = a$ and $y = 80$ to form $80 = (a - (-2))(a^2 - 2a^2 + a^2 + 16)$ (M1)

OR

substitutes $r = -2, x = a$ and $y = 80$ into $y = 16(x - r)$ (M1)

THEN

$$\frac{80}{a+2} = 16 \Rightarrow a = 3$$

roots are -2 (seen anywhere) and $3 \pm 4i$

A1A1

Note: Award A1 for -2 and A1 for $3 \pm 4i$. Do not accept coordinates.

[4 marks]

(ii) $(3, -4)$

A1

Note: Accept " $x = 3$ and $y = -4$ ".

Do not award A1FT for $(a, -4)$.

[1 mark]

(g) (i) $g'(x) = 2(x-r)(x-a) + x^2 - 2ax + a^2 + b^2$

attempts to find $g''(x)$

M1

$$g''(x) = 2(x-a) + 2(x-r) + 2x - 2a \quad (= 6x - 2r - 4a)$$

sets $g''(x) = 0$ and correctly solves for x

A1

for example, obtaining $x - r + 2(x - a) = 0$ leading to $3x = 2a + r$

$$\text{so } x = \frac{1}{3}(2a + r)$$

AG

Note: Do not award **A1** if the answer does not lead to the **AG**.

[2 marks]

(ii) point **P** is $\frac{2}{3}$ of the horizontal distance (way) from point **R** to point **A**

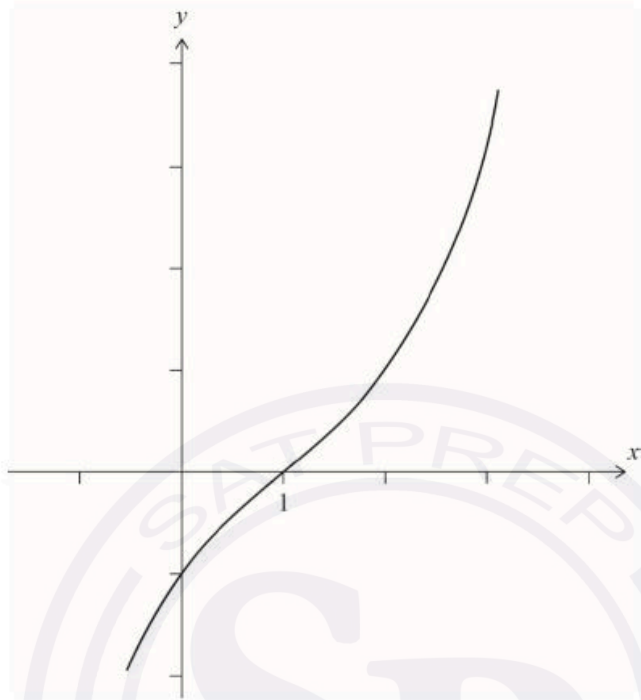
A1

Note: Accept equivalent numerical statements or a clearly labelled diagram displaying the numerical relationship.
Award **A0** for non-numerical statements such as " **P** is between **R** and **A**, closer to **A**".

[1 mark]

(h) (i) $y = (x-1)(x^2 - 2x + 5)$

(A1)



a positive cubic with no stationary points and a non-stationary point of inflexion at $x = 1$

A1

Note: Graphs may appear approximately linear. Award this **A1** if a change of concavity either side of $x = 1$ is apparent. Coordinates are not required and the y -intercept need not be indicated.

[2 marks]

(ii) $(r, 0)$

A1

[1 mark]

Total [28 marks]

Question 4

(a) (i) $P_3(n) = \frac{(3-2)n^2 - (3-4)n}{2}$ OR $P_3(n) = \frac{n^2 - (-n)}{2}$ A1

$$P_3(n) = \frac{n^2 + n}{2} \quad \text{A1}$$

Note: Award **A0A1** if $P_3(n) = \frac{n^2 + n}{2}$ only is seen.

Do not award any marks for numerical verification.

so for triangular numbers, $P_3(n) = \frac{n(n+1)}{2}$ AG

[2 marks]

(ii) **METHOD 1**

uses a table of values to find a positive integer that satisfies $P_3(n) = 351$ (M1)

for example, a list showing at least 3 consecutive terms (... 325, 351, 378...)

Note: Award **(M1)** for use of a GDC's numerical solve or graph feature.

$n = 26$ (26th triangular number) A1

Note: Award **A0** for $n = -27, 26$. Award **A0** if additional solutions besides $n = 26$ are given.

METHOD 2

attempts to solve $\frac{n(n+1)}{2} = 351$ ($n^2 + n - 702 = 0$) for n (M1)

$$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-702)}}{2} \quad \text{OR} \quad (n-26)(n+27) = 0$$

$n = 26$ (26th triangular number) A1

Note: Award **A0** for $n = -27, 26$. Award **A0** if additional solutions besides $n = 26$ are given.

[2 marks]

- (b) (i) attempts to form an expression for $P_3(n) + P_3(n+1)$ in terms of n **M1**

EITHER

$$P_3(n) + P_3(n+1) \equiv \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2}$$

$$\equiv \frac{(n+1)(2n+2)}{2} \left(\equiv \frac{2(n+1)(n+1)}{2} \right)$$

A1

OR

$$P_3(n) + P_3(n+1) \equiv \left(\frac{n^2}{2} + \frac{n}{2} \right) + \left(\frac{(n+1)^2}{2} + \frac{n+1}{2} \right)$$

$$\equiv \left(\frac{n^2 + n}{2} \right) + \left(\frac{n^2 + 2n + 1 + n + 1}{2} \right) \left(\equiv n^2 + 2n + 1 \right)$$

A1

THEN

$$\equiv (n+1)^2$$

AG

[2 marks]

- (ii) the sum of the n th and $(n+1)$ th triangular numbers is the $(n+1)$ th square number

A1

[1 mark]

- (iii)

```

X X X X X
O X X X X
O O X X X
O O O X X
O O O O X

```

A1

Note: Accept equivalent single diagrams, such as the one above, where the 4th and 5th triangular numbers and the 5th square number are clearly shown. Award **A1** for a diagram that show $P_3(4)$ (a triangle with 10 dots) and $P_3(5)$ (a triangle with 15 dots) and $P_4(5)$ (a square with 25 dots).

[1 mark]

(c) **METHOD 1**

$$8P_3(n)+1=8\left(\frac{n(n+1)}{2}\right)+1 (=4n(n+1)+1) \quad \text{A1}$$

attempts to expand their expression for $8P_3(n)+1$ (M1)

$$=4n^2+4n+1$$

$$=(2n+1)^2 \quad \text{A1}$$

and $2n+1$ is odd AG

METHOD 2

$$8P_3(n)+1=8\left((n+1)^2-P_3(n+1)\right)+1 \left(=8\left((n+1)^2-\frac{(n+1)(n+2)}{2}\right)+1\right) \quad \text{A1}$$

attempts to expand their expression for $8P_3(n)+1$ (M1)

$$8(n^2+2n+1)-4(n^2+3n+2)+1 (=4n^2+4n+1)$$

$$=(2n+1)^2 \quad \text{A1}$$

and $2n+1$ is odd AG

METHOD 3

$$8P_3(n)+1=8\left(\frac{n(n+1)}{2}\right)+1 (= (An+B)^2) \text{ (where } A, B \in \mathbb{Z}^+) \quad \text{A1}$$

attempts to expand their expression for $8P_3(n)+1$ (M1)

$$4n^2+4n+1 (=A^2n^2+2ABn+B^2)$$

now equates coefficients and obtains $B=1$ and $A=2$

$$=(2n+1)^2 \quad \text{A1}$$

and $2n+1$ is odd AG

[3 marks]

(d) EITHER

$$u_1 = 1 \text{ and } d = 3 \quad (\text{A1})$$

substitutes their u_1 and their d into $P_5(n) = \frac{n}{2}(2u_1 + (n-1)d)$ M1

$$P_5(n) = \frac{n}{2}(2 + 3(n-1)) \left(= \frac{n}{2}(2 + 3n - 3) \right) \quad (\text{A1})$$

OR

$$u_1 = 1 \text{ and } u_n = 3n - 2 \quad (\text{A1})$$

substitutes their u_1 and their u_n into $P_5(n) = \frac{n}{2}(u_1 + u_n)$ M1

$$P_5(n) = \frac{n}{2}(1 + 3n - 2) \quad (\text{A1})$$

OR

$$P_5(n) = (3(1) - 2) + (3(2) - 2) + (3(3) - 2) + \dots + 3n - 2$$

$$P_5(n) = (3(1) + 3(2) + 3(3) + \dots + 3n) - 2n \quad (= 3(1 + 2 + 3 + \dots + n) - 2n) \quad (\text{A1})$$

substitutes $\frac{n(n+1)}{2}$ into their expression for $P_5(n)$ M1

$$P_5(n) = 3 \left(\frac{n(n+1)}{2} \right) - 2n$$

$$P_5(n) = \frac{n}{2}(3(n+1) - 4) \quad (\text{A1})$$

OR

attempts to find the arithmetic mean of n terms (M1)

$$= \frac{1 + (3n - 2)}{2} \quad (\text{A1})$$

multiplies the above expression by the number of terms n

$$P_5(n) = \frac{n}{2}(1 + 3n - 2) \quad (\text{A1})$$

THEN

$$\text{so } P_5(n) = \frac{n(3n-1)}{2} \quad (\text{AG})$$

[3 marks]

(e) **METHOD 1**

forms a table of $P_3(n)$ values that includes some values for $n > 5$

(M1)

forms a table of $P_5(m)$ values that includes some values for $m > 5$

(M1)

Note: Award (M1) if at least one $P_3(n)$ value is correct. Award (M1) if at least one $P_5(m)$ value is correct. Accept as above for $(n^2 + n)$ values and $(3m^2 - m)$ values.

$n = 20$ for triangular numbers

(A1)

$m = 12$ for pentagonal numbers

(A1)

Note: Award (A1) if $n = 20$ is seen in or out of a table. Award (A1) if $m = 12$ is seen in or out of a table. Condone the use of the same parameter for triangular numbers and pentagonal numbers, for example, $n = 20$ for triangular numbers and $n = 12$ for pentagonal numbers.

210 (is a triangular number and a pentagonal number)

A1

Note: Award all five marks for 210 seen anywhere with or without working shown.

