Subject – Math AI(Higher Level)
Topic - Calculus
Year - May 2021 – Nov 2022
Paper -1
Questions

Question 1

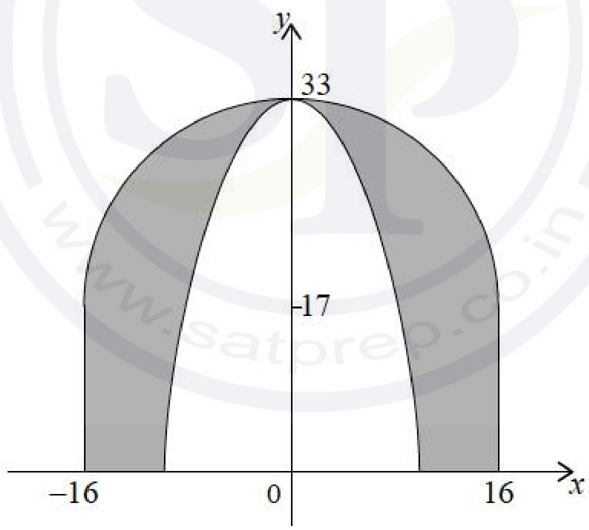
[Maximum mark: 8]

(a) The graph of $y = -x^3$ is transformed onto the graph of $y = 33 - 0.08x^3$ by a translation of a units vertically and a stretch parallel to the x-axis of scale factor b.

(i) Write down the value of a.

(ii) Find the value of b. [3]

(b) The outer dome of a large cathedral has the shape of a hemisphere of diameter $32\,\text{m}$, supported by vertical walls of height $17\,\text{m}$. It is also supported by an inner dome which can be modelled by rotating the curve $y = 33 - 0.08x^3$ through 360° about the y-axis between $y = 0$ and $y = 33$, as indicated in the diagram.



Find the volume of the space between the two domes. [5]

Question 2

[Maximum mark: 6]

The rates of change of the area covered by two types of fungi, X and Y,on a particular tree are given by the following equations, where x is the area covered by X and y is the area covered by Y.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x - 2y$$

The matrix $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ has eigenvalues of 2 and -1 with corresponding eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Initially $x = 8\,\mathrm{cm}^2$ and $y = 10\,\mathrm{cm}^2$.

(a) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $t = 0$. [2]

(b) On the following axes, sketch a possible trajectory for the growth of the two fungi, making clear any asymptotic behaviour. [4]

Question 3

[Maximum mark: 5]

A particle, A, moves so that its velocity ($v\,\mathrm{m\,s}^{-1}$) at time t is given by $v = 2\sin t,\ t \geq 0$.

The kinetic energy (E) of the particle A is measured in joules (J) and is given by $E = 5v^2$.

(a) Write down an expression for E as a function of time. [1]

(b) Hence find $\frac{\mathrm{d}E}{\mathrm{d}t}$. [2]

(c) Hence or otherwise find the first time at which the kinetic energy is changing at a rate of $5\,\mathrm{J\,s}^{-1}$. [2]

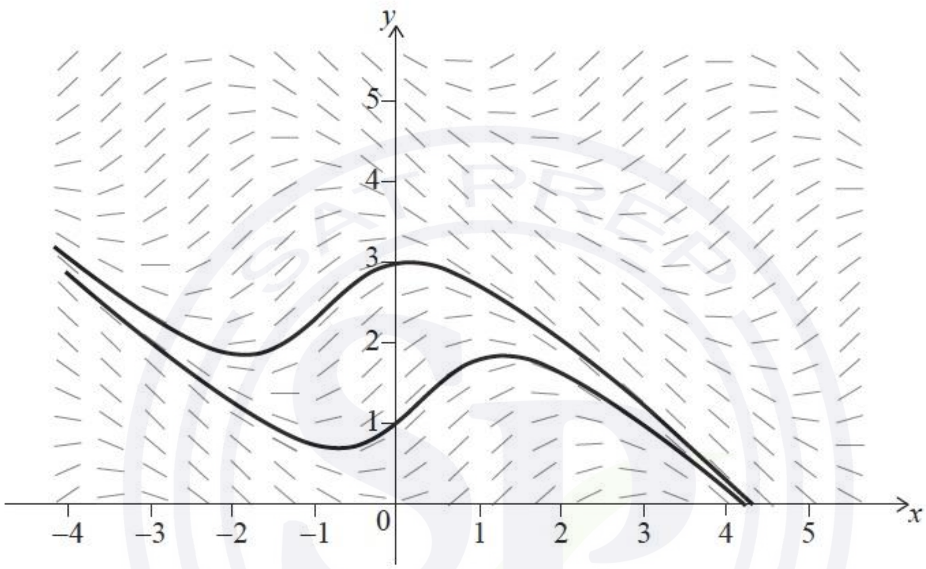
Question 4

[Maximum mark: 5]

The diagram shows the slope field for the differential equation

$$\frac{dy}{dx} = \sin(x + y),\ -4 \le x \le 5,\ 0 \le y \le 5.$$

The graphs of the two solutions to the differential equation that pass through points $(0, 1)$ and $(0, 3)$ are shown.



