Subject – Math AA(Standard Level)
Topic - Calculus
Year - May 2021 – Nov 2024
Paper -1
Questions

Question 1

[Maximum mark: 16]

Let $f(x) = \dfrac{\ln 5x}{kx}$ where $x > 0$, $k \in \mathbb{R}^+$.

(a) Show that $f'(x) = \dfrac{1 - \ln 5x}{kx^2}$. [3]

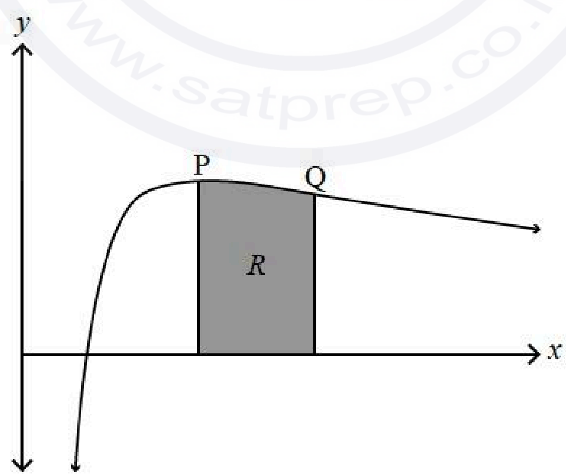
The graph of f has exactly one maximum point P.

(b) Find the x-coordinate of P. [3]

The second derivative of f is given by $f''(x) = \dfrac{2\ln 5x - 3}{kx^3}$. The graph of f has exactly one point of inflexion Q.

(c) Show that the x-coordinate of Q is $\dfrac{1}{5}e^{\frac{3}{2}}$. [3]

The region R is enclosed by the graph of f, the x-axis, and the vertical lines through the maximum point P and the point of inflexion Q.



(d) Given that the area of R is 3, find the value of k. [7]

Question 2

[Maximum mark: 16]

Let $f(x) = \frac{1}{3}x^3 + x^2 - 15x + 17$.

(a) Find $f'(x)$. [2]

The graph of f has horizontal tangents at the points where $x = a$ and $x = b$, $a < b$.

(b) Find the value of a and the value of b. [3]

(c) (i) Sketch the graph of $y = f'(x)$.

(ii) Hence explain why the graph of f has a local maximum point at $x = a$. [2]

(d) (i) Find $f''(b)$.

(ii) Hence, use your answer to part (d)(i) to show that the graph of f has a local minimum point at $x = b$. [4]

The normal to the graph of f at $x = a$ and the tangent to the graph of f at $x = b$ intersect at the point (p, q).

(e) Find the value of p and the value of q. [5]

Question 3

[Maximum mark: 5]

Let $f'(x) = \frac{8x}{\sqrt{2x^2 + 1}}$. Given that $f(0) = 5$, find $f(x)$.

Question 4

[Maximum mark: 16]

Let $y = \frac{\ln x}{x^4}$ for $x > 0$.

(a) Show that $\frac{dy}{dx} = \frac{1 - 4\ln x}{x^5}$. [3]

Consider the function defined by $f(x) = \frac{\ln x}{x^4}$ for $x > 0$ and its graph $y = f(x)$.

(b) The graph of f has a horizontal tangent at point P. Find the coordinates of P. [5]

(c) Given that $f''(x) = \frac{20\ln x - 9}{x^6}$, show that P is a local maximum point. [3]

(d) Solve $f(x) > 0$ for $x > 0$. [2]

(e) Sketch the graph of f, showing clearly the value of the x-intercept and the approximate position of point P. [3]

Question 5

[Maximum mark: 7]

Consider the functions $f(x) = -(x-h)^2 + 2k$ and $g(x) = e^{x-2} + k$ where $h, k \in \mathbb{R}$.

(a) Find $f'(x)$. [1]

The graphs of f and g have a common tangent at $x = 3$.

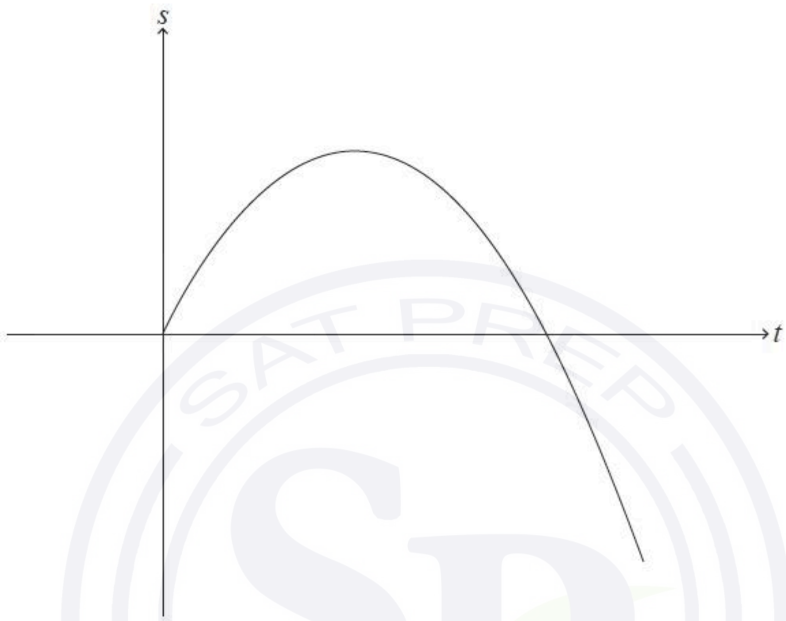
(b) Show that $h = \frac{e + 6}{2}$. [3]

