Subject – Math AA(Standard Level)
Topic - Functions
Year - May 2021 – Nov 2024
Paper -2
Answers

Question 1

(a) $100 = A_0 e^0$ **A1**

$A_0 = 100$ **AG**

[1 mark]

(b) correct substitution of values into exponential equation **(M1)**

$50 = 100e^{-5730k}$ OR $e^{-5730k} = \frac{1}{2}$

EITHER

$-5730k = \ln\frac{1}{2}$ **A1**

$\ln\frac{1}{2} = -\ln 2$ OR $-\ln\frac{1}{2} = \ln 2$ **A1**

OR

$e^{5730k} = 2$ **A1**

$5730k = \ln 2$ **A1**

THEN

$k = \frac{\ln 2}{5730}$ **AG**

Note: There are many different ways of showing that $k = \frac{\ln 2}{5730}$ which involve showing different steps. Award full marks for at least two correct algebraic steps seen.

[3 marks]

(c) if $25\,\%$ of the carbon-14 has decayed, $75\,\%$ remains ie, 75 units remain ***(A1)***

$$75 = 100e^{-\frac{\ln 2}{5730}t}$$

EITHER

using an appropriate graph to attempt to solve for t ***(M1)***

OR

manipulating logs to attempt to solve for t ***(M1)***

$$\ln 0.75 = -\frac{\ln 2}{5730}t$$

$$t = 2378.164...$$

THEN

$t = 2380$ (years) (correct to the nearest 10 years) ***A1***

[3 marks]

Total [7 marks]

Question 2

(a) attempting to find the vertex ***(M1)***

$x = 1$ OR $y = -5$ OR $f(x) = 6(x-1)^2 - 5$

range is $y \geq -5$ ***A1***

[2 marks]

(b) **METHOD 1**

$(g \circ f)(x) = -(6x^2 - 12x + 1) + c \ \left(= -\left(6(x-1)^2 - 5\right) + c\right)$ ***(A1)***

EITHER

relating to the range of f OR attempting to find $g(-5)$ ***(M1)***

$5 + c \leq 0$ ***(A1)***

OR

attempting to find the discriminant of $(g \circ f)(x)$ ***(M1)***

$144 + 24(c-1) \leq 0 \ (120 + 24c \leq 0)$ ***(A1)***

THEN

$c \leq -5$ ***A1***

[4 marks]

METHOD 2

vertical reflection followed by vertical shift ***(M1)***

new vertex is $(1, 5+c)$ ***(A1)***

$5 + c \leq 0$ ***(A1)***

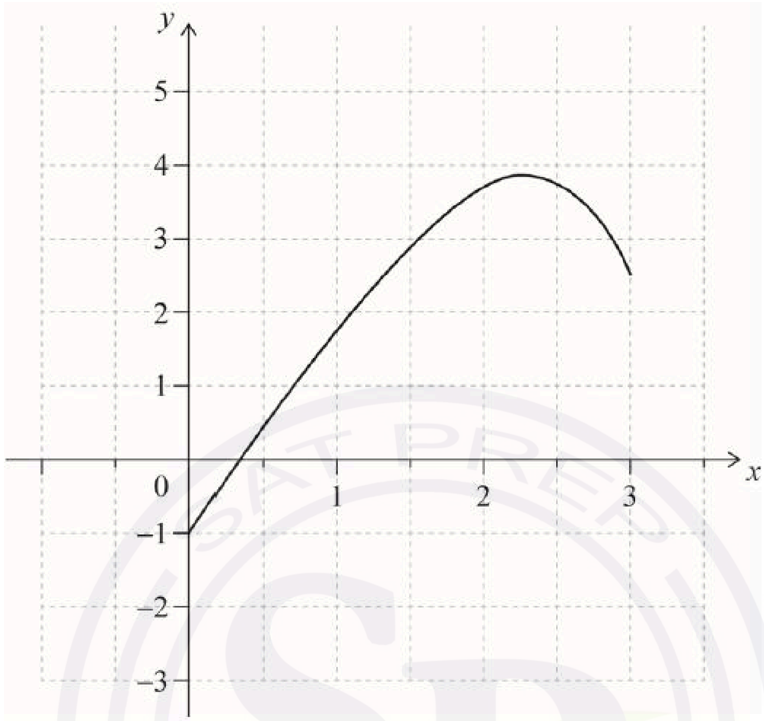
$c \leq -5$ ***A1***

[4 marks]

Total [6 marks]

Question 3

(a)



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Note: Award ***A1*** for a smooth concave down curve with generally correct shape. If first mark is awarded, award ***A1*** for local maximum and *x*-intercept in approximately correct position, award ***A1*** for endpoints at $x=0$ and $x=3$ with approximately correct y-coordinates.