For the two solutions given, the local minimum points lie on the straight line L_1.

(a) Find the equation of L_1, giving your answer in the form $y = mx + c$. [3]

For the two solutions given, the local maximum points lie on the straight line L_2.

(b) Find the equation of L_2. [2]

Question 5

[Maximum mark: 8]

A tank of water initially contains 400 litres. Water is leaking from the tank such that after 10 minutes there are 324 litres remaining in the tank.

The volume of water, V litres, remaining in the tank after t minutes, can be modelled by the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -k\sqrt{V}, \text{ where } k \text{ is a constant.}$$

(a) Show that $V = \left(20 - \frac{t}{5}\right)^2$. [6]

(b) Find the time taken for the tank to empty. [2]

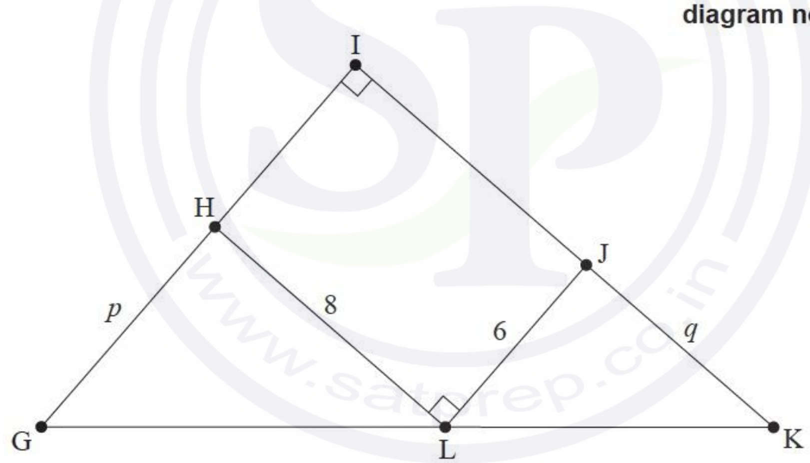
Question 6

[Maximum mark: 8]

Ellis designs a gift box. The top of the gift box is in the shape of a right-angled triangle GIK.

A rectangular section HIJL is inscribed inside this triangle. The lengths of GH, JK, HL, and LJ are $p\,\mathrm{cm}$, $q\,\mathrm{cm}$, $8\,\mathrm{cm}$ and $6\,\mathrm{cm}$ respectively.

diagram not to scale



The area of the top of the gift box is $A\,\mathrm{cm}^2$.

(a) (i) Find A in terms of p and q.

(ii) Show that $A = \frac{192}{q} + 3q + 48$. [4]

(b) Find $\frac{\mathrm{d}A}{\mathrm{d}q}$. [2]

Ellis wishes to find the value of q that will minimize the area of the top of the gift box.

(c) (i) Write down an equation Ellis could solve to find this value of q.

(ii) Hence, or otherwise, find this value of q. [2]

Question 7

[Maximum mark: 8]

A particle P moves in a straight line, such that its displacement x at time t $(t \geq 0)$ is defined by the differential equation $\dot{x} = x\cos t\left(e^{-\sin t}\right)$. At time $t = 0$, $x = \frac{1}{e}$.

(a) By using Euler's method with a step length of 0.1, find an approximate value for x when $t = 0.3$. [3]

(b) By solving the differential equation, find the percentage error in your approximation for x when $t = 0.3$. [5]

Question 8

[Maximum mark: 7]

The sides of a bowl are formed by rotating the curve $y = 6\ln x$, $0 \leq y \leq 9$, about the y-axis, where x and y are measured in centimetres. The bowl contains water to a height of $h\,\text{cm}$.

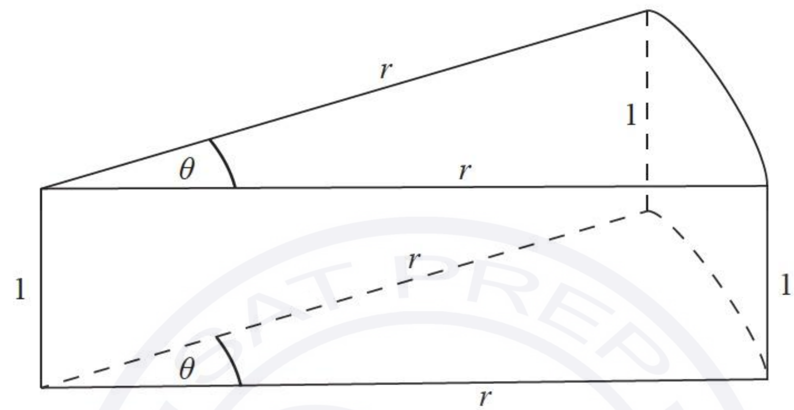
(a) Show that the volume of water, V, in terms of h is $V = 3\pi\left(e^{\frac{h}{3}} - 1\right)$. [5]

(b) Hence find the maximum capacity of the bowl in cm^3. [2]

Question 9

[Maximum mark: 9]

The following diagram shows a frame that is made from wire. The total length of wire is equal to $15\,\text{cm}$. The frame is made up of two identical sectors of a circle that are parallel to each other. The sectors have angle θ radians and radius $r\,\text{cm}$. They are connected by $1\,\text{cm}$ lengths of wire perpendicular to the sectors. This is shown in the diagram below.