(c) Hence, show that $k = e + \frac{e^2}{4}$. [3]

Question 6

[Maximum mark: 14]

Particle A travels in a straight line such that its displacement, s metres, from a fixed origin after t seconds is given by $s(t) = 8t - t^2$, for $0 \le t \le 10$, as shown in the following diagram.



Particle A starts at the origin and passes through the origin again when $t = p$.

(a) Find the value of p. [2]

Particle A changes direction when $t = q$.

(b) (i) Find the value of q.

(ii) Find the displacement of particle A from the origin when $t = q$. [4]

(c) Find the distance of particle A from the origin when $t = 10$. [2]

The total distance travelled by particle A is given by d.

(d) Find the value of d. [2]

A second particle, particle B, travels along the same straight line such that its velocity is given by $v(t) = 14 - 2t$, for $t \ge 0$.

When $t = k$, the distance travelled by particle B is equal to d.

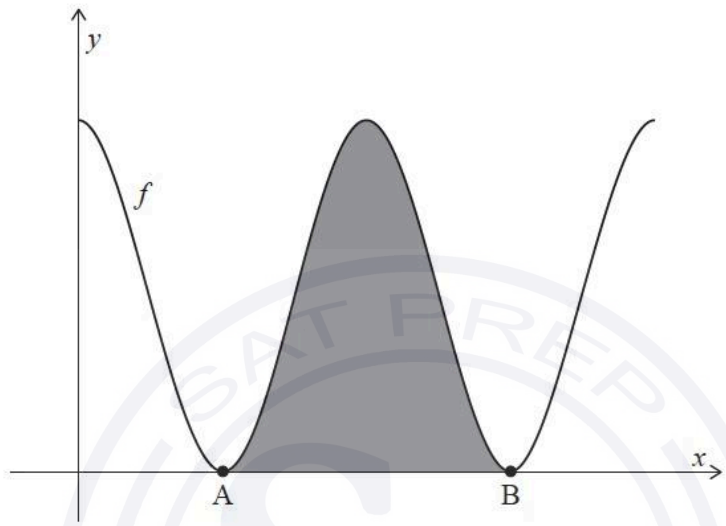
(e) Find the value of k. [4]

Question 7

[Maximum mark: 15]

Consider the function f defined by $f(x) = 6 + 6\cos x$, for $0 \leq x \leq 4\pi$.

The following diagram shows the graph of $y = f(x)$.



The graph of f touches the x-axis at points A and B, as shown. The shaded region is enclosed by the graph of $y = f(x)$ and the x-axis, between the points A and B.

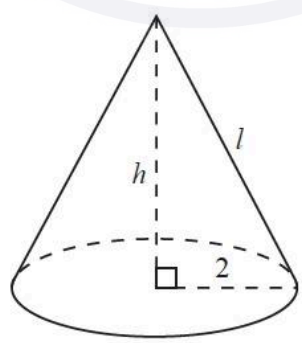
(a) Find the x-coordinates of A and B. [3]

(b) Show that the area of the shaded region is 12π. [5]

The right cone in the following diagram has a total surface area of 12π, equal to the shaded area in the previous diagram.

The cone has a base radius of 2, height h, and slant height l.

diagram not to scale



(c) Find the value of l. [3]

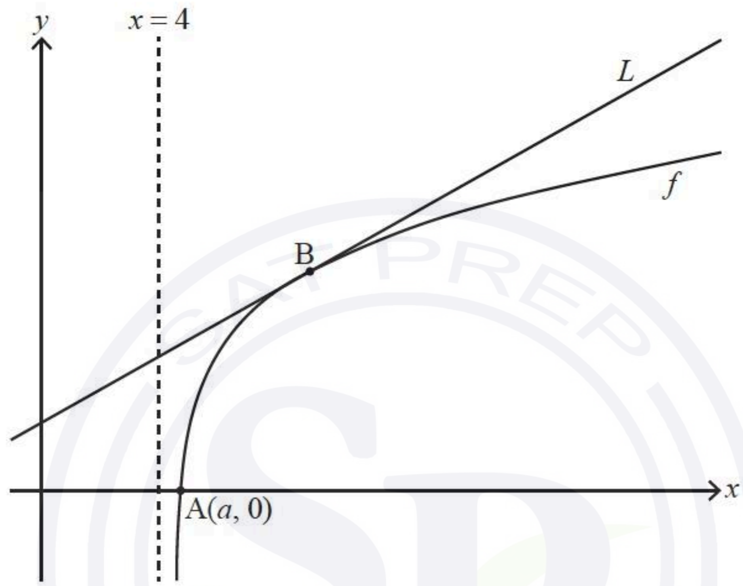
(d) Hence, find the volume of the cone. [4]

Question 8

[Maximum mark: 9]

Consider the function f defined by $f(x) = \ln(x^2 - 16)$ for $x > 4$.