[3 marks]

(b) recognizing that $f'(x)=0$ at local maximum ***(M1)***

$x = 2.33084...$

$x = 2.33$ ***A1***

[2 marks]

Total [5 marks]

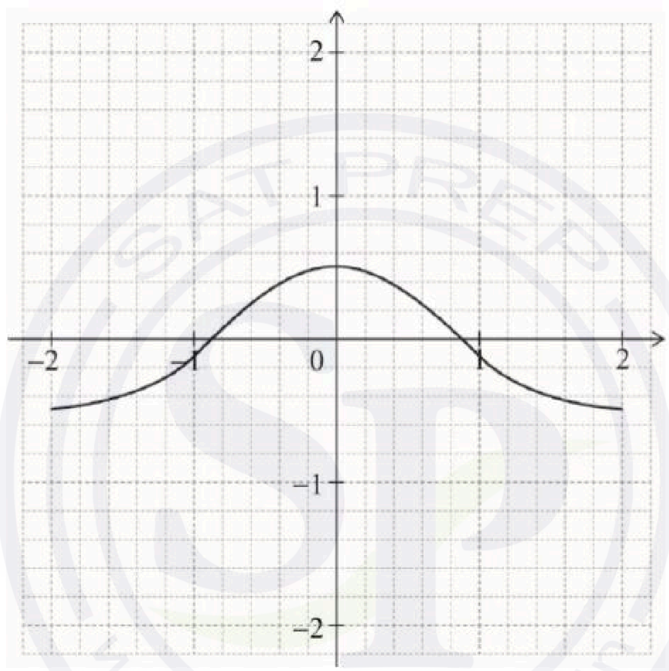
Question 4

(a) $x = -0.832554...,\ x = 0.832554...$

$x = -0.833,\ x = 0.833$ ***A1A1***

[2 marks]

(b)



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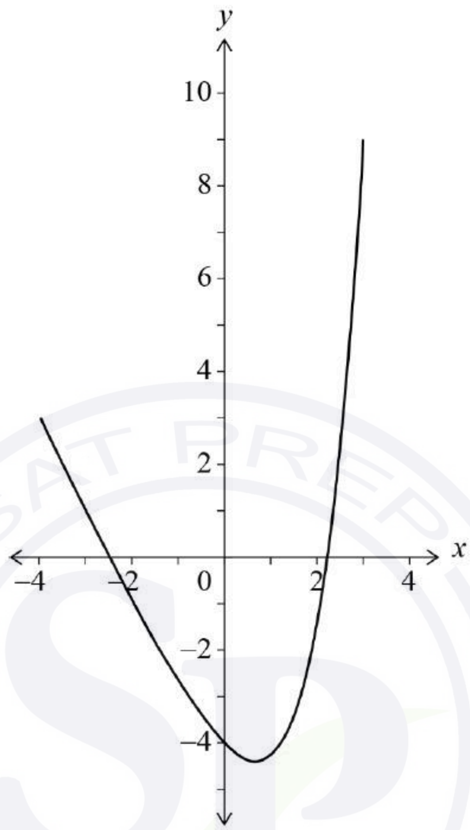
Note: Award ***A1*** for approximately correct shape. Only if this mark is awarded, award ***A1*** for approximately correct roots and maximum point and ***A1*** for approximately correct endpoints.
Allow $-1 < x \le -0.8,\ 0.8 \le x < 1$ for roots, $x = 0,\ 0.4 \le y \le 0.6$ for maximum and $x = \pm 2,\ -0.6 \le y \le -0.4$ for endpoints.

[3 marks]

Total [5 marks]

Question 5

(a)



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te: Award marks as follows:

A1 for approximately correct roots, in the intervals $-3 < x < -2$ and $2 < x < 3$.