METHOD 2**EITHER**

attempts to express $P_3(n) = P_5(m)$ as a quadratic in n (M1)

$$n^2 + n + (m - 3m^2) = 0 \text{ (or equivalent)}$$

attempts to solve their quadratic in n (M1)

$$n = \frac{-1 \pm \sqrt{12m^2 - 4m + 1}}{2} \left(= \frac{-1 \pm \sqrt{1^2 - 4(m - 3m^2)}}{2} \right)$$

OR

attempts to express $P_3(n) = P_5(m)$ as a quadratic in m (M1)

$$3m^2 - m - (n^2 + n) = 0 \text{ (or equivalent)}$$

attempts to solve their quadratic in m (M1)

$$m = \frac{1 \pm \sqrt{12n^2 + 12n + 1}}{6} \left(= \frac{1 \pm \sqrt{(-1)^2 + 12(n^2 + n)}}{6} \right)$$

THEN

$n = 20$ for triangular numbers (A1)

$m = 12$ for pentagonal numbers (A1)

210 (is a triangular number and a pentagonal number) A1

(f)

Note: Award a maximum of **R1M0M0A1M1A1A1R0** for a 'correct' proof using n and $n+1$.

$$\text{consider } n=1: P_r(1) = 1 + (1-1)(r-2) = 1 \text{ and } P_r(1) = \frac{(r-2)(1^2) - (r-4)(1)}{2} = 1$$

so true for $n=1$

R1

Note: Accept $P_r(1) = 1$ and $P_r(1) = \frac{(r-2)(1^2) - (r-4)(1)}{2} = 1$.

Do not accept one-sided considerations such as ' $P_r(1) = 1$ and so true for $n=1$ '.

Subsequent marks after this **R1** are independent of this mark can be awarded.

$$\text{Assume true for } n=k, \text{ ie. } P_r(k) = \frac{(r-2)k^2 - (r-4)k}{2}$$

M1

Note: Award **M0** for statements such as "let $n=k$ ". The assumption of truth must be clear.

Subsequent marks after this **M1** are independent of this mark and can be awarded.

Consider $n = k + 1$:

$(P_r(k+1))$ can be represented by the sum

$$\sum_{m=1}^{k+1} (1+(m-1)(r-2)) = \sum_{m=1}^k (1+(m-1)(r-2)) + (1+k(r-2)) \text{ and so}$$

$$P_r(k+1) = \frac{(r-2)k^2 - (r-4)k}{2} + (1+k(r-2)) \quad (P_r(k+1) = P_r(k) + (1+k(r-2))) \quad \text{M1}$$

$$= \frac{(r-2)k^2 - (r-4)k + 2 + 2k(r-2)}{2} \quad \text{A1}$$

$$= \frac{(r-2)(k^2 + 2k) - (r-4)k + 2}{2}$$

$$= \frac{(r-2)(k^2 + 2k + 1) - (r-2) - (r-4)k + 2}{2} \quad \text{M1}$$

$$= \frac{(r-2)(k+1)^2 - (r-4)k - (r-4)}{2} \quad \text{(A1)}$$

$$= \frac{(r-2)(k+1)^2 - (r-4)(k+1)}{2} \quad \text{A1}$$

hence true for $n = 1$ and $n = k$ true $\Rightarrow n = k + 1$ true R1

therefore true for all $n \in \mathbb{Z}^+$

Note: Only award the final **R1** if the first five marks have been awarded. Award marks as appropriate for solutions that expand both the LHS and (given) RHS of the equation.

[8 marks]
Total [27 marks]

Question 5

- (a) attempt to expand $(x - \alpha)(x - \beta)(x - \gamma)$ **M1**
- $= (x^2 - (\alpha + \beta)x + \alpha\beta)(x - \gamma)$ **OR** $= (x - \alpha)(x^2 - (\beta + \gamma)x + \beta\gamma)$ **A1**
- $(x^3 + px^2 + qx + r) = x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$ **A1**
- comparing coefficients:
- $p = -(\alpha + \beta + \gamma)$ **AG**
- $q = (\alpha\beta + \beta\gamma + \gamma\alpha)$ **AG**
- $r = -\alpha\beta\gamma$ **AG**

Note: For candidates who do not include the **AG** lines award full marks.

[3 marks]

- (b) (i) $p^2 - 2q = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ **(A1)**
- attempt to expand $(\alpha + \beta + \gamma)^2$ **(M1)**
- $= \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ or equivalent **A1**
- $= \alpha^2 + \beta^2 + \gamma^2$ **AG**

Note: Accept equivalent working from RHS to LHS.

[3 marks]

(ii) **EITHER**

attempt to expand $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2$ (M1)

$$= (\alpha^2 + \beta^2 - 2\alpha\beta) + (\beta^2 + \gamma^2 - 2\beta\gamma) + (\gamma^2 + \alpha^2 - 2\gamma\alpha) \quad \text{A1}$$

$$= 2(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= 2(p^2 - 2q) - 2q \text{ or equivalent} \quad \text{A1}$$

$$= 2p^2 - 6q \quad \text{AG}$$

OR

attempt to write $2p^2 - 6q$ in terms of α, β, γ (M1)

$$= 2(p^2 - 2q) - 2q$$

$$= 2(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \quad \text{A1}$$

$$= (\alpha^2 + \beta^2 - 2\alpha\beta) + (\beta^2 + \gamma^2 - 2\beta\gamma) + (\gamma^2 + \alpha^2 - 2\gamma\alpha) \quad \text{A1}$$

$$= (\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 \quad \text{AG}$$

Note: Accept equivalent working where LHS and RHS are expanded to identical expressions.

[3 marks]

- (c) $p^2 < 3q \Rightarrow 2p^2 - 6q < 0$
 $\Rightarrow (\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 < 0$ A1
 if all roots were real $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 \geq 0$ R1

Note: Condone strict inequality in the **R1** line.

Note: Do not award **A0R1**.

\Rightarrow roots cannot all be real AG
[2 marks]

- (d) $p^2 = (-7)^2 = 49$ and $3q = 51$ A1
 so $p^2 < 3q \Rightarrow$ the equation has at least one complex root R1

Note: Allow equivalent comparisons; e.g. checking $2p^2 < 6q$

[2 marks]

- (e) (i) use of GDC (eg graphs or tables) (M1)
 $q = 12$ A1
[2 marks]

- (ii) complex roots appear in conjugate pairs (so if complex roots occur the other root will be real OR all 3 roots will be real).
 OR
 a cubic curve always crosses the x -axis at at least one point. R1
[1 mark]

(f) (i) attempt to expand $(\alpha + \beta + \gamma + \delta)^2$ (M1)

$$(\alpha + \beta + \gamma + \delta)^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

(A1)

$$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$$(\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + \delta^2 =) p^2 - 2q$$

A1

[3 marks]

(ii) $p^2 < 2q$ OR $p^2 - 2q < 0$

A1

Note: Allow *FT* on their result from part (f)(i).

[1 mark]

(g) $4 < 6$ OR $2^2 - 2 \times 3 < 0$

R1

hence there is at least one complex root.

AG

Note: Allow *FT* from part (f)(ii) for the *R* mark provided numerical reasoning is seen.

[1 mark]

(h) (i) $(p^2 > 2q)$ ($81 > 2 \times 24$) (so) nothing can be deduced

R1

Note: Do not allow **FT** for the **R** mark.

[1 mark]

(ii) -1

A1

[1 mark]

(iii) attempt to express as a product of a linear and cubic factor

M1

$$(x+1)(x^3 - 10x^2 + 34x - 12)$$

A1A1

Note: Award **A1** for each factor. Award at most **A1A0** if not written as a product.

since for the cubic, $p^2 < 3q$ ($100 < 102$)

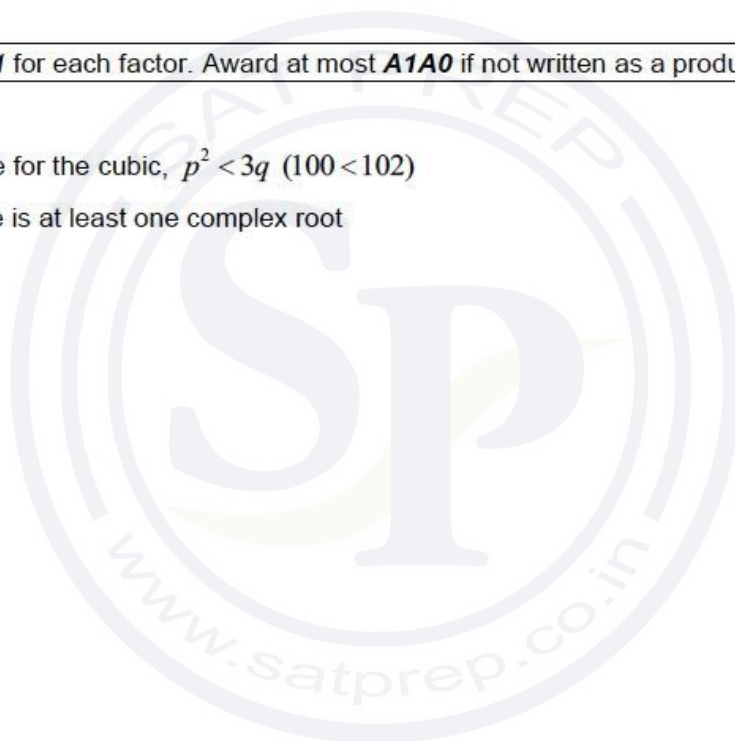
R1

there is at least one complex root

AG

[4 marks]

[Total: 27 marks]



Question 6

(a) EITHER

$$S_n = \frac{n}{2}(2 \times 1 + (n-1) \times 1)$$

A1

OR

$u_1 = 1$ and either $u_n = n$ or $d = 1$ stated explicitly

A1

OR

$1 + 2 + \dots + n$ (or equivalent) stated explicitly

A1

THEN

$$S_n = \frac{n}{2}(1+n)$$

AG

Note: Award **A0** for a numerical verification.

[1 mark]

(b) (i) 14

A1

[1 mark]

(ii) $a_1 + a_2 + a_3 = 1$

A1

$$2a_1 + 4a_2 + 8a_3 = 5$$

A1

$$3a_1 + 9a_2 + 27a_3 = 14$$

A1

Note: For the third **A** mark, award **A1FT** for $3a_1 + 9a_2 + 27a_3 = p$ where p is their answer to part (b) (i).

[3 marks]

(iii) $a_1 = \frac{1}{6}$ ($= 0.166666... \approx 0.167$), $a_2 = \frac{1}{2}$ ($= 0.5$),
 $a_3 = \frac{1}{3}$ ($= 0.333333... \approx 0.333$)

A2

Note: Award **A1** if only two of a_1, a_2, a_3 are correct.

Only award **FT** if three linear equations, each in a_1, a_2 and a_3 are stated in part (b) (ii) or (iii).

Award **A2FT** for their a_1, a_2 and a_3 .

Award **A1FT** for their a_1, a_2 and $a_3 = 0$.

[2 marks]

(c) $f'(x) = 1 + 2x + 3x^2 + \dots + nx^{n-1}$

A1

Note: Award **A1** for $f'(x) = \sum_{i=1}^n ix^{i-1}$.