The following diagram shows part of the graph of f which crosses the x-axis at point A, with coordinates $(a, 0)$. The line L is the tangent to the graph of f at the point B.



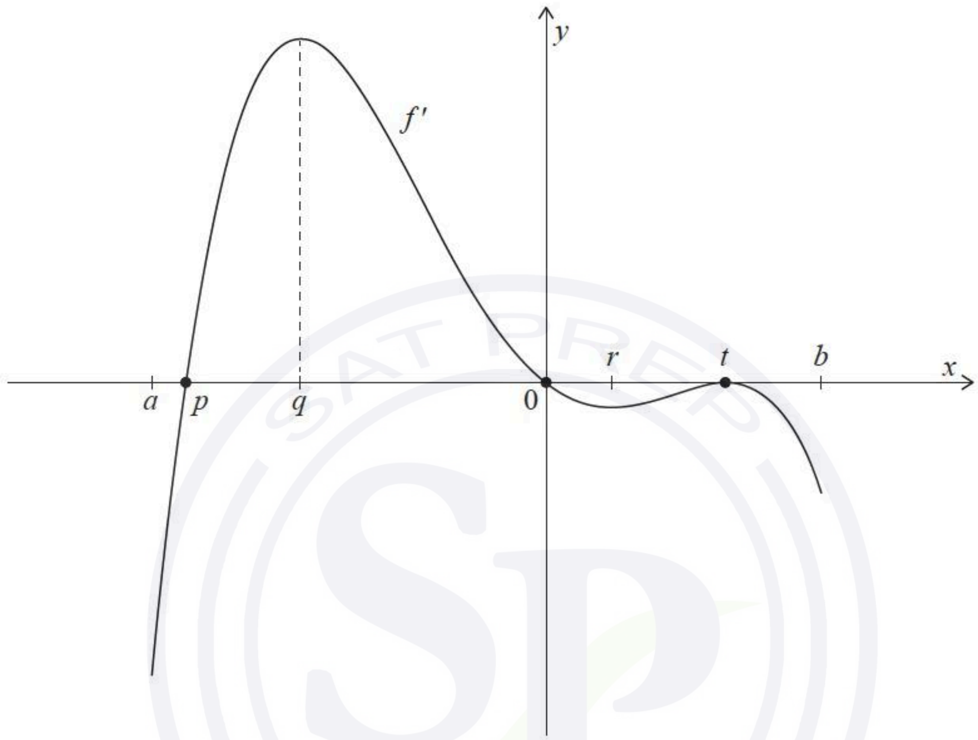
(a) Find the exact value of a. [3]

(b) Given that the gradient of L is $\frac{1}{3}$, find the x-coordinate of B. [6]

Question 9

[Maximum mark: 14]

Consider a function f with domain $a < x < b$. The following diagram shows the graph of f', the derivative of f.



The graph of f', the derivative of f, has x-intercepts at $x = p$, $x = 0$ and $x = t$. There are local maximum points at $x = q$ and $x = t$ and a local minimum point at $x = r$.

(a) Find all the values of x where the graph of f is increasing. Justify your answer. [2]

(b) Find the value of x where the graph of f has a local maximum. [1]

(c) (i) Find the value of x where the graph of f has a local minimum. Justify your answer.

(ii) Find the values of x where the graph of f has points of inflexion. Justify your answer. [5]

(d) The total area of the region enclosed by the graph of f', the derivative of f, and the x-axis is 20.

Given that $f(p) + f(t) = 4$, find the value of $f(0)$. [6]

Question 10

[Maximum mark: 16]

A particle P moves along the x-axis. The velocity of P is $v\,\text{m s}^{-1}$ at time t seconds, where $v(t) = 4 + 4t - 3t^2$ for $0 \le t \le 3$. When $t = 0$, P is at the origin O.

(a) (i) Find the value of t when P reaches its maximum velocity.

(ii) Show that the distance of P from O at this time is $\dfrac{88}{27}$ metres. [7]

(b) Sketch a graph of v against t, clearly showing any points of intersection with the axes. [4]

(c) Find the total distance travelled by P. [5]

Question 11

[Maximum mark: 7]

The function f is defined for all $x \in \mathbb{R}$. The line with equation $y = 6x - 1$ is the tangent to the graph of f at $x = 4$.

(a) Write down the value of $f'(4)$. [1]

(b) Find $f(4)$. [1]

The function g is defined for all $x \in \mathbb{R}$ where $g(x) = x^2 - 3x$ and $h(x) = f(g(x))$.

(c) Find $h(4)$. [2]

(d) Hence find the equation of the tangent to the graph of h at $x = 4$. [3]

Question 12

[Maximum mark: 4]

Given that $\dfrac{dy}{dx} = \cos\left(x - \dfrac{\pi}{4}\right)$ and $y = 2$ when $x = \dfrac{3\pi}{4}$, find y in terms of x.