A1 for y-intercept AND local minimum in approximately correct positions. Allow for y-intercept $-4.5 < y < -3.5$, and for local minimum $0.2 < x < 1.2$, $-5 < y < -4$.

A1 for approximately correct endpoints, with the left end in the intervals $-4.5 < x < -3.5,\ 2.5 < y < 3.5$ and the right end in the intervals $2.5 < x < 3.5,\ 8.5 < y < 9.5$.

[3 marks]

(b) $k = \frac{1}{3}$ ***A1***

$c = -2$ (accept translate/shift 2 (units) down) ***A1***

[2 marks]

Total [5 marks]

Question 6

(a) (i) $x=-4$ **A1**

(ii) attempt to substitute into $y=\frac{a}{c}$ OR table with large values of x OR sketch of f showing asymptotic behaviour **(M1)**

$y=4$ **A1**

[3 marks]

(b) (i) $y=\frac{4x+1}{x+4}$

attempt to interchange x and y (seen anywhere) **M1**

$xy+4y=4x+1$ OR $xy+4x=4y+1$ **(A1)**

$xy-4x=1-4y$ OR $xy-4y=1-4x$ **(A1)**

$f^{-1}(x)=\frac{1-4x}{x-4}$ (accept $y=\frac{1-4x}{x-4}$) **A1**

(ii) reflection in y-axis given by $f(-x)$ **(M1)**

$f(-x)=\frac{-4x+1}{-x+4}$ **(A1)**

reflection of their $f(-x)$ in x-axis given by $-f(-x)$ accept "now $-f(x)$" **M1**

$(-f(-x)=)\ -\frac{-4x+1}{-x+4}$

$=\frac{-4x+1}{x-4}$ OR $\frac{4x-1}{-x+4}$ **A1**

$=\frac{1-4x}{x-4}\ \left(=f^{-1}(x)\right)$ **AG**

> **Note:** If the candidate attempts to show the result using a particular coordinate on the graph of f rather than a general coordinate on the graph of f, where appropriate, award marks as follows:
> ***M0A0*** **for eg** $(2,3)\rightarrow(-2,3)$
> ***M0A0*** **for** $(-2,3)\rightarrow(-2,-3)$

[8 marks]

(c) (i) attempt to solve $f(x)=f^{-1}(x)$ using graph or algebraically ***(M1)***

$p=-1$ AND $q=1$ ***A1***

Note: Award ***(M1)A0*** if only one correct value seen.

(ii) attempt to set up an integral to find area between f and f^{-1} ***(M1)***

$\int_{-1}^{1}\left(\frac{4x+1}{x+4}-\frac{1-4x}{x-4}\right)\mathrm{d}x$ ***(A1)***

$=0.675231...$

$=0.675$ ***A1***

[5 marks]

Total [16 marks]

Question 7

recognition that initial population is 15000 (seen anywhere) ***(A1)***

$P(0)=15000$ OR 0.11×15000 OR 0.89×15000

population after 11% decrease is $15000\times0.89\,(=13350)$ ***(A1)***

recognizing that $t=8$ on 1 January 2022 (seen anywhere) ***(A1)***

substitution of their value of t for 1 January 2022 and their value of $P(8)$ into the model ***(M1)***

$15000\times0.89=15000\mathrm{e}^{8k}$ OR $13350=15000\mathrm{e}^{8k}$

$k=\frac{\ln 0.89}{8}\,(-0.014566)$ ***(A1)***

substitution of $t=2041-2014(=27)$ and their value for k into the model ***(M1)***

$P(27)=15000e^{-0.0145...\times27}$

$10122.3\ldots$

$P(27)=10100\ (10122)$ ***A1***

Total [7 marks]

