Question 1

[Maximum mark: 6]

Professor Vinculum investigated the migration season of the Bulbul bird from their natural wetlands to a warmer climate.

He found that during the migration season their population, P could be modelled by $P = 1350 + 400(1.25)^{-t}$, $t \geq 0$, where t is the number of days since the start of the migration season.

(a)   Find the population of the Bulbul birds,

   (i)   at the start of the migration season.

   (ii)   in the wetlands after 5 days.   [3]

(b)   Calculate the time taken for the population to decrease below 1400.   [2]

(c)   According to this model, find the smallest possible population of Bulbul birds during the migration season.   [1]

Question 2

[Maximum mark: 5]

The perimeter of a given square P can be represented by the function $P(A) = 4\sqrt{A}$, $A \geq 0$, where A is the area of the square. The graph of the function P is shown for $0 \leq A \leq 25$.
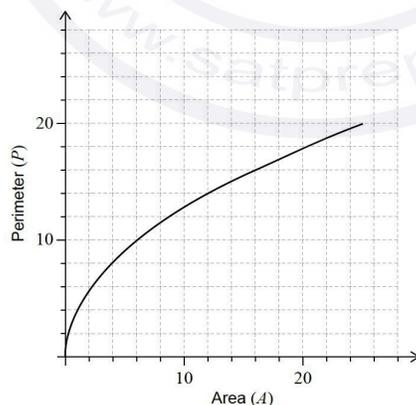
(a)   Write down the value of $P(25)$.   [1]

(b)   On the axes above, draw the graph of the inverse function, P^{-1}.   [3]

(c)   In the context of the question, explain the meaning of $P^{-1}(8) = 4$.   [1]

Question 3

[Maximum mark: 7]

The graph of the function $f(x) = \ln x$ is translated by $\begin{pmatrix} a \\ b \end{pmatrix}$ so that it then passes through the points $(0, 1)$ and $(e^3, 1 + \ln 2)$.

Find the value of a and the value of b.

Question 4

[Maximum mark: 6]

Professor Wei observed that students have difficulty remembering the information presented in his lectures.

He modelled the percentage of information retained, R, by the function $R(t) = 100\,e^{-pt}$, $t \geq 0$, where t is the number of days after the lecture.

He found that 1 day after a lecture, students had forgotten 50% of the information presented.

(a)   Find the value of p. [2]

(b)   Use this model to find the percentage of information retained by his students 36 hours after Professor Wei's lecture. [2]

Based on his model, Professor Wei believes that his students will always retain some information from his lecture.

(c)   State a mathematical reason why Professor Wei might believe this. [1]

(d)   Write down one possible limitation of the **domain** of the model. [1]

Question 5

[Maximum mark: 7]

A geometric transformation $T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix}$ is defined by

$$T: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 7 & -10 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -5 \\ 4 \end{pmatrix}.$$

(a)   Find the coordinates of the image of the point $(6, -2)$. [2]

(b)   Given that $T: \begin{pmatrix} p \\ q \end{pmatrix} \mapsto 2\begin{pmatrix} p \\ q \end{pmatrix}$, find the value of p and the value of q. [3]

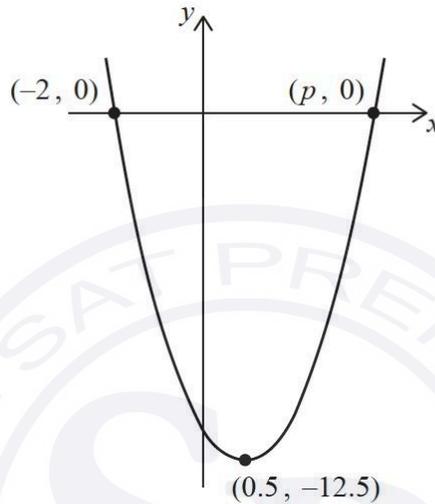(c)   A triangle L with vertices lying on the xy plane is transformed by T.