Question 13

[Maximum mark: 15]

(a) (i) Expand and simplify $(1-a)^3$ in ascending powers of a.

(ii) By using a suitable substitution for a, show that
$1-3\cos 2x+3\cos^2 2x-\cos^3 2x=8\sin^6 x$. [6]

Consider $f(x)=4\cos x\left(1-3\cos 2x+3\cos^2 2x-\cos^3 2x\right)$.

(b) (i) Show that $\int_0^m f(x)\,\mathrm{d}x=\frac{32}{7}\sin^7 m$, where m is a positive real constant.

(ii) It is given that $\int_m^{\frac{\pi}{2}} f(x)\,\mathrm{d}x=\frac{127}{28}$, where $0\le m\le\frac{\pi}{2}$. Find the value of m. [9]

Question 14

[Maximum mark: 5]

Consider the curve with equation $y=(2x-1)\mathrm{e}^{kx}$, where $x\in\mathbb{R}$ and $k\in\mathbb{Q}$.

The tangent to the curve at the point where $x=1$ is parallel to the line $y=5\mathrm{e}^{k}x$.

Find the value of k.

Question 15

[Maximum mark: 5]

(a) The expression $\frac{3\sqrt{x}-5}{\sqrt{x}}$ can be written as $3-5x^{p}$. Write down the value of p. [1]

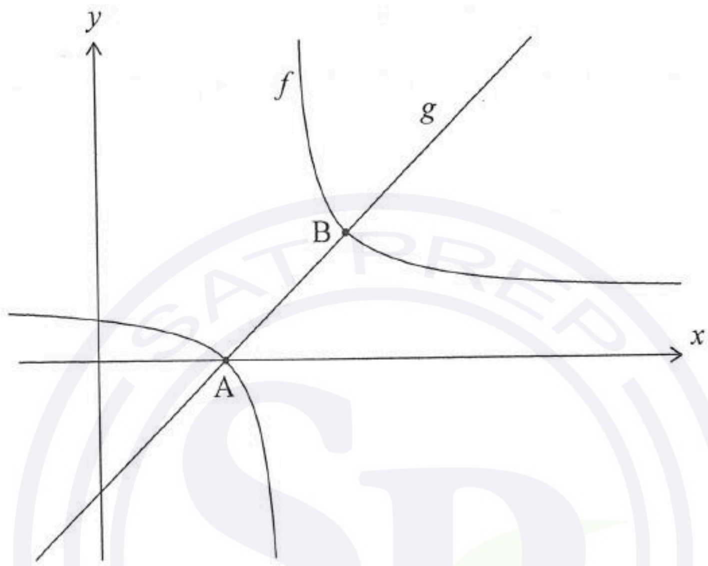
(b) Hence, find the value of $\int_1^9\left(\frac{3\sqrt{x}-5}{\sqrt{x}}\right)\mathrm{d}x$. [4]

Question 16

[Maximum mark: 15]

Consider the functions $f(x) = \frac{1}{x-4} + 1$, for $x \neq 4$, and $g(x) = x - 3$ for $x \in \mathbb{R}$.

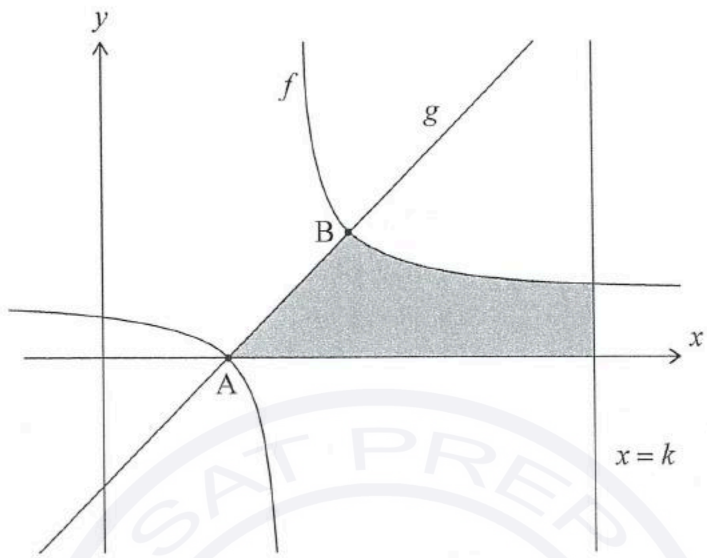
The following diagram shows the graphs of f and g.



The graphs of f and g intersect at points A and B. The coordinates of A are $(3, 0)$.

(a) Find the coordinates of B. [5]

In the following diagram, the shaded region is enclosed by the graph of f, the graph of g, the x-axis, and the line $x = k$, where $k \in \mathbb{Z}$.



The area of the shaded region can be written as $\ln(p) + 8$, where $p \in \mathbb{Z}$.