$\Rightarrow x f'(x) = x + 2x^2 + 3x^3 + \dots + nx^n$

AG

[1 mark]

(d) (i) **METHOD 1**

$$f_2(x) = xf_1'(x)$$

$$f_1'(x) = 1^2 + 2^2x + (3^2x^2) + \dots + n^2x^{n-1} \quad (= 1 + 4x + (9x^2) + \dots + n^2x^{n-1}) \quad \mathbf{A1}$$

Note: Award **A1** for

$$xf_1'(x) = x(1^2 + 2^2x + (3^2x^2) + \dots + n^2x^{n-1}) \quad (= x(1 + 4x + (9x^2) + \dots + n^2x^{n-1}))$$

$$xf_1'(x) = 1^2x + 2^2x^2 + (3^2x^3) + \dots + n^2x^n \quad (= x + 4x^2 + (9x^3) + \dots + n^2x^n) \quad \mathbf{A1}$$

Note: Award **A1** for $f_1'(x) = \sum_{i=1}^n i^2x^{i-1}$ and **A1** for $xf_1'(x) = x \sum_{i=1}^n i^2x^{i-1}$.

The second **A1** is dependent on the first **A1**.

Award a maximum of **A0A1** if a general term is not considered.

$$= \sum_{i=1}^n i^2x^i \quad \mathbf{AG}$$

METHOD 2

$$f_2(x) = x \frac{d}{dx}(xf_1'(x))$$

$$= x(f_1'(x) + xf_1''(x)) \quad (= xf_1'(x) + x^2f_1''(x)) \quad \mathbf{A1}$$

$$= x \sum_{i=1}^n ix^{i-1} + x^2 \sum_{i=1}^n i(i-1)x^{i-2}$$

$$= \sum_{i=1}^n ix^i + \sum_{i=1}^n i(i-1)x^i \quad \left(= \sum_{i=1}^n (i+i^2-i)x^i \right) \quad \mathbf{A1}$$

$$= \sum_{i=1}^n i^2x^i \quad \mathbf{AG}$$

[2 marks]

(ii) consider $q = 1$

$$f_1(x) = x + 2x^2 + \dots + nx^n \text{ (reference to part (c)) and } f_1(x) = \sum_{i=1}^n ix^i$$

R1

$$\text{assume true for } q = k, (f_k(x) = \sum_{i=1}^n i^k x^i)$$

M1

Note: Do not award **M1** for statements such as “let $q = k$ ” or “ $q = k$ is true”.
Subsequent marks after this **M1** are independent of this mark and can be awarded.

consider $q = k + 1$

$$f_{k+1}(x) = xf'_k(x)$$

M1

$$= x \sum_{i=1}^n i^{k+1} x^{i-1} \text{ OR } x(1 + 2^{k+1}x + 3^{k+1}x^2 + \dots + n^{k+1}x^{n-1})$$

A1

Note: Award the above **M1** if $f_{k+1}(x) = x \sum_{i=1}^n i^{k+1} x^{i-1}$ or $xf'_k(x) = x \sum_{i=1}^n i^{k+1} x^{i-1}$ (or equivalent) is stated.

$$= \sum_{i=1}^n i^{k+1} x^i \text{ OR } x + 2^{k+1}x^2 + 3^{k+1}x^3 + \dots + n^{k+1}x^n$$

A1

since true for $q = 1$ and true for $q = k + 1$ if true for $q = k$, hence true for all $q (\in \mathbb{Z}^+)$

R1

Note: To obtain the final **R1**, three of the previous five marks must have been awarded.

[6 marks]

(iii) $f_q(1) = 1^q + 2^q + 3^q + \dots + n^q$

$$= \sum_{i=1}^n i^q \left(= \sum_{i=1}^n 1^i i^q \right)$$

A1

[1 mark]

(e) uses $S_n = \frac{u_1(r^n - 1)}{r - 1}$ with $r = x$ and $u_1 = 1$

M1

clear indication there are $(n + 1)$ terms

R1

$$f(x) = \frac{x^{n+1} - 1}{x - 1}$$

AG

[2 marks]

(f) **METHOD 1**

$$f_1(x) = xf'(x)$$

$$= x \frac{(x-1)(n+1)x^n - 1 \times (x^{n+1} - 1)}{(x-1)^2}$$

M1A1

Note: Award **M1** for attempting to use the quotient or the product rule to find $f'(x)$.

$$= x \frac{(nx + x - n - 1)x^n - (x^{n+1} - 1)}{(x-1)^2} \left(= x \frac{nx^{n+1} - nx^n - x^n + 1}{(x-1)^2} \right)$$

A1

Note: Award **A1** for any correct manipulation of the derivative that leads to the **AG**.

$$= \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$$

AG

METHOD 2

attempts to form $(x-1)f_1(x)$

M1

$$(x-1)f_1(x) = nx^{n+1} - (x + x^2 + x^3 + \dots + x^n)$$

$$f_1(x) = \frac{1}{x-1} \left(nx^{n+1} - \left(\frac{x^{n+1} - 1}{x-1} - 1 \right) \right)$$

A1

$$f_1(x) = \frac{1}{x-1} \left(\frac{nx^{n+1}(x-1) - x^{n+1} + x}{x-1} \right) \left(= \frac{1}{x-1} \left(\frac{nx^{n+2} - nx^{n+1} - x^{n+1} + x}{x-1} \right) \right)$$

A1

$$f_1(x) = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$$

AG

[3 marks]

(g) (i) $\lim_{x \rightarrow 1} f_1(x) = \frac{n - (n+1) + 1}{0} \left(= \frac{0}{0} \right)$

R1

Note: Only award **R1** for sufficient simplification of the numerator, for example, as shown above.

Do not award **R1** if $\lim_{x \rightarrow 1}$ is not referred to or stated.

[1 mark]



(ii) attempts to differentiate both the numerator and the denominator

M1

$$\lim_{x \rightarrow 1} \frac{n(n+2)x^{n+1} - (n+1)^2 x^n + 1}{2(x-1)}$$

A1

Note: Award **A1** for $\left(\lim_{x \rightarrow 1} \frac{n(n+2)x^{n+1} - n(n+1)x^n - (n+1)x^n + 1}{2(x-1)} \right)$. This form can be used in subsequent work.

(l'Hôpital's rule applies again since)

$$\lim_{x \rightarrow 1} \frac{n(n+2)x^{n+1} - (n+1)^2 x^n + 1}{2(x-1)} = \frac{0}{0}$$

R1

Note: Do not award **R1** if $\lim_{x \rightarrow 1}$ is not referred to or stated.

Subsequent marks are independent of this **R** mark.

attempts to differentiate both the numerator and the denominator

M1

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{n(n+2)(n+1)x^n - n(n+1)^2 x^{n-1}}{2} \\ &= \frac{n(n+2)(n+1) - n(n+1)^2}{2} \left(= \frac{n^3 + 3n^2 + 2n - (n^3 + 2n^2 + n)}{2} \right) \\ &= \frac{n(n+1)((n+2) - (n+1))}{2} \left(= \frac{n^2 + n}{2} \right) \\ &= \frac{1}{2}n(n+1) \end{aligned}$$

A1

AG

[5 marks]

Total [28 marks]

Question 7

(a) $m = 2, c = -1$

$$r = \frac{1}{2}$$

(A1)

$$2, \frac{1}{2}, -1$$

EITHER

$$d \left(= \frac{1}{2} - 2 = -1 - \frac{1}{2} \right) = -\frac{3}{2}$$

A1

OR

this sequence has a common difference of $-\frac{3}{2}$

A1

OR

the (arithmetic) mean of 2 and -1 is $\frac{1}{2}$

A1

THEN

hence $L(x) = 2x - 1$ is an AS-linear function

AG

[2 marks]

(b) (i) $(L(r) = 0 \Rightarrow) mr + c = 0$

A1

$$r = -\frac{c}{m}$$

AG

Note: Award **A0** for numerical verification from $L(x) = 2x - 1$ in part (a).

[1 mark]

(ii) **METHOD 1**

EITHER

attempts to use $(d =) r - m = c - r$

(M1)

Note: Award **(M1)** for attempting to use $(d =) m - r = r - c$.

$$(d =) -\frac{c}{m} - m = c - \left(-\frac{c}{m}\right)$$

A1

Note: Award **A1** for $(d =) -\frac{c}{m} - m = \frac{c - m}{2}$.

removes the denominator m from their expression involving m and c

(M1)

$$m^2 + cm + 2c = 0 \text{ (or equivalent)}$$

OR

attempts to use $\frac{m + c}{2} = r$

(M1)

$$m + c = -\frac{2c}{m}$$

A1

removes the denominator m from their expression involving m and c

(M1)

$$m^2 + cm + 2c = 0 \text{ (or equivalent)}$$

OR

attempts to use $c = m + 2d$

(M1)

$$c = m + 2\left(-\frac{c}{m} - m\right)$$

A1

Note: Award **A1** for $c = m + 2\left(c - \left(-\frac{c}{m}\right)\right)$.

removes the denominator m from their expression involving m and c

(M1)

$$m^2 + cm + 2c = 0 \text{ (or equivalent)}$$

OR

attempts to use $r = m + d$ and $c = m + 2d$ ($c = m + 2(r - m)$) **(M1)**

$m^2 + dm + m + 2d = 0$ (or equivalent) **A1**

substitutes $d = \frac{c - m}{2}$ into their expression involving m and d **(M1)**

$m^2 + cm + 2c = 0$ (or equivalent)

THEN

$c(m + 2) = -m^2 \Rightarrow c = -\frac{m^2}{m + 2}$ **A1**

Note: Award **A1** for a convincing demonstration that $c = -\frac{m^2}{m + 2}$.

so $L(x) = mx - \frac{m^2}{m + 2}$ **AG**

Note: Do not accept working backwards from the **AG**.

METHOD 2

considers $L(x) = mx - mr$

attempts to use $(d =)r - m = c - r$ **(M1)**

Note: Award **(M1)** for attempting to use $(d =)m - r = r - c$.

$(d =)r - m = -mr - r$ **A1**

attempts to express r in terms of m **(M1)**

$2r + mr = m \Rightarrow r = \frac{m}{m + 2}$ **A1**

so $L(x) = mx - \frac{m^2}{m + 2}$ **AG**

Note: Do not accept working backwards from the **AG**.

[4 marks]

(iii) $m \neq -2$ ($m \neq 0$)

A1

[1 mark]

(c)

attempts to find an integer value of m

(M1)

e.g. uses the result that $m+2$ exactly divides 2 OR uses a table OR uses a graph and slider OR uses systematic trial and error

Note: Award (M1) for solving $m^2 = k(m+2)$ for m or solving $mr - \frac{m^2}{m+2} = 0$ for m or solving $m^2 + cm + 2c = 0$ for m .

$m = -4$ OR $m = -3$

(A1)

$-4, 2, 8$ OR $-3, 3, 9$

$L(x) = -4x + 8, L(x) = -3x + 9$

A1

Note: Award (M1)(A1)A0 for $-4x+8$ and $-3x+9$.

[3 marks]

(d) (i) $-\frac{b}{a}$

A1

[1 mark]

(ii) $\frac{c}{a}$

A1

[1 mark]

(e) (i) $b - a$

A1

Note: Award marks as appropriate in parts (e) (ii) and (iii) for use of $r_1 - r_2 = a - b$.