(a) Show that $r = \dfrac{6}{2+\theta}$. [2]

The faces of the frame are covered by paper to enclose a volume, V.

(b) (i) Find an expression for V in terms of θ.

(ii) Find the expression $\dfrac{\mathrm{d}V}{\mathrm{d}\theta}$.

(iii) Solve algebraically $\dfrac{\mathrm{d}V}{\mathrm{d}\theta} = 0$ to find the value of θ that will maximize the volume, V. [7]

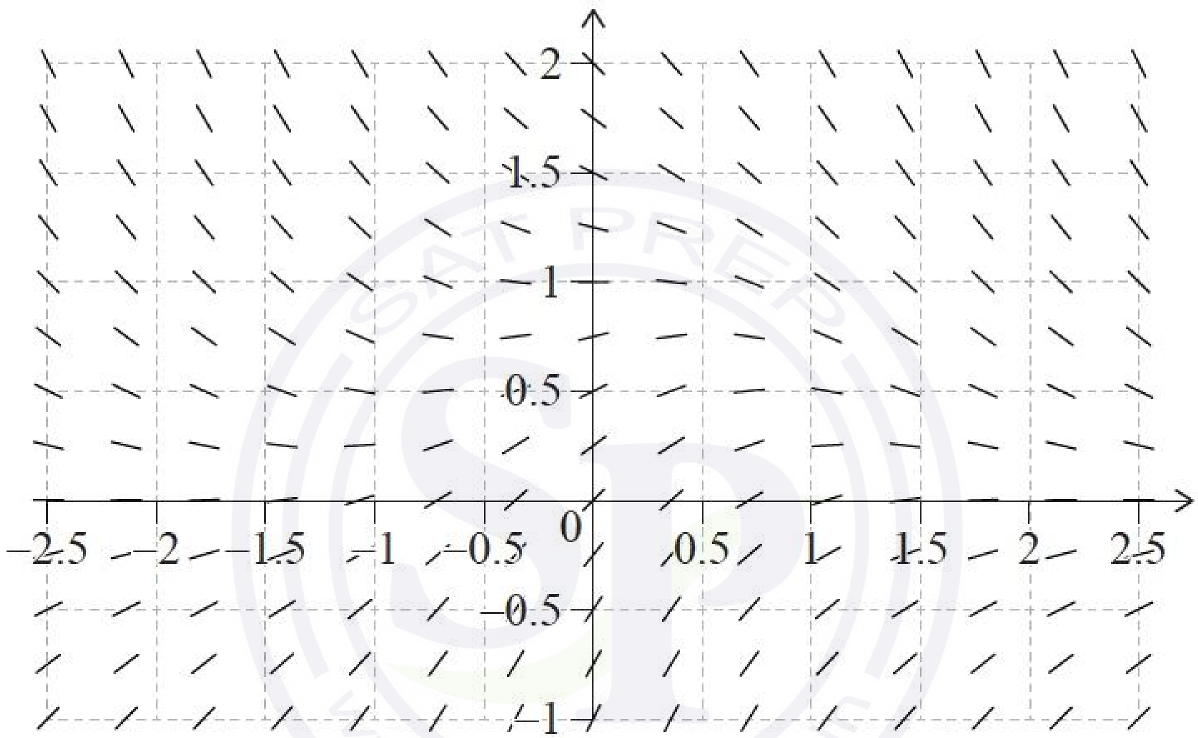
Question 10

[Maximum mark: 7]

The slope field for the differential equation $\frac{dy}{dx} = e^{-x^2} - y$ is shown in the following two graphs.

(a) Calculate the value of $\frac{dy}{dx}$ at the point $(0, 1)$. [1]