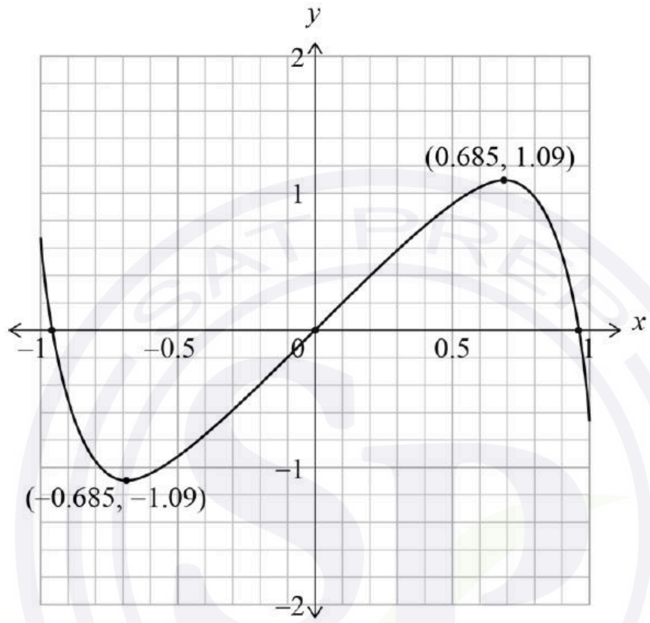
Question 8

(a) attempt to substitute g into f **(M1)**

$(f \circ g)(x) = 2\tan x - \tan^3 x$ **A1**

[2 marks]

(b)



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Note: ***A1*** for approximately correct odd function passing through the origin with a maximum above $y = 1$ and a minimum below $y = -1$.

A1 for endpoints at $x = \pm 1$ and y in the intervals $[0.6, 0.8]$ and $[-0.8, -0.6]$

A1 for maximum in approximately correct position and labelled $(0.685, 1.09)$ AND minimum in approximately correct position and labelled $(-0.685, -1.09)$. For approximate position, allow $-0.8 \le x \le -0.6$, $-1.2 \le y \le -1$ for minimum and $0.6 \le x \le 0.8$, $1 \le y \le 1.2$ for maximum. If the candidate gives the coordinates of extrema below their sketch, only award this mark if extrema are marked in the correct interval (eg by a dot).

[3 marks]

Total [5 marks]

Question 9

(a) $A(0) = 500$ (mg) ***A1***

[1 mark]

(b) $280 = 500e^{-3k}$ ***(A1)***

$k = 0.193272...$

$k = 0.193 \ \left(= -\frac{1}{3}\ln\left(\frac{280}{500}\right)\right)$ ***A1***

[2 marks]

(c) $500e^{-0.193272...T} = 140$ ***(A1)***

$T = 6.58636...$

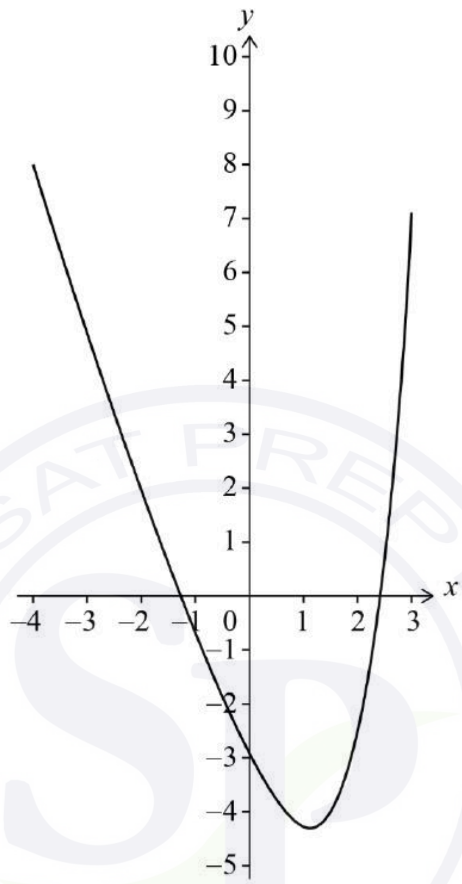
$T = 6.59$ (h) ***A1***

[2 marks]

Total [5 marks]

Question 10

(a)



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: Award marks as follows:

A1 for approximately correct roots, in the intervals $-2 < x < -1$ and $2 < x < 3$.

A1 for y-intercept AND local minimum in approximately correct positions. Allow for y-intercept $-3.5 < y < -2.5$, and for local minimum $0.5 < x < 1.5$, $-5 < y < -4$.

A1 for approximately correct endpoints, with the left end in the intervals $-4.5 < x < -3.5$, $7.5 < y < 8.5$ and the right end in the intervals $2.5 < x < 3.5$, $6.5 < y < 7.5$.