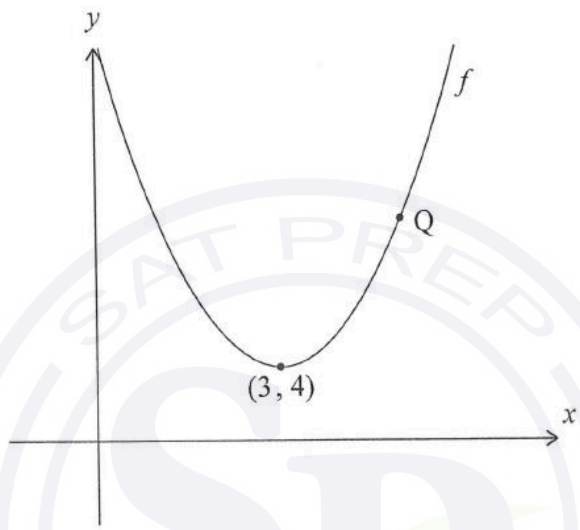
(b) Find the value of k and the value of p. [10]

Question 17

[Maximum mark: 15]

The following diagram shows part of the graph of a quadratic function f.

The graph of f has its vertex at $(3, 4)$, and it passes through point Q as shown.



(a) Write down the equation of the axis of symmetry. [1]

(b) The function can be written in the form $f(x) = a(x - h)^2 + k$.

(i) Write down the values of h and k.

(ii) Point Q has coordinates $(5, 12)$. Find the value of a. [4]

The line L is tangent to the graph of f at Q.

(c) Find the equation of L. [4]

Now consider another function $y = g(x)$. The derivative of g is given by $g'(x) = f(x) - d$, where $d \in \mathbb{R}$.

(d) Find the values of d for which g is an increasing function. [3]

(e) Find the values of x for which the graph of g is concave-up. [3]

Question 18

[Maximum mark: 15]

The function f is defined by $f(x) = \cos^2 x - 3\sin^2 x,\ 0 \le x \le \pi$.

(a) Find the roots of the equation $f(x) = 0$. [5]

(b) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the points on the graph of $y = f(x)$ where $f'(x) = 0$. [7]

(c) Sketch the graph of $y = f(x)$, clearly showing the coordinates of any points where $f'(x) = 0$ and any points where the graph meets the coordinate axes. [3]

Question 19

[Maximum mark: 5]

The derivative of the function f is given by $f'(x) = \dfrac{6x}{x^2+1}$.

The graph of $y = f(x)$ passes through the point $(1, 5)$. Find an expression for $f(x)$.

Question 20

[Maximum mark: 4]

The function g is defined by $g(x) = e^{x^2+1}$, where $x \in \mathbb{R}$.

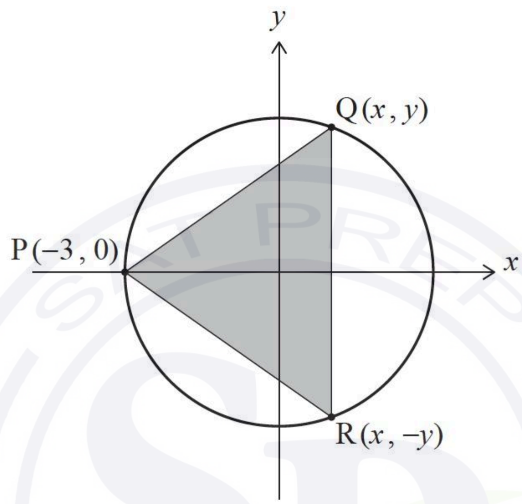
Find $g'(-1)$.

Question 21

[Maximum mark: 14]

A circle with equation $x^2 + y^2 = 9$ has centre $(0, 0)$ and radius 3.

A triangle, PQR, is inscribed in the circle with its vertices at $\mathrm{P}(-3, 0)$, $\mathrm{Q}(x, y)$ and $\mathrm{R}(x, -y)$, where Q and R are variable points in the first and fourth quadrants respectively. This is shown in the following diagram.



(a) For point Q, show that $y = \sqrt{9 - x^2}$. [1]

(b) Hence, find an expression for A, the area of triangle PQR, in terms of x. [3]

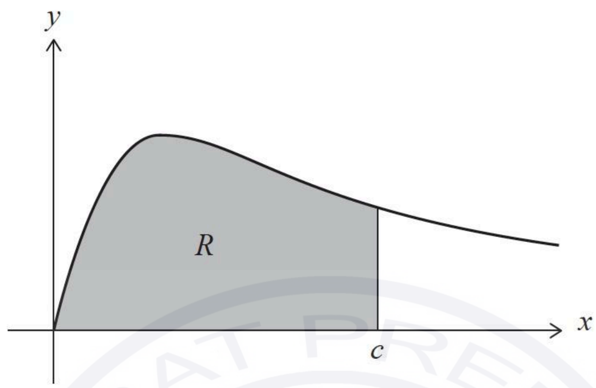
(c) Show that $\dfrac{\mathrm{d}A}{\mathrm{d}x} = \dfrac{9 - 3x - 2x^2}{\sqrt{9 - x^2}}$. [4]

(d) Hence or otherwise, find the y-coordinate of R such that A is a maximum. [6]

Question 22

[Maximum mark: 6]

The following diagram shows part of the graph of $y = \frac{x}{x^2 + 2}$ for $x \geq 0$.