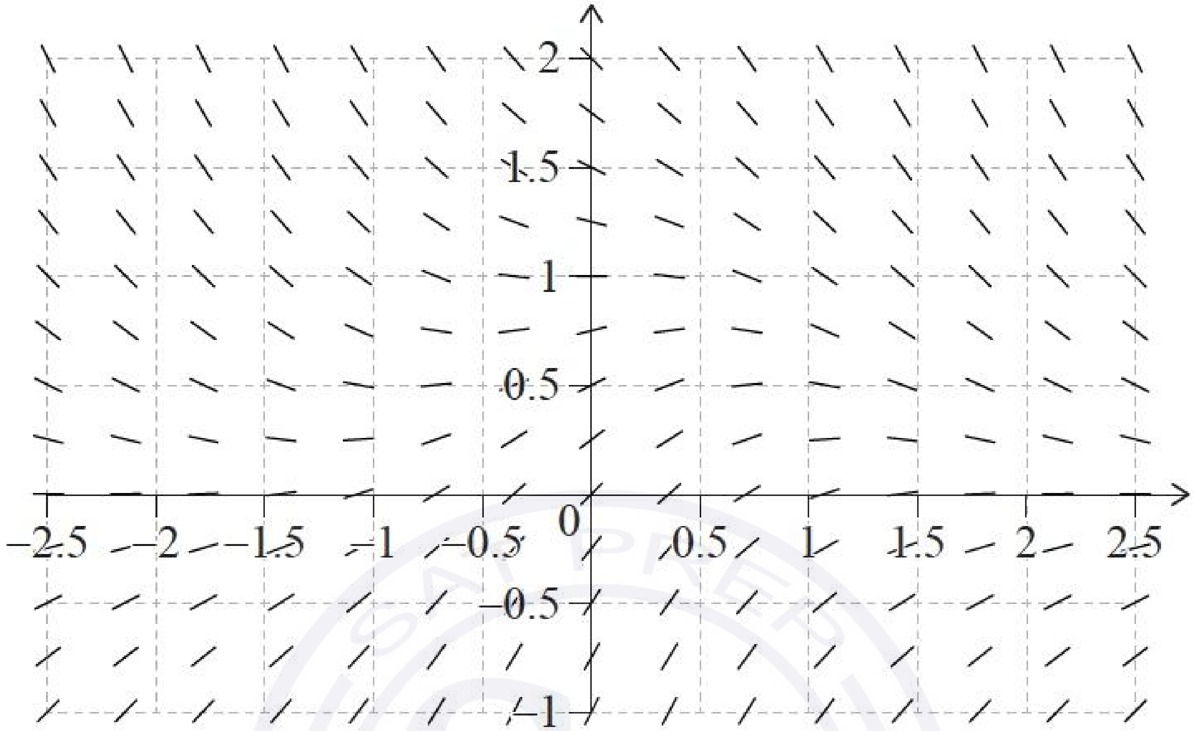
(b) Sketch, on the first graph, a curve that represents the points where $\frac{dy}{dx} = 0$. [2]



(c) On the second graph,

(i) sketch the solution curve that passes through the point $(0, 0)$.

(ii) sketch the solution curve that passes through the point $(0, 0.75)$. [4]



2
1.5
1
0.5
0
−0.5
−1
−2.5
−2
−1.5
−1
−0.5
0.5
1
1.5
2
2.5

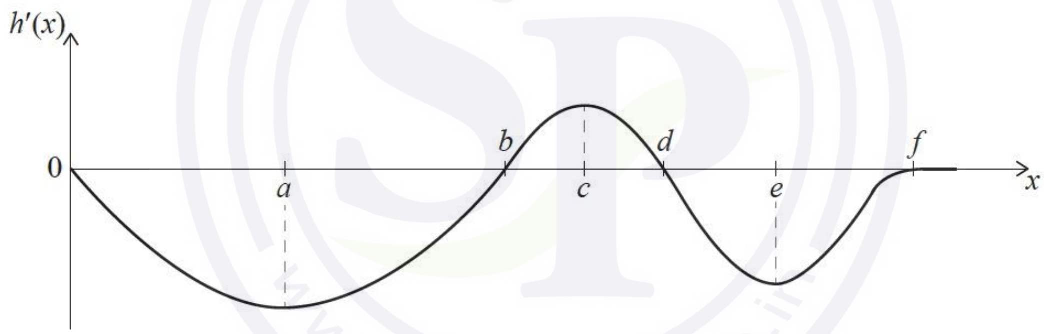
Question 11

[Maximum mark: 5]

Juri skis from the top of a hill to a finishing point at the bottom of the hill. She takes the shortest route, heading directly to the finishing point (F).

Let $h(x)$ define the height of the hill above F at a horizontal distance x from the starting point at the top of the hill.

The graph of the **derivative** of $h(x)$ is shown below. The graph of $h'(x)$ has local minima and maxima when x is equal to a, c and e. The graph of $h'(x)$ intersects the x-axis when x is equal to b, d, and f.