Explain why both L and its image will have exactly the same area. [2]

Question 6

[Maximum mark: 7]

Consider the function $f(x) = ax^2 + bx + c$. The graph of $y = f(x)$ is shown in the diagram. The vertex of the graph has coordinates $(0.5, -12.5)$. The graph intersects the x-axis at two points, $(-2, 0)$ and $(p, 0)$.

diagram not to scale

(a) Find the value of p. [1]

(b) Find the value of

    (i) a.

    (ii) b.

    (iii) c. [5]

(c) Write down the equation of the axis of symmetry of the graph. [1]

Question 7

[Maximum mark: 7]
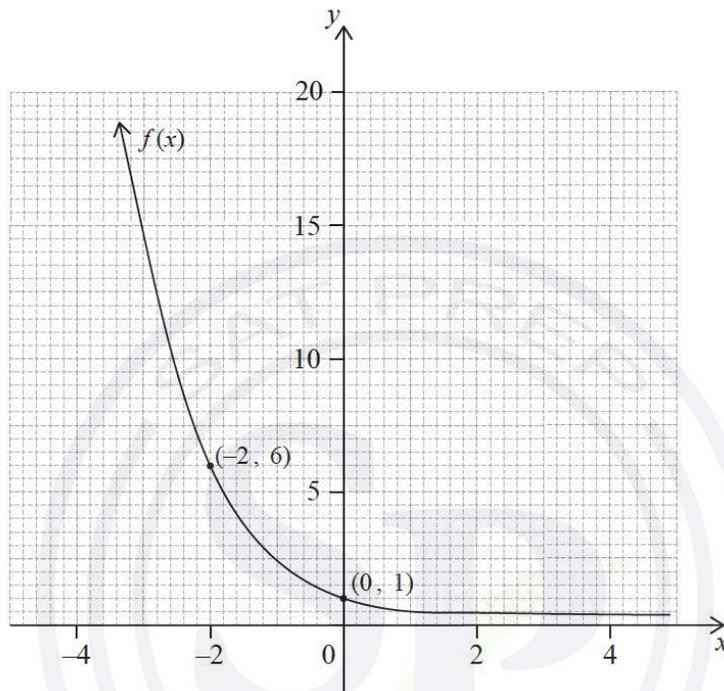
A function is defined by $f(x) = 2 - \dfrac{12}{x+5}$ for $-7 \le x \le 7$, $x \ne -5$.

(a) Find the range of f. [3]

(b) Find an expression for the inverse function $f^{-1}(x)$. The domain is not required. [3]

(c) Write down the range of $f^{-1}(x)$. [1]

Question 8

[Maximum mark: 4]

The graph of $y = f(x)$ is given on the following set of axes. The graph passes through the points $(-2, 6)$ and $(0, 1)$, and has a horizontal asymptote at $y = 0$.

Let $g(x) = 2 f(x - 2) + 4$.

(a)   Find $g(0)$.   [2]

(b)   On the same set of axes draw the graph of $y = g(x)$, showing any intercepts and asymptotes.   [2]

Question 9

[Maximum mark: 7]

Let the function $h(x)$ represent the height in centimetres of a cylindrical tin can with diameter x cm.

$$h(x) = \frac{640}{x^2} + 0.5 \text{ for } 4 \leq x \leq 14.$$

(a)   Find the range of h.   [3]

The function h^{-1} is the inverse function of h.

(b)   (i)   Find $h^{-1}(10)$.

        (ii)   In the context of the question, interpret your answer to part (b)(i).

        (iii)   Write down the range of h^{-1}.   [4]

Question 10

[Maximum mark: 5]

The height of a baseball after it is hit by a bat is modelled by the function

$$h(t) = -4.8t^2 + 21t + 1.2$$

where $h(t)$ is the height in metres above the ground and t is the time in seconds after the ball was hit.