[1 mark]

(ii) attempts to eliminate r_2

M1

$$2r_1 = -\frac{b}{a} - (b - a) \Rightarrow 2r_1 = \frac{a^2 - ab - b}{a} \text{ (or equivalent)}$$

A1

Note: Award **A1** for a correct alternative form of $\pm r_1$ or $\pm 2r_1$.

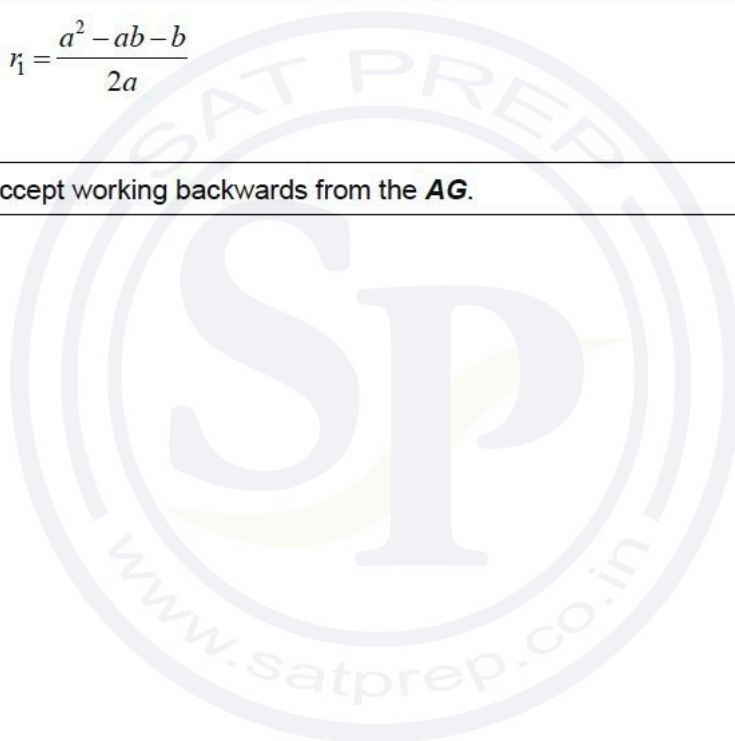
$$\text{so } r_1 = \frac{a^2 - ab - b}{2a}$$

AG

Note: Do not accept working backwards from the **AG**.

[2 marks]

\



(iii) **METHOD 1**

EITHER

$$(r_1 =) \frac{a+b}{2} \quad \text{(A1)}$$

attempts to equate two expressions for either r_1 or $2r_1$ in terms of a and b **M1**

$$\frac{a+b}{2} = \frac{a^2-ab-b}{2a} \quad \text{OR} \quad a+b = \frac{a^2-ab-b}{a}$$

OR

$$b-r_1 = r_1 - a \quad \text{(A1)}$$

attempts to use $b-r_1 = r_1 - a$ with $r_1 = \frac{a^2-ab-b}{2a}$ **M1**

$$b - \left(\frac{a^2-ab-b}{2a} \right) = \frac{a^2-ab-b}{2a} - a$$

OR

$$(r_1 =) a+d \quad \text{(A1)}$$

attempts to use $r_1 = a+d$ with $r_1 = \frac{a^2-ab-b}{2a}$ and $d = \frac{b-a}{2}$ **M1**

$$\frac{a^2-ab-b}{2a} = a + \frac{b-a}{2}$$

THEN

$$2a^2 + 2ab = 2a^2 - 2ab - 2b \quad \text{OR} \quad a^2 + ab = a^2 - ab - b$$

$$4ab + 2b = 0 \quad \text{OR} \quad 2ab + b = 0$$

$$2b(2a+1) = 0 \quad \text{OR} \quad b(2a+1) = 0 \quad \text{A1}$$

Note: Award **(A1)M1** for any valid approach that correctly leads to $2ab+b=0$ (or equivalent).
Do not accept numerical verification from the **AG**.

$$\text{so } b=0 \text{ or } a = -\frac{1}{2} \quad \text{AG}$$

METHOD 2

$$(b =) a + 2d \text{ OR } (r_1 =) a + d$$

(A1)

attempts to equate two expressions for either r_1 or $2r_1$ in terms of a and d

M1

$$a + d = \frac{a^2 - a(a + 2d) - (a + 2d)}{2a} \text{ OR } 2(a + d) = \frac{a^2 - a(a + 2d) - (a + 2d)}{a}$$

$$2a^2 + 4ad + a + 2d = 0$$

$$(2a + 1)(a + 2d) = 0$$

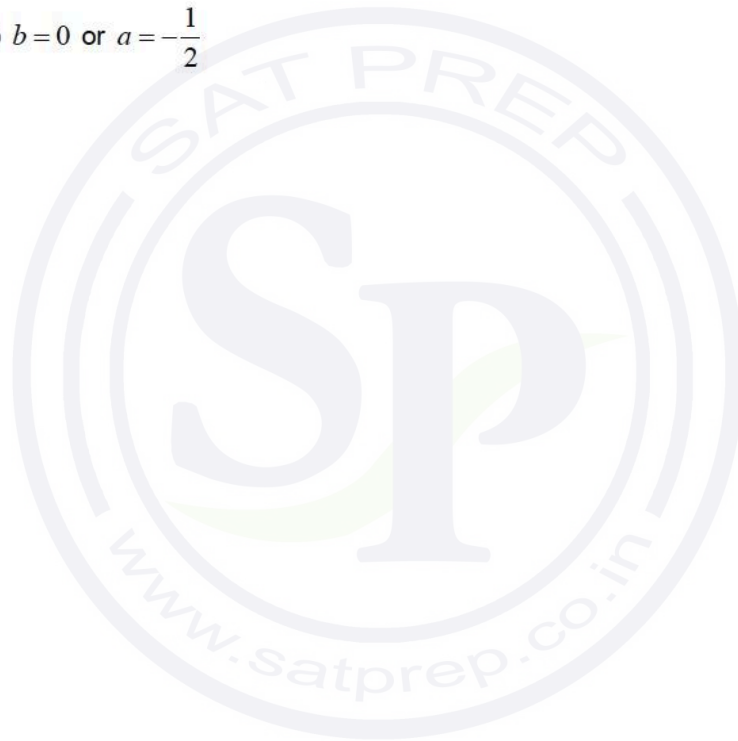
A1

Note: Do not accept numerical verification from the **AG**.

$$\text{so } b = 0 \text{ or } a = -\frac{1}{2}$$

AG

[3 marks]



(f) **METHOD 1**

$$r_1 = \frac{a}{2} \text{ OR } r_2 = -\frac{a}{2} \text{ OR } d = -\frac{a}{2} \quad (\text{A1})$$

$$c = -a \quad (\text{A1})$$

attempts to find the values of a (M1)

EITHER

the roots of $ax^2 - a = 0$ are ± 1 and $\frac{a}{2} = \pm 1$

OR

substitutes $x = \pm \frac{a}{2}$ into $ax^2 - a = 0$ giving $\frac{a^3}{4} - a = 0$

OR

$$(r_1 r_2) \frac{c}{a} = -\frac{a^2}{4} \Rightarrow c = -\frac{a^3}{4} \text{ and so } -a = -\frac{a^3}{4} \Rightarrow \frac{a^3}{4} - a = 0$$

Note: Award (M1) for attempting to find the values of a from their arithmetic sequence expressed in terms of a .

OR

$$c - r_2 = r_2 - b \Rightarrow -\frac{a^3}{4} - \left(-\frac{a}{2}\right) = -\frac{a}{2} \Rightarrow \frac{a^3}{4} - a = 0$$

THEN

$$a = \pm 2 \quad (\text{A1})$$

$$(r_1 = \pm 1, b = 0, r_2 = \mp 1, c = \mp 2)$$

$$\text{so } Q(x) = 2x^2 - 2, Q(x) = -2x^2 + 2 \quad \text{A1}$$

Note: Award A0 for $2x^2 - 2$ and $-2x^2 + 2$.
Award (A1)(A1)(M1)(A0)A0 for obtaining either $a = 2$ or $a = -2$.

METHOD 2

$$r_1 = -d \text{ OR } r_2 = d \text{ OR } a = -2d$$

(A1)

$$c = 2d$$

(A1)attempts to find the values of d **(M1)****EITHER**the roots of $-2dx^2 + 2d = 0$ are ± 1 **OR**substitutes $x = \pm d$ into $-2dx^2 + 2d = 0$ giving $-2d^3 + 2d = 0$ **OR**attempts to use $r_1 r_2 = \frac{c}{a}$ to form $-d^2 = \frac{2d}{-2d}$

Note: Award **(M1)** for attempting to find the values of d from their arithmetic sequence expressed in terms of d .

THEN

$$d = \pm 1$$

(A1)

$$(a = \pm 2, r_1 = \pm 1, b = 0, r_2 = \mp 1, c = \mp 2)$$

$$\text{so } Q(x) = 2x^2 - 2, \quad Q(x) = -2x^2 + 2$$

A1

Note: Award **A0** for $2x^2 - 2$ and $-2x^2 + 2$.
Award **(A1)(A1)(M1)(A0)A0** for obtaining either $d = 1$ or $d = -1$.

METHOD 3

$a = 2r_1$ OR $r_2 = -r_1$ OR $d = -r_1$ (A1)

$c = -2r_1$ (A1)

attempts to find the values of r_1 (M1)

EITHER

the roots of $2r_1x^2 - 2r_1 = 0$ are ± 1

OR

substitutes $x = \pm r_1$ into $2r_1x^2 - 2r_1 = 0$ giving $2r_1^3 - 2r_1 = 0$

OR

attempts to use $r_1r_2 = \frac{c}{a}$ to form $-r_1^2 = \frac{-2r_1}{2r_1}$

Note: Award (M1) for attempting to find the values of r_1 from their arithmetic sequence expressed in terms of r_1 .

THEN

$r_1 = \pm 1$ (A1)

$(a = \pm 2, r_1 = \pm 1, b = 0, r_2 = \mp 1, c = \mp 2)$

so $Q(x) = 2x^2 - 2, Q(x) = -2x^2 + 2$ A1

Note: Award A0 for $2x^2 - 2$ and $-2x^2 + 2$.
Award (A1)(A1)(M1)(A0)A0 for obtaining either $r_1 = 1$ or $r_1 = -1$.

[5 marks]

(g) (i) attempts to express r_1 in terms of b with $a = -\frac{1}{2}$

(M1)

Note: Do not award **(M1)** if $a = \frac{1}{2}$ is used.