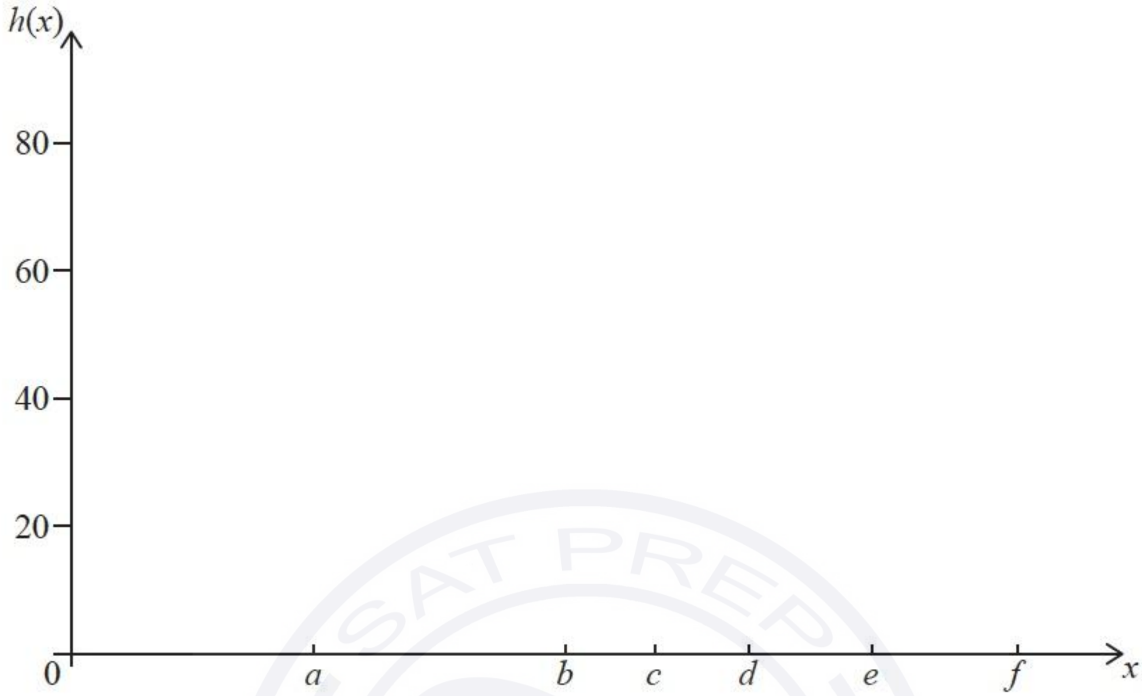


(a) (i) Identify the x value of the point where $|h'(x)|$ has its maximum value.

(ii) Interpret this point in the given context. [2]

Juri starts at a height of 60 metres and finishes at F, where $x = f$.

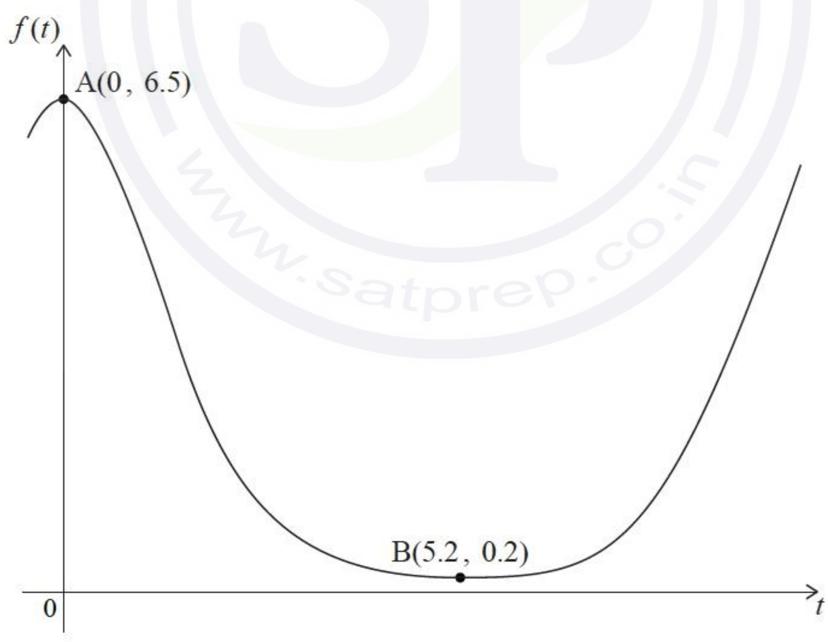
(b) Sketch a possible diagram of the hill on the following pair of coordinate axes. [3]



Question 12

[Maximum mark: 8]

A function f is of the form $f(t) = pe^{q\cos(rt)}$, p, q, $r \in \mathbb{R}^+$. Part of the graph of f is shown.



The points A and B have coordinates $A(0, 6.5)$ and $B(5.2, 0.2)$, and lie on f.

The point A is a local maximum and the point B is a local minimum.

Find the value of p, of q and of r.

Question 13

[Maximum mark: 7]

The wind chill index W is a measure of the temperature, in $^\circ\mathrm{C}$, felt when taking into account the effect of the wind.

When Frieda arrives at the top of a hill, the relationship between the wind chill index and the speed of the wind v in kilometres per hour ($\mathrm{km\,h^{-1}}$) is given by the equation

$$W = 19.34 - 7.405v^{0.16}$$

(a) Find an expression for $\frac{\mathrm{d}W}{\mathrm{d}v}$. [2]

When Frieda arrives at the top of a hill, the speed of the wind is 10 kilometres per hour and increasing at a rate of $5\,\mathrm{km\,h^{-1}\,minute^{-1}}$.