(a)   Write down the height of the ball above the ground at the instant it is hit by the bat.   [1]

(b)   Find the value of t when the ball hits the ground.   [2]

(c)   State an appropriate domain for t in this model.   [2]
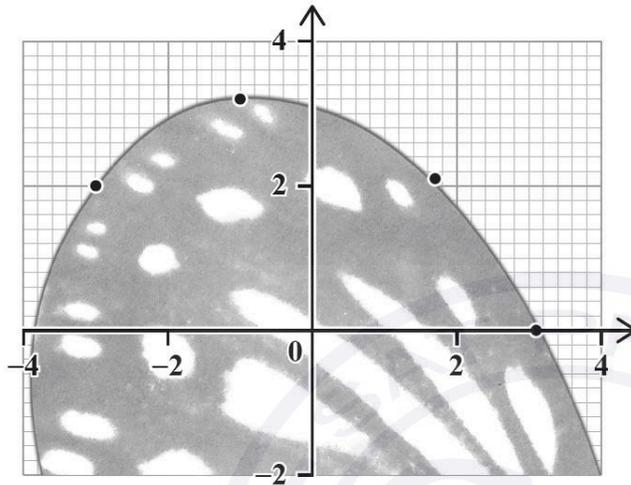
Question 11

[Maximum mark: 5]

The function $f(x) = \ln\left(\frac{1}{x-2}\right)$ is defined for $x > 2$, $x \in \mathbb{R}$.

(a)   Find an expression for $f^{-1}(x)$. You are not required to state a domain.   [3]

(b)   Solve $f(x) = f^{-1}(x)$.   [2]

Question 12

[Maximum mark: 5]

Gloria wants to model the curved edge of a butterfly wing. She inserts a photo of the wing into her graphing software and finds the coordinates of four points on the edge of the wing.

x	y
-3	2
-1	3.2
1.7	2.1
3.1	0

Gloria thinks a cubic curve will be a good model for the butterfly wing.

(a)   Find the equation of the cubic regression curve for this data. [2]

For the photo of a second butterfly wing, Gloria finds the equation of the regression curve is $y = 0.0083x^3 - 0.075x^2 - 0.58x + 2.2$.

Gloria realizes that her photo of the second butterfly is an enlargement of the life-size butterfly, scale factor 2 and centred on $(0,\ 0)$.

(b)   Find the equation of the cubic curve that models the life-size wing. [3]

Question 13

[Maximum mark: 5]

Celeste heated a cup of coffee and then let it cool to room temperature. Celeste found the coffee's temperature, T, measured in °C, could be modelled by the following function,

$$T(t) = 71\,e^{-0.0514t} + 23,\ t \geq 0,$$

where t is the time, in minutes, after the coffee started to cool.

(a)   Find the coffee's temperature 16 minutes after it started to cool. [2]

(b)   Write down the room temperature. [1]

(c)   Given that $T^{-1}(50) = k$, find the value of k. [2]

**Question 14**

[Maximum mark: 7]

In a baseball game, Sakura is the batter standing beside home plate. The ball is thrown towards home plate along a path that can be modelled by the following function.

$$y = -0.045x + 2$$

In this model, x is the horizontal distance of the ball from the point the ball is thrown and y is the vertical height of the ball above the ground. Both measured in metres.

The outcome of the throw is called a strike if the height of the ball is between $0.53\,\text{m}$ and $1.24\,\text{m}$ at some point while it travels over home plate. The length of home plate is $0.43\,\text{m}$.