[3 marks]

(b) $k = \frac{1}{2}$ ***A1***

$c = -3$ (accept translate/shift 3 (units) down) ***A1***

[2 marks]

Total [5 marks]

Question 11

(a) $I = 2\times10^{-6}\left(=\frac{1}{500000}\right)$ (units) ***A1***

[1 mark]

(b) substitutes their doubled I-value from part (a) into L ***(M1)***

$L = 10\log_{10}\left(2\times10^{-6}\times10^{12}\right)\ (=63.0102...)$

$= 63.0$ (decibels) ***A1***

Note: Accept $60+10\log_{10}2$ (decibels) as a final answer. Do not award the final ***A1*** for $L=0$ (from $I=10^{-12}$).

[2 marks]

(c) $115 = 10\log_{10}\left(I\times10^{12}\right)$ ***(A1)***

attempts to solve for I ***(M1)***

$I = \frac{10^{11.5}}{10^{12}}$ (or equivalent) $(=0.316227...)$

$I = 0.316$ (units) ***A1***

Note: Accept exact final answers such as $10^{-0.5}$ and $\frac{1}{\sqrt{10}}$.

[3 marks]

Total [6 marks]

Question 12

(a) **METHOD 1**

attempt to interchange x and y ***M1***

Note: This ***M1*** may be awarded at any stage in the working.

attempt to rearrange using definition of natural log or take the natural log of both sides ***M1***

$\frac{2x}{3} = e^{y-2} \Rightarrow \ln\left(\frac{2x}{3}\right) = y - 2$ OR $x = \frac{3}{2}e^{y-2} \Rightarrow \ln(x) = \ln\left(\frac{3}{2}\right) + y - 2$ ***A1***

$y = 2 + \ln\left(\frac{2x}{3}\right)$

so $f^{-1}(x) = 2 + \ln\left(\frac{2x}{3}\right)$ ***AG***

[3 marks]

METHOD 2

attempt to verify that $(f \circ f^{-1})(x) = x$ ***M1***

$(f \circ f^{-1})(x) = \frac{3}{2}e^{\ln\left(\frac{2x}{3}\right)+2-2} \left(= \frac{3}{2}e^{\ln\left(\frac{2x}{3}\right)}\right)$

attempt to use definition of natural log ***M1***

$(f \circ f^{-1})(x) = \frac{3}{2} \times \frac{2x}{3}$ ***A1***

$(f \circ f^{-1})(x) = x$ ***AG***

[3 marks]

(b) $(0.264456\ldots, 0.264456\ldots)$ AND $(2.51799\ldots, 2.51799\ldots)$ ***(A1)***

Note: Award ***A1*** for $0.264456\ldots$ and $2.51799\ldots$ seen.

attempt to put their values in distance formula or use of the isosceles right-angled triangle ***(M1)***

$\sqrt{(2.51799\ldots - 0.264456\ldots)^2 + (2.51799\ldots - 0.264456\ldots)^2}$ OR

$\sqrt{2} \times (2.51799\ldots - 0.264456\ldots)$

$= 3.18689\ldots$

$= 3.19$ ***A1***

[3 marks]

(c) (i) $g(x) = -\dfrac{3}{2}e^{x-2} + 5$ OR $g(x) = -f(x) + 5$ ***A1A1***

Note: Award ***A1*** for each correct term.

(ii) $0 \le x \le 4$ ***A1***

[3 marks]

(d) $\dfrac{3}{2}e^{x-2} = -\dfrac{3}{2}e^{x-2} + 5$ OR $f(x) = -f(x) + 5$

attempt to collect together terms in e^{x-2} or $f(x)$ ***(M1)***

$3e^{x-2} = 5$ OR $2f(x) = 5$

$e^{x-2} = \dfrac{5}{3}$ OR $x = f^{-1}\left(\dfrac{5}{2}\right)$ ***(A1)***

$x = 2 + \ln\left(\dfrac{5}{3}\right)$ ***A1***

$\left(a = 2, b = \dfrac{5}{3}\right)$

Note: Award ***A1*** for each correct term given in exact form.