EITHER

uses $r_1 = \frac{a+b}{2}$

OR

uses $r_1 = \frac{a^2 - ab - b}{2a}$

OR

uses $r_1 - a = b - r_1$

THEN

$$r_1 = \frac{2b-1}{4} \left(= \frac{b}{2} - \frac{1}{4}, = \frac{b-\frac{1}{2}}{2} \right)$$

A1

[2 marks]

(ii) **METHOD 1**

EITHER

substitutes their expression for r_1 with $a = -\frac{1}{2}$ into $Q(x)(=0)$ **(M1)**

$$Q\left(\frac{2b-1}{4}\right)(=0) \Rightarrow -\frac{1}{2}\left(\frac{2b-1}{4}\right)^2 + b\left(\frac{2b-1}{4}\right) + c(=0)$$

OR

$$r_2 = \frac{6b+1}{4} \left(= \frac{3b}{2} + \frac{1}{4} \right)$$

substitutes their expression for r_2 with $a = -\frac{1}{2}$ into $Q(x)(=0)$ **(M1)**

$$Q\left(\frac{6b+1}{4}\right)(=0) \Rightarrow -\frac{1}{2}\left(\frac{6b+1}{4}\right)^2 + b\left(\frac{6b+1}{4}\right) + c(=0)$$

THEN

$$c = \frac{4b+1}{2} \left(= 2b + \frac{1}{2} \right) \text{ (seen anywhere)} \quad \mathbf{A1}$$

$$4b^2 + 20b + 5 = 0$$

attempts to solve their quadratic in b **(M1)**

$$b = \frac{-5 \pm 2\sqrt{5}}{2} \quad \mathbf{A1}$$

substitutes into $c = \frac{4b+1}{2}$

$$c = \frac{-9 \pm 4\sqrt{5}}{2} \quad \mathbf{A1}$$

Note: Award **A0A0** for b and c expressed as decimal values.

Note: Award a maximum of **(M1)A1(M1)A0A0FT** for **FT** from part (g) (i).

METHOD 2

substitutes their expressions for r_1 and r_2 with $a = -\frac{1}{2}$ into $Q(x)$ **(M1)**

$$-\frac{1}{2}\left(x - \left(\frac{2b-1}{4}\right)\right)\left(x - \left(\frac{6b+1}{4}\right)\right)$$

$$-\frac{1}{2}x^2 + bx - \frac{3}{8}b^2 + \frac{1}{8}b + \frac{1}{32}$$

$$c = \frac{4b+1}{2} \left(= 2b + \frac{1}{2} \right) \text{ (seen anywhere)} \quad \mathbf{A1}$$

$$2b + \frac{1}{2} = -\frac{3}{8}b^2 + \frac{1}{8}b + \frac{1}{32}$$

$$4b^2 + 20b + 5 = 0$$

attempts to solve their quadratic in b **(M1)**

$$b = \frac{-5 \pm 2\sqrt{5}}{2} \quad \mathbf{A1}$$

substitutes into $c = \frac{4b+1}{2}$

$$c = \frac{-9 \pm 4\sqrt{5}}{2} \quad \mathbf{A1}$$

Note: Award **A0A0** for b and c expressed as decimal values.

Note: Award a maximum of **(M1)A1(M1)A0A0FT** for **FT** from part (g) (i).

METHOD 3

$$r_2 = \frac{6b+1}{4} \left(= \frac{3b}{2} + \frac{1}{4} \right)$$

substitutes their expressions for r_1 and r_2 with $a = -\frac{1}{2}$ into $r_1 r_2 = \frac{c}{a}$ **(M1)**

$$\left(\frac{2b-1}{4} \right) \left(\frac{6b+1}{4} \right) = \frac{c}{-\frac{1}{2}} \text{ (or equivalent)}$$

EITHER

$$c = \frac{4b+1}{2} \left(= 2b + \frac{1}{2} \right) \text{ (seen anywhere)} \quad \mathbf{A1}$$

OR

$$-\frac{1}{2} \left(\frac{2b-1}{4} \right) \left(\frac{6b+1}{4} \right) - \left(\frac{6b+1}{4} \right) = \frac{2b-1}{4} - \left(-\frac{1}{2} \right) \quad \mathbf{A1}$$

THEN

$$4b^2 + 20b + 5 = 0$$

attempts to solve their quadratic in b **(M1)**

$$b = \frac{-5 \pm 2\sqrt{5}}{2} \quad \mathbf{A1}$$

substitutes into $c = \frac{4b+1}{2}$

$$c = \frac{-9 \pm 4\sqrt{5}}{2} \quad \mathbf{A1}$$

Note: Award **A0A0** for b and c expressed as decimal values.

Note: Award a maximum of **(M1)A1(M1)A0A0FT** for **FT** from part (g) (i).

METHOD 4

attempts to equate two expressions for r_1 with $a = -\frac{1}{2}$ **(M1)**

$$\frac{-b \pm \sqrt{b^2 + 2c}}{-1} = \frac{2b-1}{4} \left(\pm \sqrt{b^2 + 2c} = \frac{2b+1}{4} \right)$$

$$c = \frac{4b+1}{2} \left(= 2b + \frac{1}{2} \right) \text{ (seen anywhere)} \quad \text{A1}$$

$$12b^2 - 4b - 1 + 32 \left(2b + \frac{1}{2} \right) = 0 \quad (4b^2 + 20b + 5 = 0)$$

attempts to solve their quadratic in b **(M1)**

$$b = \frac{-5 \pm 2\sqrt{5}}{2} \quad \text{A1}$$

substitutes into $c = \frac{4b+1}{2}$

$$c = \frac{-9 \pm 4\sqrt{5}}{2} \quad \text{A1}$$

Note: Award **A0A0** for b and c expressed as decimal values.

Note: Award a maximum of **(M1)A1(M1)A0A0FT** for **FT** from part (g) (i).

METHOD 6

$$r_1 = d - \frac{1}{2} \text{ and } r_2 = 3d - \frac{1}{2}$$

substitutes their expressions for r_1 and r_2 in terms of d with $a = -\frac{1}{2}$ into $r_1 r_2 = \frac{c}{a}$ **(M1)**

$$\left(d - \frac{1}{2}\right)\left(3d - \frac{1}{2}\right) = \frac{c}{-\frac{1}{2}} \text{ (or equivalent)}$$

$$c = 4d - \frac{1}{2} \text{ (seen anywhere)}$$

A1

$$4d^2 + 8d - 1 = 0$$

attempts to solve their quadratic in d

(M1)

$$d = \frac{-2 \pm \sqrt{5}}{2}$$

$$b = \frac{-5 \pm 2\sqrt{5}}{2}$$

A1

substitutes into $c = \frac{4b+1}{2}$

$$c = \frac{-9 \pm 4\sqrt{5}}{2}$$

A1

Note: Award **A0A0** for b and c expressed as decimal values.

Note: Award a maximum of **(M1)A1(M1)A0A0FT** for **FT** from part (g) (i).

[5 marks]

Total [31 marks]

Question 8

- (a) varies the value of b with $a = 3$ **(M1)**

Note: The **(M1)** in part (a) can also be awarded for a correct answer to either part (b)(i) or (b)(ii).

Award **(M1)** for evidence that $b = 0$ case is considered/determined.

$b = -4, 0$ **A1**

[2 marks]

- (b) (i) $b < -4$ or $b > 0$ **A1**

[1 mark]

(ii) $-4 < b < 0$ **A1**

[1 mark]

- (c) (i) $b = 0, 4$ **A1**

[1 mark]

(ii) $b < 0$ or $b > 4$ **A1**

[1 mark]

(iii) $0 < b < 4$ **A1**

[1 mark]

(d) one point of zero gradient is located on either side (of the x -axis) (or equivalent) **A1**

[1 mark]

(e) **METHOD 1**

$$\frac{dy}{dx} = 3x^2 + 2ax \quad \text{(A1)}$$

attempts to solve their $\frac{dy}{dx} = 0$ for x **M1**

$$x(3x + 2a) = 0 \text{ OR } x = \frac{-2a \pm \sqrt{4a^2}}{6} \text{ OR } x + \frac{a}{3} = \pm \frac{a}{3}$$

$$x = -\frac{2}{3}a, 0 \quad \text{A1}$$

when $x = 0$, $y = b$ and so $P(0, b)$ is a point of zero gradient **AG**

Note: The following two marks are independent of the first three marks.

substitutes their expression for x in terms of a into $y = x^3 + ax^2 + b$ **(M1)**

$$y = \left(-\frac{2}{3}a\right)^3 + a\left(-\frac{2}{3}a\right)^2 + b$$

$$y = -\frac{8}{27}a^3 + \frac{4}{9}a^3 + b \left(y = -\frac{8}{27}a^3 + \frac{12}{27}a^3 + b \right) \quad \text{A1}$$

so $Q\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$ is a point of zero gradient **AG**

[5 marks]

METHOD 2

$$\frac{dy}{dx} = 3x^2 + 2ax \quad \text{(A1)}$$

substitutes either $x=0$ or $x = -\frac{2}{3}a$ into their $\frac{dy}{dx}$ **M1**

when $x=0$, $\frac{dy}{dx} = 0$ and $y = b$ so $P(0, b)$ is a point of zero gradient **AG**

$$\begin{aligned} \frac{dy}{dx} &= 3\left(-\frac{2}{3}a\right)^2 + 2a\left(-\frac{2}{3}a\right) \\ &= \frac{4}{3}a^2 - \frac{4}{3}a^2 (=0) \left(= 3\left(\frac{4}{9}a^2\right) - \frac{4}{3}a^2 (=0), = \frac{12}{9}a^2 - \frac{4}{3}a^2 (=0)\right) \end{aligned} \quad \text{A1}$$

and so $\frac{dy}{dx} = 0$ when $x = -\frac{2}{3}a$ **AG**

Note: The following two marks are independent of the first three marks.
--

substitutes $x = -\frac{2}{3}a$ into $y = x^3 + ax^2 + b$ **(M1)**

$$\begin{aligned} y &= \left(-\frac{2}{3}a\right)^3 + a\left(-\frac{2}{3}a\right)^2 + b \\ y &= -\frac{8}{27}a^3 + \frac{4}{9}a^3 + b \left(y = -\frac{8}{27}a^3 + \frac{12}{27}a^3 + b\right) \end{aligned} \quad \text{A1}$$

so $Q\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$ is a point of zero gradient **AG**

[5 marks]

(f) (i) $\frac{d^2y}{dx^2} = 6x + 2a$ **A1**

when $x = 0$, $\frac{d^2y}{dx^2} = 2a$ ($a > 0$) and so (P) is a (local) minimum (point) **R1**

when $x = -\frac{2}{3}a$, $\frac{d^2y}{dx^2} = -2a$ ($a > 0$) and so (Q) is a (local) maximum (point) **R1**

[3 marks]

(ii) (P and Q are) both above (the x - axis) **A1**

Note: Award **A1** if it is made clear that both points are above (the x - axis).
Accept a labelled sketch that clearly shows this information.

[1 mark]

(g) (i) (P) is a (local) maximum (point) and is above (the x - axis) **A1**

[1 mark]

(ii) (Q is below the x - axis when) $\frac{4}{27}a^3 + b < 0$ **A1**

Note: Award **A1** for an equivalent correct inequality, eg. $\frac{4}{27}a^3 < -b$.
Accept a labelled sketch that clearly shows this information.