(b) Find the rate of change of W at this time. [5]

Question 14

[Maximum mark: 8]

(a) (i) Expand $\left(\frac{1}{u}+1\right)^2$.

(ii) Find $\int\left(\frac{1}{(x+2)}+1\right)^2 \mathrm{d}x$.

[4]

The region bounded by $y = \frac{1}{(x+2)}+1$, $x=0$, $x=2$ and the x-axis is rotated through 2π about the x-axis to form a solid.

(b) Find the volume of the solid formed. Give your answer in the form $\frac{\pi}{4}(a+b\ln(c))$, where a, b, $c \in \mathbb{Z}$. [4]

Question 15

[Maximum mark: 6]

Consider the curve $y = 2x(4-\mathrm{e}^x)$.

(a) Find

(i) $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(ii) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. [4]

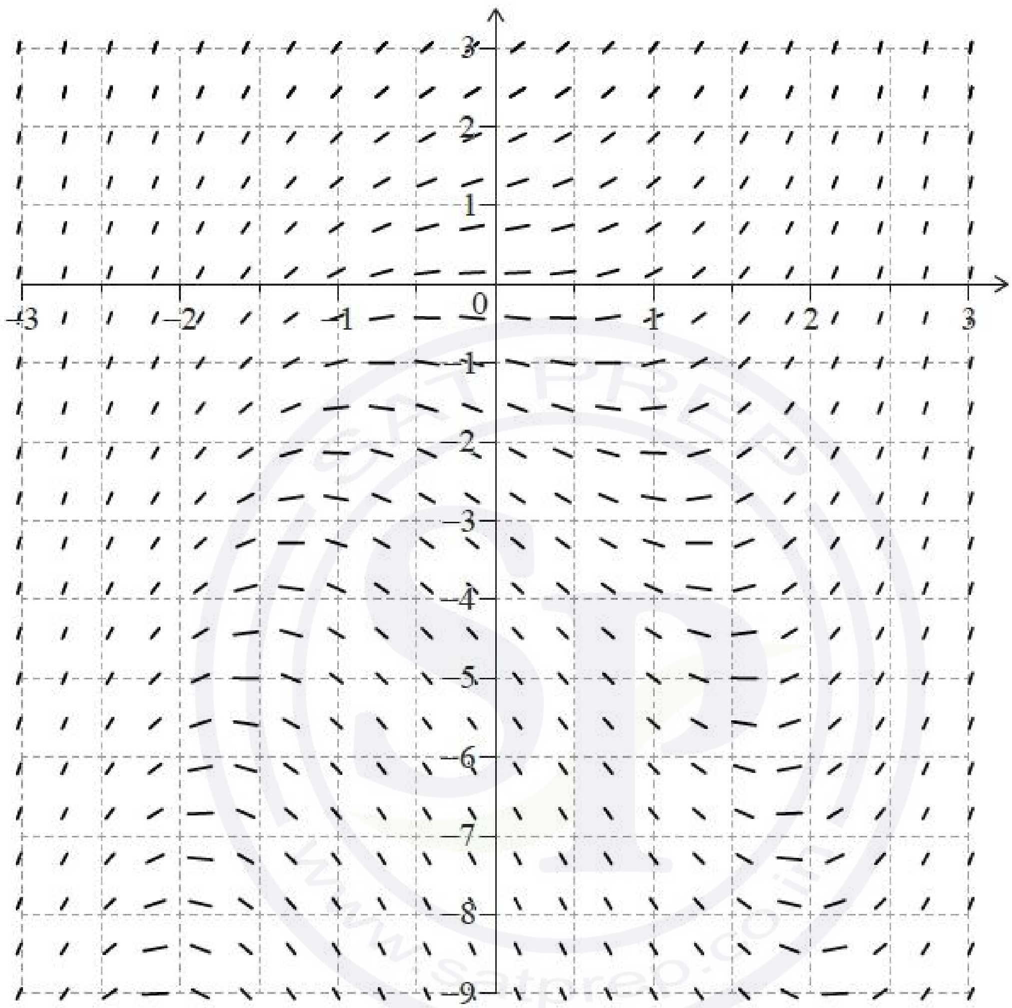
The curve has a point of inflexion at (a, b).

(b) Find the value of a. [2]

Question 16

[Maximum mark: 4]

A slope field for the differential equation $\frac{dy}{dx} = x^2 + \frac{y}{2}$ is shown.



Some of the solutions to the differential equation have a local maximum point and a local minimum point.

(a) (i) Write down the equation of the curve on which all these maximum and minimum points lie.

(ii) Sketch this curve on the slope field. [2]

The solution to the differential equation that passes through the point $(0, -2)$ has both a local maximum point and a local minimum point.