[3 marks]

Total [12 marks]

Question 13

METHOD 1

recognition that $4x^2 - rx + r - 1$ must be greater than zero (seen anywhere) ***R1***

(discriminant =) $(-r)^2 - 4(4)(r-1)$ $\left(= r^2 - 16r + 16\right)$ (seen anywhere) ***(A1)***

$1.07179...$ $\left(= 8 - 4\sqrt{3}\right)$ AND $14.9282...$ $\left(= 8 + 4\sqrt{3}\right)$ (seen anywhere) ***(A1)***

recognition that discriminant of $4x^2 - rx + r - 1$ is less than zero ***(M1)***

$1.07 < r < 14.9$ $(8 - 4\sqrt{3} < r < 8 + 4\sqrt{3})$ ***A1***

Note: Accept $1.08 \le r \le 14.9$.

METHOD 2

recognition that $4x^2 - rx + r - 1$ must be greater than zero (seen anywhere) ***R1***

EITHER

minimum when $x = \frac{r}{8} \Rightarrow (y =)\, 4\left(\frac{r}{8}\right)^2 - r\left(\frac{r}{8}\right) + r - 1 \ (> 0)$ ***(A1)***

attempt to solve their inequality for y (must be in terms of r and r^2) ***(M1)***

OR

$x < 1 \Rightarrow r > \frac{4x^2 - 1}{x - 1}$ OR $x > 1 \Rightarrow r < \frac{4x^2 - 1}{x - 1}$ ***(A1)***

attempt to find local minumum AND local maximum of $r = \frac{4x^2 - 1}{x - 1}$ ***(M1)***

THEN

$(r >) 1.07179...$ $\left(= 8 - 4\sqrt{3}\right)$ AND $(r <) 14.9282...$ $\left(= 8 + 4\sqrt{3}\right)$ (seen anywhere) ***(A1)***

$1.07 < r < 14.9$ $\left(8 - 4\sqrt{3} < r < 8 + 4\sqrt{3}\right)$ ***A1***

Note: Accept $1.08 \le r \le 14.9$.

[5 marks]

Question 14

(a) (i) $f(0)=-11$ **A1**

(ii) $-1.80650...$

$f(20)=-1.81\left(=11\sqrt{20}-51=22\sqrt{5}-51\right)$ **A1**

[2 marks]

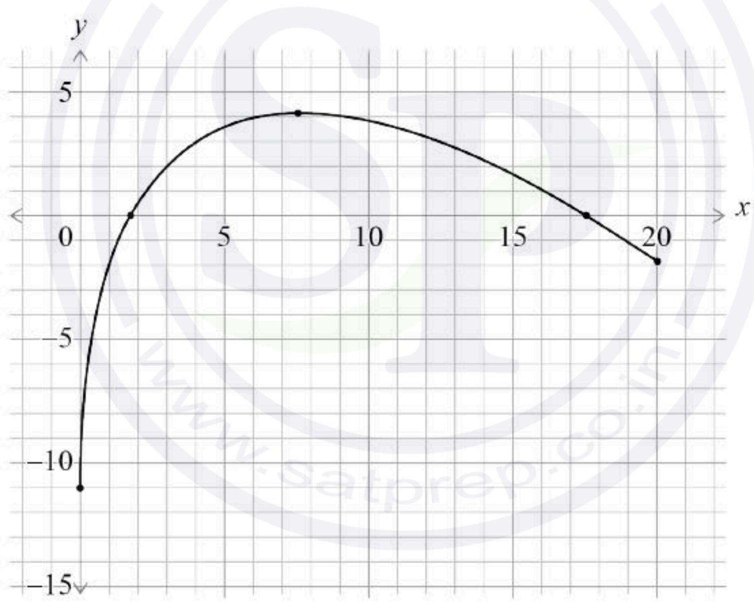
(b) attempt to find at least one root ***(M1)***

$x=1.72622...$ and $x=17.5237...$

$x=1.73$ and $x=17.5$ **A1**

[2 marks]

(c)



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[3 marks]

Total [7 marks]

Question 15

METHOD 1

recognition that $3x^2 - rx + r - 2$ must be greater than zero (seen anywhere) ***R1***

(discriminant =) $(-r)^2 - 4(3)(r-2)$ $\left(= r^2 - 12r + 24\right)$ (seen anywhere) ***(A1)***

$2.53589...$ $\left(= 6 - 2\sqrt{3}\right)$ AND $9.46410...$ $\left(= 6 + 2\sqrt{3}\right)$ (seen anywhere) ***(A1)***

recognition that discriminant of $3x^2 - rx + r - 2$ is less than zero ***(M1)***

$2.54 < r < 9.46$ $\left(6 - 2\sqrt{3} < r < 6 + 2\sqrt{3}\right)$ ***A1***

Note: Accept $2.54 \le r \le 9.46$.