[1 mark]

(h) **METHOD 1**

attempts to factorize $4a^3b + 27b^2 (< 0)$ **(M1)**

$27b\left(\frac{4}{27}a^3 + b\right) (< 0)$ OR $b(4a^3 + 27b) (< 0)$ **A1**

$b > 0$ and $\frac{4}{27}a^3 + b < 0$ or $b < 0$ and $\frac{4}{27}a^3 + b > 0$ **A1**

Note: Only award this **A1** if both cases are stated.

Award **A1** for stating that exactly one of b and $\frac{4}{27}a^3 + b$ is less than zero (or equivalent).

when b and $\frac{4}{27}a^3 + b$ have opposite sign, P and Q are located on either side (of the x -axis) (or equivalent) **R1**

Note: Accept labelled sketches that clearly show this information.

P and Q are located on either side (of the x -axis) if (and only if) the curve has exactly three x -axis intercepts **R1**

if $4a^3b + 27b^2 < 0$, then the graph of $y = x^3 + ax^2 + b$ has exactly three x -axis intercepts **AG**

Note: For proving the converse, award a maximum of 3 marks (likely to be similar steps but presented in reverse; 2nd **A1** line not necessary in reverse method).

METHOD 2

attempts to factorize $4a^3b + 27b^2 (< 0)$ **(M1)**

$27b\left(\frac{4}{27}a^3 + b\right) (< 0)$ OR $b(4a^3 + 27b) (< 0)$ **A1**

either $b > 0$ and $\frac{4}{27}a^3 + b < 0$ or $b < 0$ and $\frac{4}{27}a^3 + b > 0$ **A1**

Note: Only award this **A1** if both cases are stated.

Award **A1** for stating that exactly one of b and $\frac{4}{27}a^3 + b$ is less than zero (or equivalent).

$b > 0$ and $\frac{4}{27}a^3 + b < 0$, $\left(\Rightarrow 0 < b < -\frac{4}{27}a^3\right) \Rightarrow a < 0$ and hence three x -axis intercepts **R1**

$b < 0$ and $\frac{4}{27}a^3 + b > 0$, $\left(\Rightarrow -\frac{4}{27}a^3 < b < 0\right) \Rightarrow a > 0$ and hence three x -axis intercepts **R1**

Note: Accept labelled sketches that clearly show this information.

if $4a^3b + 27b^2 < 0$, then the graph of $y = x^3 + ax^2 + b$ has exactly three x -axis intercepts **AG**

Note: For proving the converse, award a maximum of 3 marks (likely to be similar steps but presented in reverse; 2nd **A1** line not necessary in reverse method).

[5 marks]

Total [24 marks]

Question 9

(a) $T_2 = 1.1(6000) - 500$ OR $T_2 = 6600 - 500$
 $= 6100$

A2

AG

[2 marks]

(b) (i) $T_3 = 1.1(6100) - 500$ ($T_3 = 1.1(1.1(6000) - 500) - 500$) OR $T_3 = 6710 - 500$
 $= 6210$

A1

AG

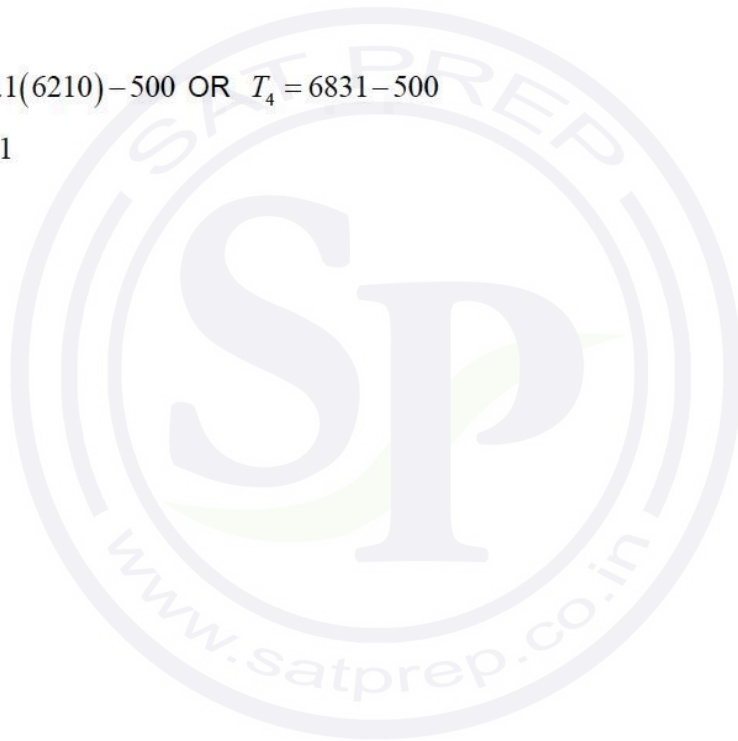
[1 mark]

(ii) $T_4 = 1.1(6210) - 500$ OR $T_4 = 6831 - 500$
 $= 6331$

(A1)

A1

[2 marks]



(c) (i) **EITHER**

attempts to eliminate the denominator

(M1)

$$T_n = 6000(1.1)^{n-1} - 5000\left((1.1)^{n-1} - 1\right) \left(= (6000 - 5000)(1.1)^{n-1} + 5000 \right)$$

A1

Note: Accept correct equivalent forms, eg. $T_n = 6000(1.1)^{n-1} - 5000(1.1)^{n-1} + 5000$.

OR

attempts to express T_n with a common denominator of 0.1

(M1)

$$T_n = \frac{600(1.1)^{n-1} - 500\left((1.1)^{n-1} - 1\right)}{0.1} \left(= \frac{600(1.1)^{n-1} - 500(1.1)^{n-1} + 500}{0.1} \right)$$

A1

Note: Accept correct equivalent forms.

THEN

$$T_n = 1000(1.1)^{n-1} + 5000$$

AG

[2 marks]

(c) (ii) **EITHER**

attempts to find T_6 using the explicit formula for T_n

(M1)

$$T_6 = 1000(1.1)^{6-1} + 5000 \text{ OR } T_6 = 6610.51$$

OR

attempts to find T_6 using $T_{n+1} = 1.1T_n - 500$

(M1)

$$T_4 = 6331, T_5 = 6464.1, T_6 = 6610.51$$

OR

attempts to find T_6 using a finance solver application

(M1)

$$N = 5, I = 10\%, PV = -6000 \text{ and } PMT = 500$$

THEN

$$6611$$

A1

Note: Accept 6610.

[2 marks]

(d) (i) **METHOD 1**

attempts to use the explicit formula for D_n to find an expression for $D_{n+1} - D_n$ **(M1)**

$$= 1.1D_n - 750 - (-1500(1.1)^{n-1} + 7500)$$

$$= 1.1(-1500(1.1)^{n-1} + 7500) - 750 - (-1500(1.1)^{n-1} + 7500) \quad \mathbf{A1}$$

EITHER

$$= -1650(1.1)^{n-1} + 8250 - 750 + 1500(1.1)^{n-1} - 7500 \quad \mathbf{A1}$$

OR

$$= 1500(1.1)^{n-1} - 1500(1.1)^n$$

$$= 1500(1.1)^{n-1}(1 - 1.1) \quad (= 1500(1.1)^{n-1}(-0.1)) \quad \mathbf{A1}$$

THEN

$$D_{n+1} - D_n = -150(1.1)^{n-1} \quad \mathbf{AG}$$

METHOD 2

attempts to express $D_{n+1} - D_n$ in terms of D_n **(M1)**

$$D_{n+1} - D_n = 1.1D_n - 750 - D_n$$

$$= 0.1D_n - 750 \quad \mathbf{(A1)}$$

$$= 0.1(-1500(1.1)^{n-1} + 7500) - 750 \quad (= -150(1.1)^{n-1} + 750 - 750) \quad \mathbf{A1}$$

$$D_{n+1} - D_n = -150(1.1)^{n-1} \quad \mathbf{AG}$$

[3 marks]

(ii) $D_{n+1} - D_n < 0$ (because $(1.1)^{n-1} > 0$ and so $-150(1.1)^{n-1} < 0$)

R1

Note: Award **R1** for responses such as:

'The difference is negative'.

$$-150(1.1)^{n-1} < 0$$

$$D_n = D_{n+1} + 150(1.1)^{n-1}$$

Accept responses such as:

$D_n - D_{n+1} > 0$ or 'the difference is positive' as long as $D_n - D_{n+1}$ is referred to.

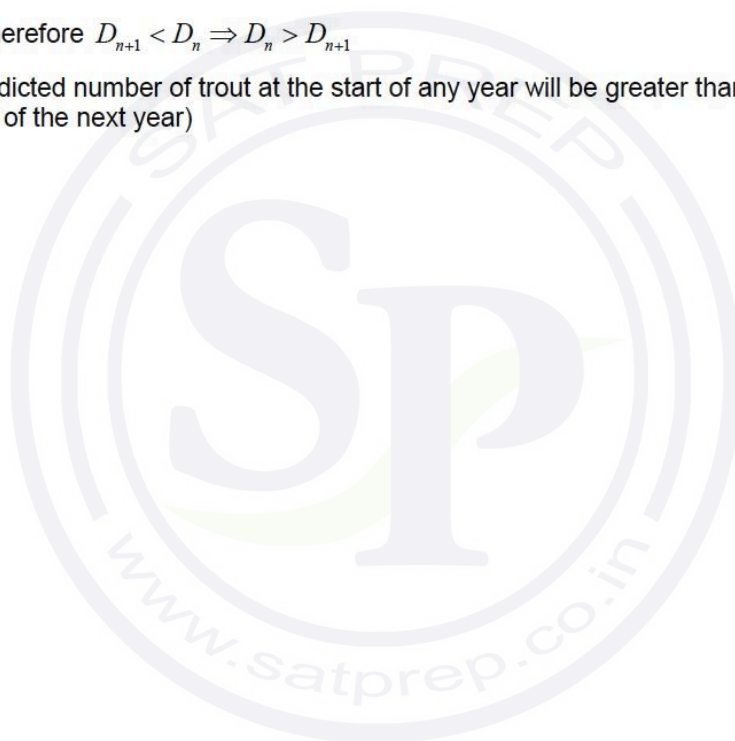
Do not accept arguments based on specific numerical examples.

$$\text{therefore } D_{n+1} < D_n \Rightarrow D_n > D_{n+1}$$

AG

(and hence the predicted number of trout at the start of any year will be greater than the predicted number at the start of the next year)

[1 mark]



(e) **METHOD 1**

attempts (numerically, graphically, algebraically or by trial & improvement) to find the least value of n such that $D_n \leq 0$ ($D_n = 0$ OR $D_n < 0$)

(M1)

Note: Award **(M1)** for attempting to obtain $-1500(1.1)^{n-1} \leq -7500$ or

$-1500(1.1)^{n-1} = -7500$ (or equivalent) and then attempting to take the log of both sides.

For the use of trial & improvement, award **(M1)**, for considering values of n either side of $D_n = 0$.