The shaded region R is bounded by the curve, the x-axis and the line $x = c$.

The area of R is $\ln 3$.

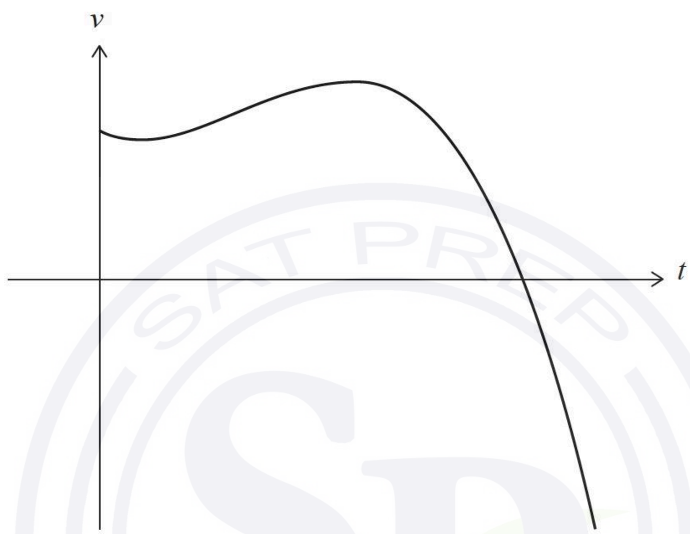
Find the value of c.

Question 23

[Maximum mark: 17]

An object moves along a straight line. Its velocity, $v\,\text{m s}^{-1}$, at time t seconds is given by $v(t) = -t^3 + \frac{7}{2}t^2 - 2t + 6$, for $0 \le t \le 4$. The object first comes to rest at $t = k$.

The graph of v is shown in the following diagram.



At $t = 0$, the object is at the origin.

(a) Find the displacement of the object from the origin at $t = 1$. [5]

(b) Find an expression for the acceleration of the object. [2]

(c) Hence, find the greatest speed reached by the object before it comes to rest. [5]

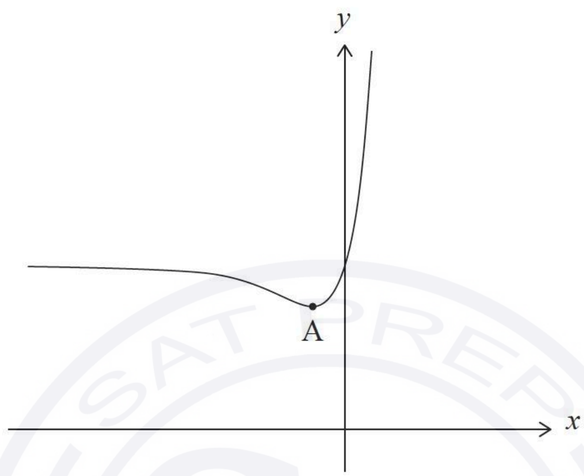
(d) Find the greatest speed reached by the object for $0 \le t \le 4$. [2]

(e) Write down an expression that represents the distance travelled by the object while its speed is increasing. Do not evaluate the expression. [3]

Question 24

[Maximum mark: 13]

The function h is defined by $h(x) = 2xe^x + 3$, for $x \in \mathbb{R}$. The following diagram shows part of the graph of h, which has a local minimum at point A.



(a) Find the value of the y-intercept. [2]

(b) Find $h'(x)$. [2]

(c) Hence, find the coordinates of A. [5]

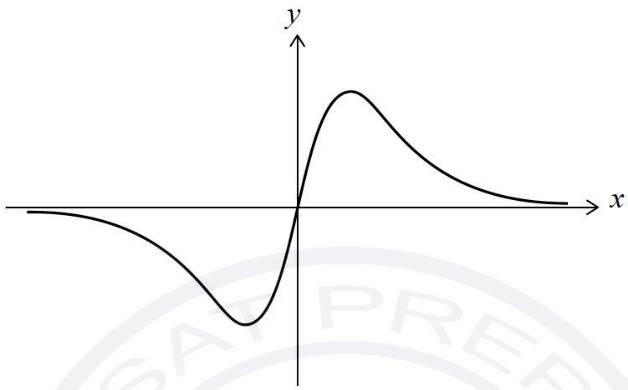
(d) (i) Show that $h''(x) = (2x + 4)e^x$.

(ii) Find the values of x for which the graph of h is concave-up. [4]

Question 25

[Maximum mark: 13]

Consider the function f defined by $f(x)=\frac{8x}{\left(x^2+1\right)^3}$, where $x \in \mathbb{R}$. The graph of f is shown in the following diagram.



(a) Show that $f'(x)=\frac{8\left(1-5x^2\right)}{\left(x^2+1\right)^4}$. [4]

(b) Find $\int f(x)dx$. [4]

Consider a function $g(x)$ defined for $x \in \mathbb{R}$. The derivative of g is such that $g'(x)=f'(x)$, for all $x \in \mathbb{R}$.

Let R be the region enclosed by the graph of f, the graph of g, the line $x=0$ and the line $x=3$. The area of R is $\frac{27}{2}$.