METHOD 2

recognition that $3x^2 - rx + r - 2$ must be greater than zero (seen anywhere) ***R1***

EITHER

minimum when $x = \frac{r}{6} \Rightarrow (y =)\, 3\left(\frac{r}{6}\right)^2 - r\left(\frac{r}{6}\right) + r - 2\ (> 0)$ ***(A1)***

attempt to solve their inequality for y (must be in terms of r and r^2) ***(M1)***

OR

$x < 1 \Rightarrow r > \frac{3x^2 - 2}{x - 1}$ OR $x > 1 \Rightarrow r < \frac{3x^2 - 2}{x - 1}$ ***(A1)***

attempt to find local minumum AND local maximum of $r = \frac{3x^2 - 2}{x - 1}$ ***(M1)***

THEN

$(r >)\, 2.53589...$ $\left(= 6 - 2\sqrt{3}\right)$ AND $(r <)\, 9.46410...$ $\left(= 6 + 2\sqrt{3}\right)$ (seen anywhere) ***(A1)***

$2.54 < r < 9.46$ $\left(6 - 2\sqrt{3} < r < 6 + 2\sqrt{3}\right)$ ***A1***

Note: Accept $2.54 \le r \le 9.46$.

[5 marks]

Question 16

(a) (i) $f(0) = -7$ **A1**

(ii) $-12.7782...$

$f(60) = -12.8 \left(= 7\sqrt{60} - 67 = 14\sqrt{15} - 67\right)$ **A1**

[2 marks]

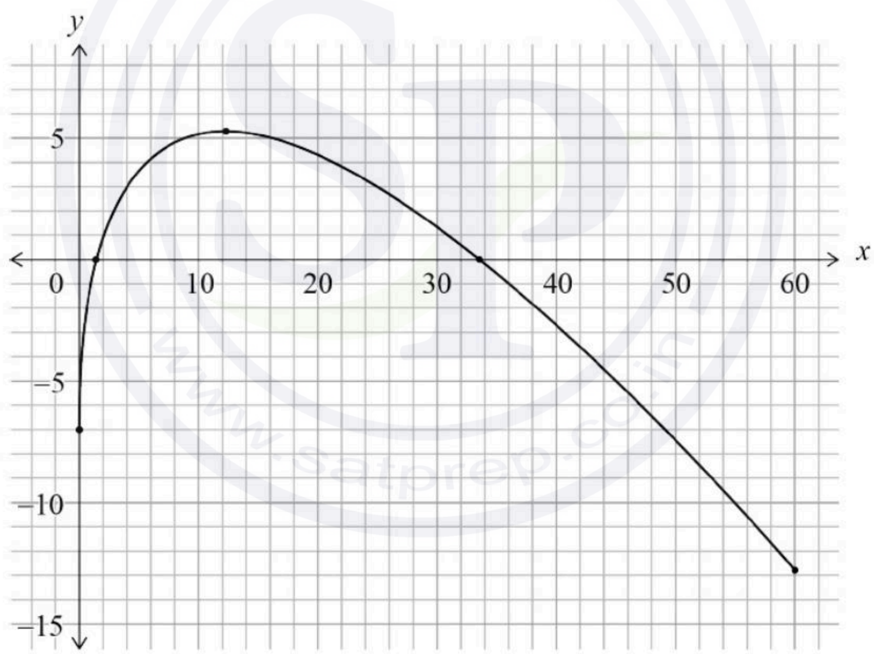
(b) attempt to find at least one root ***(M1)***

$x = 1.46098...$ and $x = 33.5390...$

$x = 1.46$ and $x = 33.5$ **A1**

[2 marks]

(c)



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[3 marks]

Total [7 marks]