EITHER

$$17.8863... \left(= \log_{1.1} 5 + 1, = \frac{\ln 5}{\ln 1.1} + 1 \right) \text{ (critical value of } n \text{ seen anywhere)}$$

(A2)

OR

$$D_{17} = 607.540... \text{ (} 607.540... < 750 \text{)}$$

(A2)

OR

$$D_{18} = -81.7054... (< 0)$$

(A2)

THEN

trout disappears from the lake during (at the end of) the 17th year

A1

Note: Award **A1** for $n = 18$ OR ('during the) 18th year' or 'start of the 18th year'.

Note: Award as above for $D_{n+1} \leq 0$ ($D_{n+1} = 0$ OR $D_{n+1} < 0$).

METHOD 2

$S_n = \sum (D_{n+1} - D_n)$ is the total decrease in the number of trout by the end of year n or start of year $n+1$

attempts (numerically, graphically, algebraically or by trial & improvement) to find the least value of n such that $S_n \leq -6000$ ($S_n = -6000$ OR $S_n < -6000$) (M1)

$$\frac{-150((1.1)^n - 1)}{1.1 - 1} (= -1500((1.1)^n - 1)) \leq -6000$$

Note: Award (M1) for attempting to obtain $-1500((1.1)^n - 1) \leq -6000$ or

$-1500((1.1)^n - 1) = -6000$ (or equivalent) and then attempting to take the log of both sides.

For the use of trial & improvement, award (M1), for considering values of n either side of $S_n = -6000$.

$$n = 16.8863... \quad \text{(A2)}$$

trout disappears from the lake during (at the end of) the 17th year A1

Note: Award A1 for $n = 18$ OR ('during the) 18th year' or 'start of the 18th year'.

[4 marks]

(f) **METHOD 1**

$$C_n = (6000 - 10d)(1.1)^{n-1} + 10d \quad \text{(A1)}$$

EITHER

$$6000 - 10d = 0 \text{ (recognizes that } C_n \text{ is independent of } n) \quad \text{(M1)}$$

OR

$$10d = 6000 \text{ (sets } C_n = 6000 \text{ and compares coefficients)} \quad \text{(M1)}$$

THEN

$$d = 600 \quad \text{A1}$$

METHOD 2**EITHER**

$$C_{n+1} - C_n (= 0)$$

$$6000(1.1)^n - 10d((1.1)^n - 1) - (6000(1.1)^{n-1} - 10d((1.1)^{n-1} - 1)) (= 0) \quad (\mathbf{A1})$$

OR

$$0.1C_n - d (= 0) \text{ (similar to part (d) (i) method 2)}$$

$$0.1(6000(1.1)^{n-1} - 10d((1.1)^{n-1} - 1)) - d (= 0) \quad (\mathbf{A1})$$

THEN

attempts to expand to eliminate the linear terms in d and then attempts to factorize (\mathbf{M1})

$$600(1.1)^{n-1} - d(1.1)^{n-1} (= 0) \quad ((600 - d)(1.1)^{n-1} (= 0))$$

$$d = 600 \quad \mathbf{A1}$$

METHOD 3

considers $C_{n+1} = 1.1C_n - d$ and $C_1 = 6000$ (\mathbf{A1})

EITHER

$$6000 = 1.1 \times 6000 - d \quad (C_1 = C_2 = 6000) \quad (\mathbf{A1})$$

OR

systematically varies the value of d (\mathbf{M1})

THEN

$$d = 600 \quad \mathbf{A1}$$

METHOD 4**EITHER**

considers a specific $C_n = C_{n+1}$, for example, $C_2 = C_3$

(A1)

$$6000(1.1) - 10d(1.1 - 1) = 6000(1.1)^2 - 10d((1.1)^2 - 1)$$

attempts to solve their $C_n = C_{n+1}$ for d

(M1)

for example, $1.1d = 660$

OR

considers a specific $C_n = 6000$, for example, $C_2 = 6000$

(A1)

$$6000(1.1)^{2-1} - 10d((1.1)^{2-1} - 1) = 6000$$

attempts to solve their $C_n = 6000$ for d

(M1)

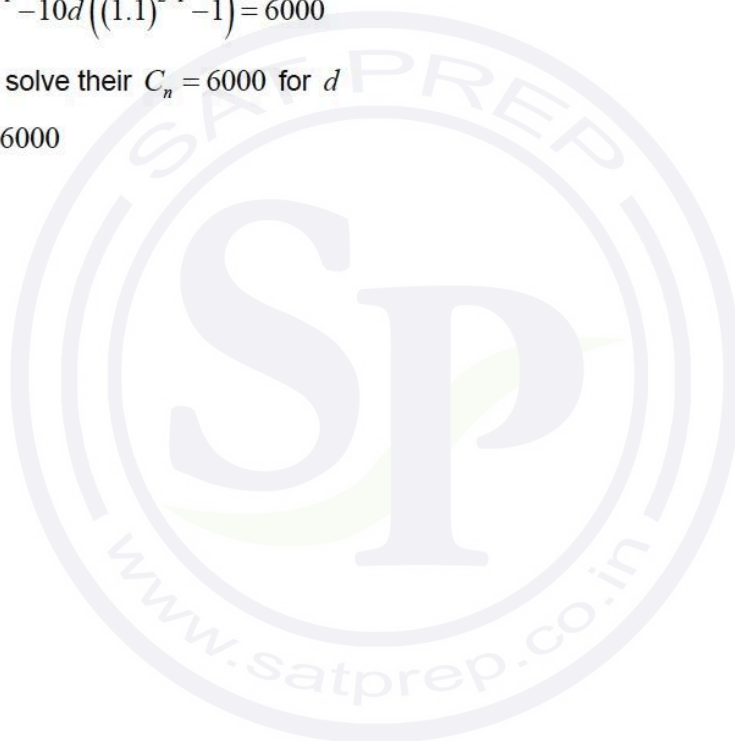
$$6600 - d = 6000$$

THEN

$$d = 600$$

A1

[3 marks]



(g) $n=1$: LHS = u_1 and RHS = $\left(u_1 r^{1-1} - \frac{d(r^{1-1}-1)}{r-1}\right) = u_1 r^0 - \frac{d(r^0-1)}{r-1} = u_1$ **R1**

LHS = RHS and so true for $n=1$

Note: Award **R0** for considering $n=2$.

Subsequent marks after this **R1** are independent of this mark and can be awarded.

assume true for $n=k$ $\left(u_k = u_1 r^{k-1} - \frac{d(r^{k-1}-1)}{r-1}\right)$ **M1**

Note: The assumption of truth must be apparent. Do not award **M1** for statements such as "let $n=k$ " or "assume that $n=k$ is true". Subsequent marks after this **M1** are independent of this mark and can be awarded.

attempts to substitute for u_k in $u_{k+1} = ru_k - d$ **M1**

$$= r \left(u_1 r^{k-1} - \frac{d(r^{k-1}-1)}{r-1} \right) - d$$

$$= u_1 r^k - \frac{dr(r^{k-1}-1)}{r-1} - \frac{d(r-1)}{r-1} \left(= u_1 r^k - \frac{dr^k - dr}{r-1} - d \right)$$
A1

$$= u_1 r^k - \frac{dr^k + dr - dr - d}{r-1} \left(= u_1 r^k - \frac{dr^k - d}{r-1} \right)$$
A1

$$= u_1 r^k - \frac{d(r^k-1)}{r-1}$$
A1

since true for $n=1$ and true for $n=k+1$ if true for $n=k$, hence true for all $n \in \mathbb{Z}^+$ **R1**

Note: To obtain the final **R1**, four of the previous six marks must have been awarded.

[7 marks]

Total [27 marks]

Question 10

(a) (i) attempts to solve

(M1)

EITHER

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2}$$

OR

$$(x+2)^2 - 3 (= 0)$$

OR

$$(x+2+\sqrt{3})(x+2-\sqrt{3})(= 0)$$

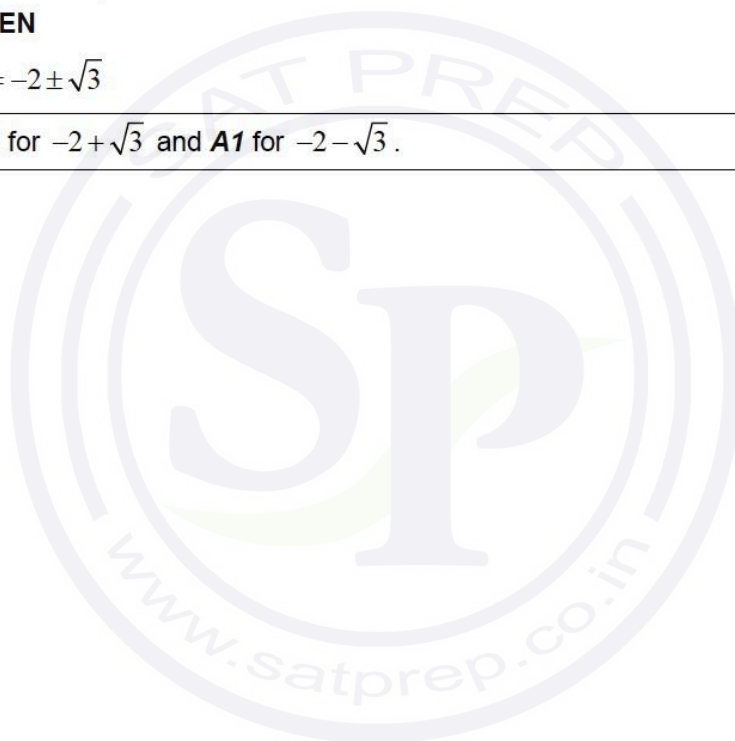
THEN

$$x = -2 \pm \sqrt{3}$$

A1A1

Note: Award **A1** for $-2 + \sqrt{3}$ and **A1** for $-2 - \sqrt{3}$.

[3 marks]



(ii) **METHOD 1**

attempts to find their $\frac{1}{-2-\sqrt{3}} \times \frac{-2+\sqrt{3}}{-2+\sqrt{3}}$ **(M1)**

Note: Only award **(M1)** if the roots are in the form $s \pm \sqrt{t}$.

$$= \frac{-2+\sqrt{3}}{(-2)^2 - (\sqrt{3})^2} \text{ and } (-2)^2 - (\sqrt{3})^2 = 1$$

A1

hence the roots form a reciprocal pair **AG**

Note: Award as above for attempting to show that $\frac{1}{-2+\sqrt{3}} = -2-\sqrt{3}$.

METHOD 2

considers the product of their roots **(M1)**

Note: Only award **(M1)** if the roots are in the form $s \pm \sqrt{t}$.

$$(-2+\sqrt{3})(-2-\sqrt{3})=1$$

A1

hence the roots form a reciprocal pair **AG**

METHOD 3

considers $\frac{c}{a} (= \frac{1}{1})$ **(M1)**

the product of roots is 1 **A1**

hence the roots form a reciprocal pair **AG**

[2 marks]

(b) roots are $\pm i \left(= i, \frac{1}{i}, = \text{cis} \left(\pm \frac{\pi}{2} \right), = e^{\pm i \frac{\pi}{2}} \right)$ (A1)

EITHER

$$\frac{1}{-i} \times \frac{i}{i} = i \text{ OR } \frac{1}{-i} = i \quad \text{A1}$$

Note: Award as above for showing that $\frac{1}{i} = -i$ OR stating that $\frac{1}{i} = -i$.