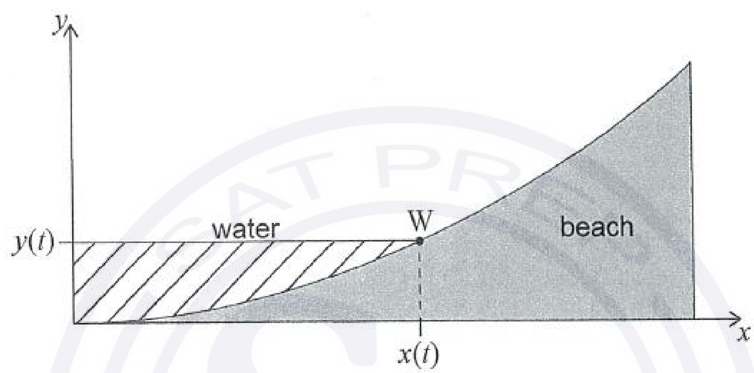
(b) On the slope field, sketch the solution to the differential equation that passes through $(0, -2)$. [2]

Question 17

[Maximum mark: 8]

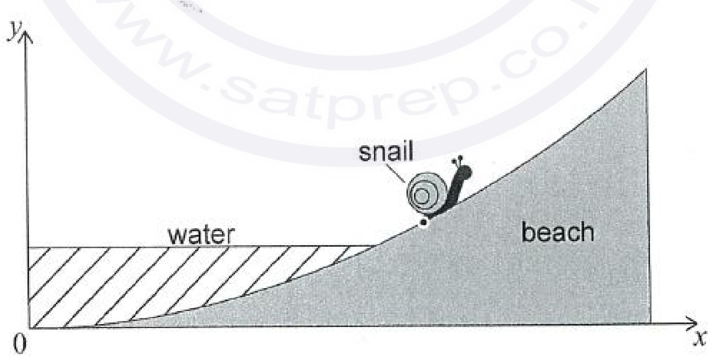
The cross-section of a beach is modelled by the equation $y = 0.02x^2$ for $0 \le x \le 10$ where y is the height of the beach (in metres) at a horizontal distance x metres from an origin. t is the time in hours after low tide.

At $t = 0$ the water is at the point $(0, 0)$. The height of the water rises at a rate of 0.2 metres per hour. The point $\mathrm{W}(x(t), y(t))$ indicates where the water level meets the beach at time t.



(a) When W has an x-coordinate equal to 1, find the horizontal component of the velocity of W. [3]

A snail is modelled as a single point. At $t = 0$ it is positioned at $(1, 0.02)$. The snail travels away from the incoming water at a speed of 1 metre per hour in the direction along the curve of the cross-section of the beach. The following diagram shows this for a value of t, such that $t > 0$.



(b) (i) Find the time taken for the snail to reach the point $(10, 2)$.

(ii) Hence show that the snail reaches the point $(10, 2)$ before the water does. [5]

Question 18

[Maximum mark: 7]

The position vector of a particle, P, relative to a fixed origin O at time t is given by

$$\vec{OP} = \begin{pmatrix} \sin(t^2) \\ \cos(t^2) \end{pmatrix}.$$

(a) Find the velocity vector of P. [2]

(b) Show that the acceleration vector of P is never parallel to the position vector of P. [5]

Question 19

[Maximum mark: 4]

The shape of a vase is formed by rotating a curve about the y-axis.

The vase is $10\,\text{cm}$ high. The internal radius of the vase is measured at $2\,\text{cm}$ intervals along the height:

Height (cm)	Radius (cm)
0	4
2	6
4	8
6	7
8	3
10	5

Use the trapezoidal rule to estimate the volume of water that the vase can hold.

Question 20

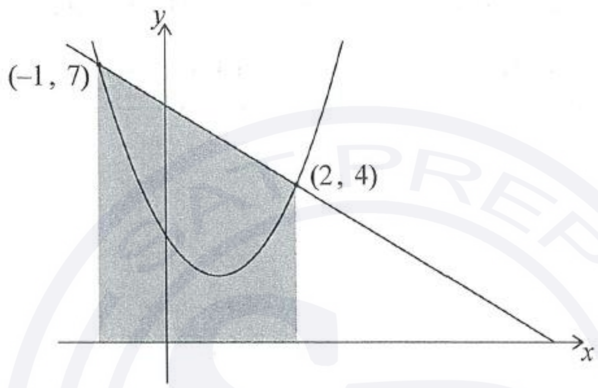
[Maximum mark: 7]

The graphs of $y = 6 - x$ and $y = 1.5x^2 - 2.5x + 3$ intersect at $(2, 4)$ and $(-1, 7)$, as shown in the following diagrams.