(c) Find the two possible expressions for $g(x)$. [5]

Question 26

[Maximum mark: 17]

The derivative of a function f is given by $f'(x) = \dfrac{2x+2}{x^2+2x+2}$, for $x \in \mathbb{R}$.

(a) (i) Show that $x^2 + 2x + 2 > 0$ for all values of x.

(ii) Hence, find the values of x for which f is increasing. [3]

(b) (i) Write down the value of x for which $f'(x) = 0$.

(ii) Show that $f''(x) = \dfrac{-2x^2 - 4x}{(x^2+2x+2)^2}$.

(iii) Hence, justify that the value of x found in part (b)(i) corresponds to a local minimum point on the graph of f. [7]

It is given that $f(2) = 3 + \ln 10$.

(c) Find an expression for $f(x)$. [4]

(d) Find the equation of the normal to the graph of f at $(2, 3 + \ln 10)$. [3]

Question 27

[Maximum mark: 14]

Consider the curve with equation $y = x^3 - x^2 - x + 1$.

(a) Find

(i) $\dfrac{dy}{dx}$;

(ii) $\dfrac{d^2y}{dx^2}$. [3]

The curve has a local maximum at A.

(b) Find the coordinates of A, using your answer to part (a)(ii) to justify your answer. [6]

The curve has a point of inflexion at B.

(c) Find the x-coordinate of B. [2]

The line L is the normal to the curve at the point $(0, 1)$.

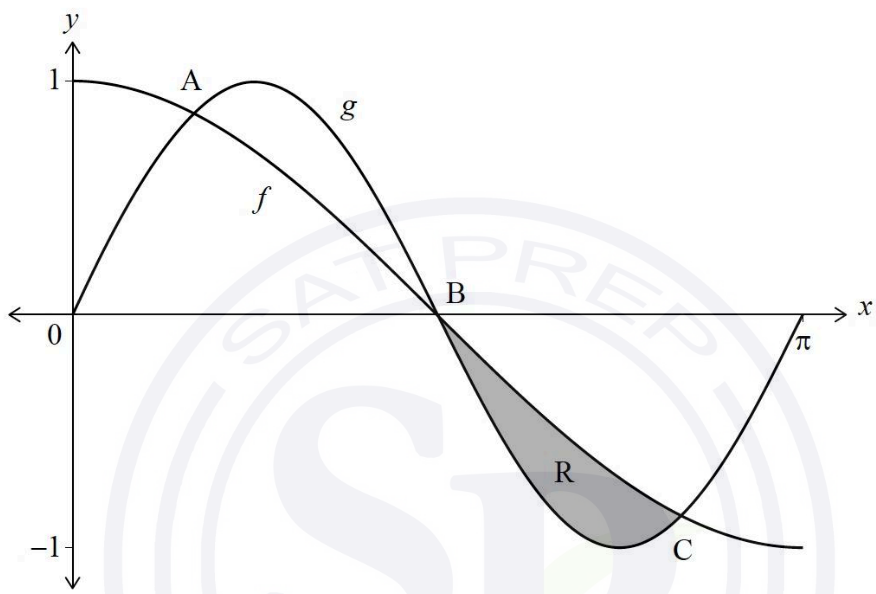
(d) Find the equation of L. [3]

Question 28

[Maximum mark: 7]

Consider the functions $f(x) = \cos x$ and $g(x) = \sin 2x$, where $0 \le x \le \pi$.

The graph of f intersects the graph of g at the point A, the point $\mathrm{B}\left(\frac{\pi}{2}, 0\right)$ and the point C as shown on the following diagram.



(a) Find the x-coordinate of point A and the x-coordinate of point C. [3]

The shaded region R is enclosed by the graph of f and the graph of g between the points B and C.

(b) Find the area of R. [4]

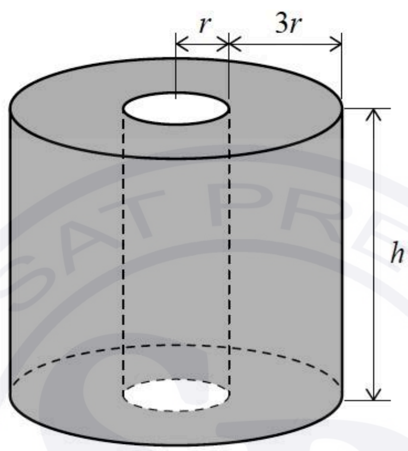
Question 29

[Maximum mark: 17]

Consider a cylinder of radius $4r$ and height h. A smaller cylinder of radius r is removed from the centre to form a hollow cylinder. This is shown in the following diagram.

All lengths are measured in centimetres.

diagram not to scale



The total surface area of the hollow cylinder, in cm^2, is given by S.

The volume of the hollow cylinder, in cm^3, is given by V.

(a) Show that $S = 30\pi r^2 + 10\pi rh$. [3]

(b) The total surface area of the hollow cylinder is $240\pi\,\text{cm}^2$.

Show that $V = 360\pi r - 45\pi r^3$. [6]

(c) Find an expression for $\frac{\text{d}V}{\text{d}r}$. [2]

The hollow cylinder has its maximum volume when $r = p\sqrt{\frac{2}{3}}$, where $p \in \mathbb{Z}^+$.