OR

$$i \times -i = 1 \text{ OR } i \times \frac{1}{i} = 1 \quad \text{A1}$$

OR

the product of roots is 1 (A1)

THEN

hence the roots form a reciprocal pair (AG)

[2 marks]

(c) $p\left(\frac{1}{x}\right) = \frac{a}{x^2} + \frac{b}{x} + a$ (A1)

$$x^2 p\left(\frac{1}{x}\right) = x^2 \left(\frac{a}{x^2} + \frac{b}{x} + a \right) \quad \text{A1}$$

$$= ax^2 + bx + a$$

$$= p(x) \quad \text{AG}$$

Note: Award as above for showing that $\frac{p(x)}{x^2} = p\left(\frac{1}{x}\right) = \frac{a}{x^2} + \frac{b}{x} + a$.

[2 marks]

(d) **METHOD 1**

$$p(\alpha) = 0 \Rightarrow \alpha^n p\left(\frac{1}{\alpha}\right) = 0$$

A1

EITHER

$$\alpha^n \neq 0 \text{ (as } \alpha \neq 0) \Rightarrow p\left(\frac{1}{\alpha}\right) = 0$$

R1

OR

$$\alpha \neq 0 \text{ (hence } \alpha^n \neq 0) \Rightarrow p\left(\frac{1}{\alpha}\right) = 0$$

R1

THEN

so $\frac{1}{\alpha}$ is also a root

AG

Note: The **R1** is dependent on the **A1**.

[2 marks]

METHOD 2

$$p\left(\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha}\right)^n p\left(\frac{1}{\frac{1}{\alpha}}\right)$$

$$p\left(\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha}\right)^n p(\alpha)$$

A1

for $\alpha \neq 0$ (seen anywhere), $p(\alpha) = 0 \Rightarrow \left(\frac{1}{\alpha}\right)^n p(\alpha) = 0$ and so $p\left(\frac{1}{\alpha}\right) = 0$

R1

so $\frac{1}{\alpha}$ is also a root

AG

Note: The **R1** is dependent on the **A1**.

[2 marks]

(e) **METHOD 1**

$$p(x) = x^n p\left(\frac{1}{x}\right)$$

$$q(x) = x^m q\left(\frac{1}{x}\right)$$

(A1)

attempts to substitute for $p(x)$ and $q(x)$ into $f(x) = p(x)q(x)$

(M1)

$$f(x) = x^n p\left(\frac{1}{x}\right) x^m q\left(\frac{1}{x}\right) \left(= x^{n+m} p\left(\frac{1}{x}\right) q\left(\frac{1}{x}\right) \right)$$

A1

$$= x^{n+m} f\left(\frac{1}{x}\right) \text{ (and } n+m \text{ is the degree of } f \text{)}$$

A1

hence f is a palindromic polynomial.

AG

METHOD 2

$$p\left(\frac{1}{x}\right) = \frac{p(x)}{x^n}$$

$$q\left(\frac{1}{x}\right) = \frac{q(x)}{x^m}$$

(A1)

attempts to substitute for $p\left(\frac{1}{x}\right)$ and $q\left(\frac{1}{x}\right)$ into $f\left(\frac{1}{x}\right) = p\left(\frac{1}{x}\right)q\left(\frac{1}{x}\right)$

(M1)

$$f\left(\frac{1}{x}\right) = \frac{p(x)}{x^n} \frac{q(x)}{x^m} \left(= \frac{p(x)q(x)}{x^{n+m}} \right)$$

A1

$$x^{n+m} f\left(\frac{1}{x}\right) = p(x)q(x) \text{ and } f(x) = p(x)q(x) \text{ (and } n+m \text{ is the degree of } f \text{)}$$

A1

hence f is a palindromic polynomial.

AG

[4 marks]

METHOD 3

for example, $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_{n-1} x + a_n$ AND

$$q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_{m-1} x + b_m \quad \text{A1}$$

attempts to multiply out their $p(x)q(x)$ (M1)

Note: Only award (M1) for polynomials of degree n and m respectively with different corresponding coefficients.

$$p(x)q(x) = a_n b_m x^{m+n} + (a_{n-1} b_m + a_n b_{m-1}) x^{m+n-1} + \dots + (a_{n-1} b_m + a_n b_{m-1}) x + a_n b_m \quad \text{A1A1}$$

Note: Award A1 for the first and second terms and A1 for the penultimate and last terms.

hence f is a palindromic polynomial AG

[4 marks]

(f) **METHOD 1**

attempts to expand the product of two quadratic factors (M1)

EITHER

$$x^4 + ux^3 + vx^3 + uvx^2 + 2x^2 + ux + vx + 1 \quad \text{(A1)}$$

OR

$$x^4 + (u+v)x^3 + (uv+2)x^2 + (u+v)x + 1 \quad \text{(A1)}$$

THEN

equates coefficients and forms a system of two equations in u and v (M1)

$$u+v=2 \text{ and } uv+2=-1 \text{ (} uv=-3\text{)} \quad \text{A1}$$

attempts to solve their system of two equations (M1)

$$u=-1, v=3 \text{ OR } (f(x)=(x^2-x+1)(x^2+3x+1)) \quad \text{A1}$$

Note: Candidates must attempt to form and solve a system of equations in u and v . Hence, award no marks for working backwards from either decimal or exact values of roots given by a GDC.

Award A0 for $u=-1, v=3$ and $u=3, v=-1$ both specified.

METHOD 2

attempts to divide one of $(x^2 + ux + 1)$ or $(x^2 + vx + 1)$ into $f(x)$ (M1)

EITHER

for example, the quotient is $x^2 + (2-u)x + u^2 - 2u - 2$ A1

sets their quotient equal to $x^2 + vx + 1$ (M1)

$$x^2 + (2-u)x + u^2 - 2u - 2 = x^2 + vx + 1$$

$$u^2 - 2u - 2 = 1 \text{ and } 2 - u = v \quad \text{A1}$$

attempts to solve their equations for u and then v (or vice versa) (M1)

OR

for example, the remainder is $u(-u^2 + 2u + 3)x + (-u^2 + 2u + 3)$ A1

sets their remainder equal to zero (M1)

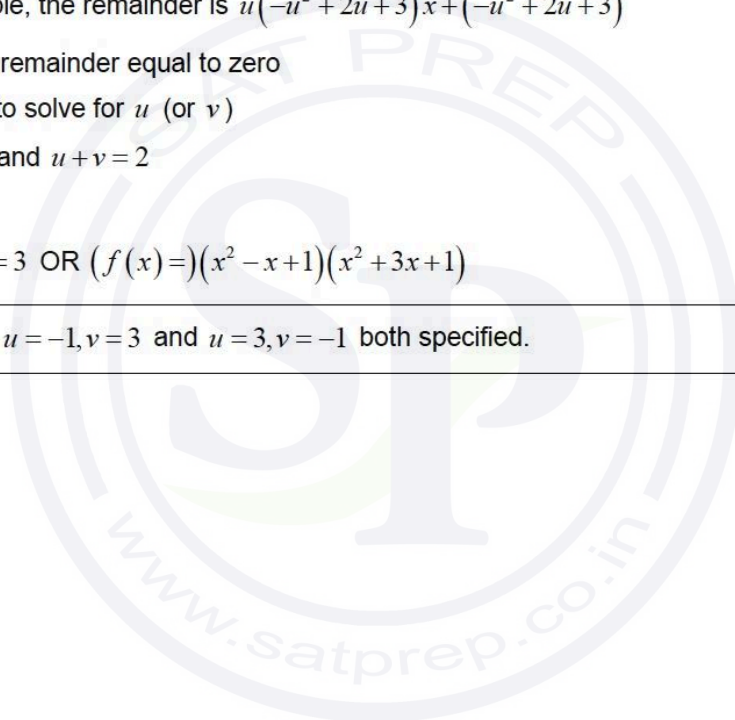
attempts to solve for u (or v) (M1)

$$u = -1, 3 \text{ and } u + v = 2 \quad \text{A1}$$

THEN

$$u = -1, v = 3 \text{ OR } (f(x) = (x^2 - x + 1)(x^2 + 3x + 1)) \quad \text{A1}$$

Note: Award **A0** for $u = -1, v = 3$ and $u = 3, v = -1$ both specified.



METHOD 3considers $f(1)$, for example**(M1)**

$$(u+2)(v+2)=5$$

A1considers $f(2)$, for example**(M1)**

$$(2u+5)(2v+5)=33$$

A1

attempts to solve their system of two equations

(M1)

$$u = -1, v = 3 \text{ OR } (f(x) = (x^2 - x + 1)(x^2 + 3x + 1))$$

A1**Note:** Award **A0** for $u = -1, v = 3$ and $u = 3, v = -1$ both specified.**[6 marks]**(g) attempts to solve $x^4 + 2x^3 - x^2 + 2x + 1 = 0$ **(M1)****Note:** Award **(M1)** for attempting to solve one of their quadratic equations.

$$\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, -\frac{3}{2} \pm \frac{\sqrt{5}}{2}$$

A1A1**Note:** Award **A1** for $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ and **A1** for $-\frac{3}{2} \pm \frac{\sqrt{5}}{2}$.**[3 marks]**

(h) **METHOD 1**

consider $p(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_2x^2 + a_1x + a_0 (= 0)$

attempts to find $p(-1)$ **(M1)**

$$p(-1) = (-1)^n + a_1(-1)^{n-1} + a_2(-1)^{n-2} + \dots + a_2(-1)^2 + a_1(-1) + 1$$
 (A1)

$$= (-1) + a_1 - a_2 + \dots + a_2 - a_1 + 1$$
 A1

(since n is odd), $p(x)$ has

EITHER

$n+1$ terms so all terms cancel giving $p(-1) = 0$ **R1**

OR

an even number of terms so all terms cancel giving $p(-1) = 0$ **R1**

THEN

hence -1 is always a root **AG**

METHOD 2

considers the product of roots of a polynomial equation of degree n **(M1)**

$$\frac{(-1)^n a_0}{a_n}$$

product of roots is $(-1)^n$ (seen anywhere) **A1**

(for n odd) there are $\frac{n-1}{2}$ reciprocal pairs of roots **A1**

product of each reciprocal pair is 1 and $(-1)^n = -1$ (since n is odd) **R1**

hence -1 is always a root **AG**

METHOD 3

consider $p(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_2x^2 + a_1x + a_0 (= 0)$

considers $p(x) = x^n p\left(\frac{1}{x}\right)$ with $x = -1$ **(M1)**

$p(-1) = (-1)^n p(-1)$ **A1**

$p(-1) = -p(-1)$ ($2p(-1) = 0$) (since n is odd) **A1**

hence $p(-1) = 0$ **R1**

hence -1 is always a root **AG**

[4 marks]

Total [28 marks]