In **diagram 1**, the region enclosed by the lines $y = 6 - x$, $x = -1$, $x = 2$ and the x-axis has been shaded.

diagram not to scale

Diagram 1

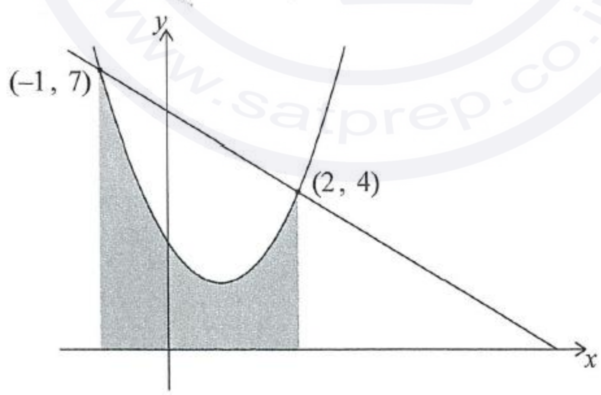


(a) Calculate the area of the shaded region in **diagram 1**. [2]

In **diagram 2**, the region enclosed by the curve $y = 1.5x^2 - 2.5x + 3$, and the lines $x = -1$, $x = 2$ and the x-axis has been shaded.

diagram not to scale

Diagram 2



(b) (i) Write down an integral for the area of the shaded region in **diagram 2**.

(ii) Calculate the area of this region. [3]

(c) Hence, determine the area enclosed between $y = 6 - x$ and $y = 1.5x^2 - 2.5x + 3$. [2]

Question 21

[Maximum mark: 5]

An electrical circuit contains a capacitor. The charge on the capacitor, q Coulombs, at time t seconds, satisfies the differential equation

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + 5\frac{\mathrm{d}q}{\mathrm{d}t} + 20q = 200.$$

Initially $q = 1$ and $\frac{\mathrm{d}q}{\mathrm{d}t} = 8$.

Use Euler's method with $h = 0.1$ to estimate the maximum charge on the capacitor during the first second.

Question 22

[Maximum mark: 9]

A particle moves such that its velocity, v metres per second, at time t seconds, is given by $v = t\sin(t^2)$.

(a) Find an expression for the acceleration of the particle. [2]

(b) Hence, or otherwise, find its greatest acceleration for $0 \le t \le 8$. [2]

The particle starts at the origin.

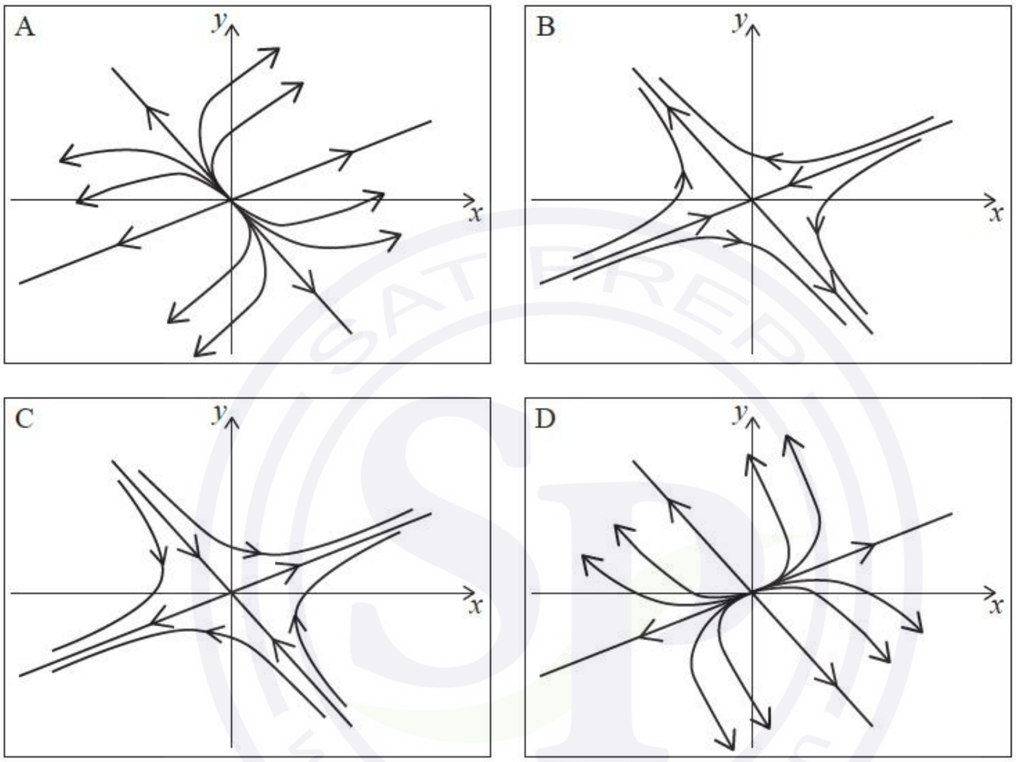
(c) Find an expression for the displacement of the particle. [3]

(d) Hence show that the particle never has a negative displacement. [2]

Question 23

[Maximum mark: 5]

Four possible phase portraits for the coupled differential equations $\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$ are shown, labelled A, B, C and D.



The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has eigenvalues λ_1 and λ_2.

(a) Complete the following table by writing down the letter of the phase portrait that best matches the description. [3]

Description	Phase portrait
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = -3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	
$\lambda_1 = -2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	