(d) Find the value of p. [3]

(e) Hence, find this maximum volume, giving your answer in the form $q\pi\sqrt{\frac{2}{3}}$, where $q \in \mathbb{Z}^+$. [3]

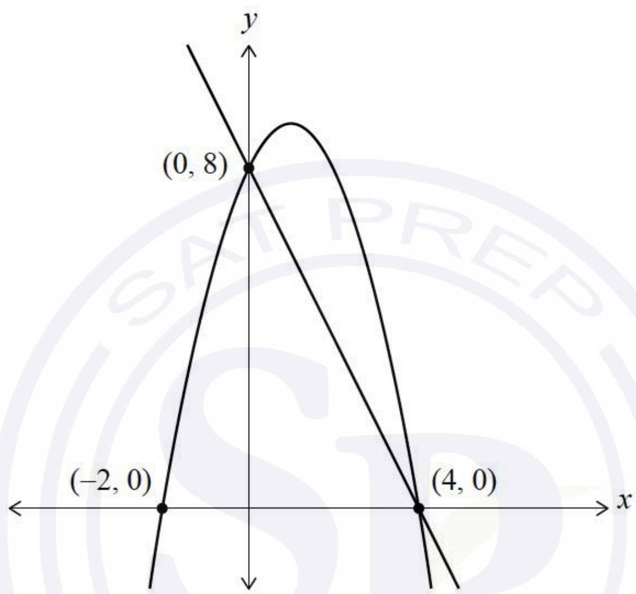
Question 30

[Maximum mark: 16]

The following diagram shows parts of the graphs of two functions f and g.

The graph of f is linear, has an x-intercept at $(4, 0)$ and a y-intercept at $(0, 8)$.

The graph of g has x-intercepts at $(-2, 0)$ and $(4, 0)$ and a y-intercept at $(0, 8)$.



(a) Write down the equation for f in the form $f(x) = mx + c$. [2]

The function g is given by $g(x) = -x^2 + bx + 8$, where b is a real constant.

(b) Find the value of b. [3]

(c) Show that the area of the region enclosed by the graph of f and the graph of g can be represented by the definite integral $\int_0^4 (-x^2 + 4x)\mathrm{d}x$. [2]

(d) Hence, find the area of the region enclosed by the graph of f and the graph of g. [4]

Point P is on the graph of g. The tangent to the graph of g at P is parallel to the graph of f.

(e) Find the coordinates of P. [5]

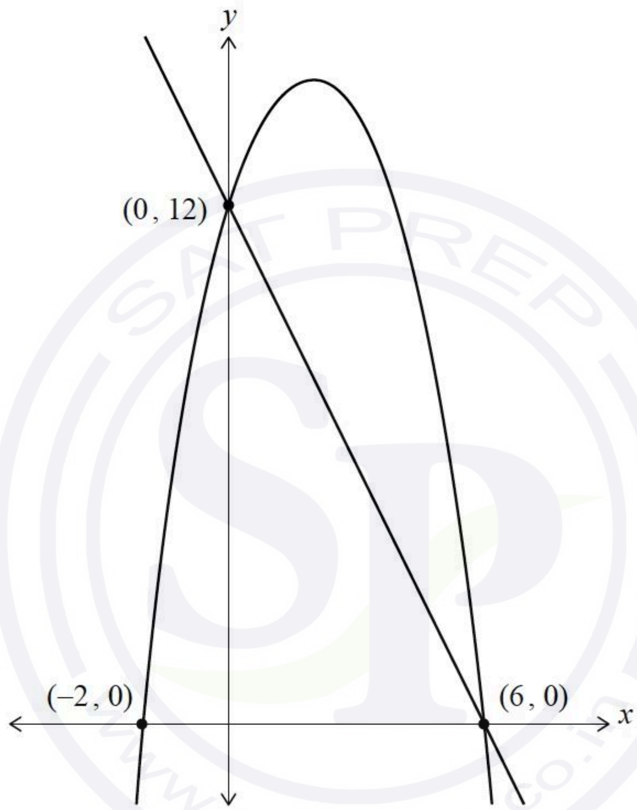
Question 31

[Maximum mark: 16]

The following diagram shows parts of the graphs of two functions f and g.

The graph of f is linear, has an x-intercept at $(6, 0)$ and a y-intercept at $(0, 12)$.

The graph of g has x-intercepts at $(-2, 0)$ and $(6, 0)$ and a y-intercept at $(0, 12)$.



(a) Write down the equation for f in the form $f(x) = mx + c$. [2]

The function g is given by $g(x) = -x^2 + bx + 12$, where b is a real constant.

(b) Find the value of b. [3]

(c) Show that the area of the region enclosed by the graph of f and the graph of g can be represented by the definite integral $\int_0^6 (-x^2 + 6x)\,dx$. [2]

(d) Hence, find the area of the region enclosed by the graph of f and the graph of g. [4]

Point P is on the graph of g. The tangent to the graph of g at P is parallel to the graph of f.

(e) Find the coordinates of P. [5]