diagram not to scale
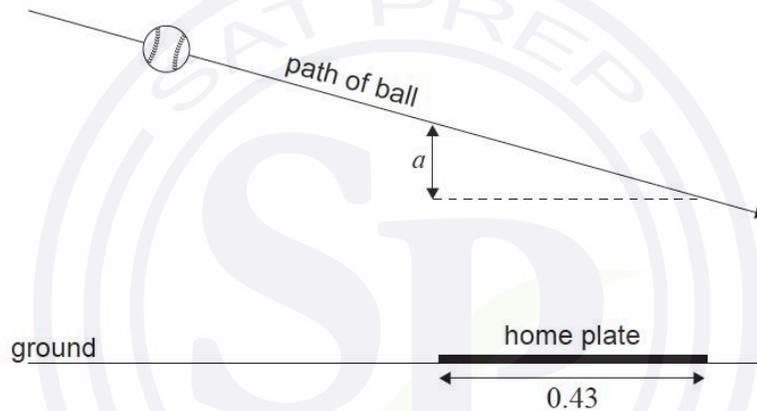
When the ball reaches the front of home plate, the height of the ball above the ground is $1.25\,\text{m}$. The height of the ball changes by a metres as the ball travels over the length of home plate.

(a) (i)   Find the value of a.

     (ii) Justify why this throw is a strike.   [4]

On the next throw, Sakura hits the ball towards a wall that is 5 metres high. The horizontal distance of the wall from the point where the ball was hit is 96 metres. The path of the ball after it is hit can be modelled by the function $h(d)$.

$$h(d) = -0.01d^2 + 1.04d + 0.66, \text{ for } h, d > 0$$

In this model, h is the height of the ball above the ground and d is the horizontal distance of the ball from the point where it was hit. Both h and d are measured in metres.

(b) Determine whether the ball will go over the wall. Justify your answer.   [3]

Question 15

A player throws a basketball. The height of the basketball is modelled by

$$h(t) = -4.75t^2 + 8.75t + 1.5, \ t \geq 0,$$

where h is the height of the basketball above the ground, in metres, and t is the time, in seconds, after it was thrown.

(a)   Find how long it takes for the basketball to reach its maximum height.   [2]

(b)   Assuming that no player catches the basketball, find how long it would take for the basketball to hit the ground.   [2]

Another player catches the basketball when it is at a height of 1.2 metres.

(c)   Find the value of t when this player catches the basketball.   [2]

(d)   Write down one limitation of using $h(t)$ to model the height of the basketball.   [1]
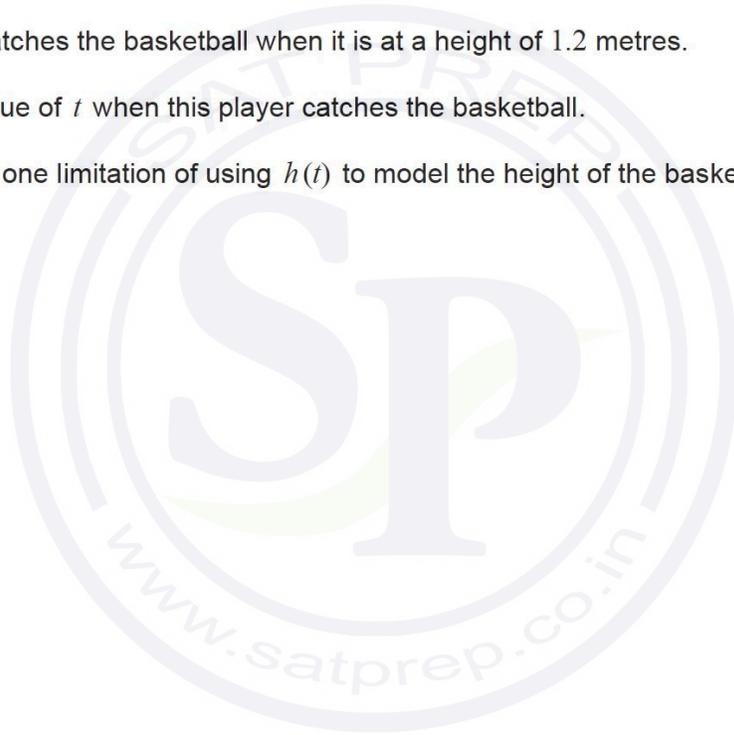
Question 16

[Maximum mark: 9]

A sports player on a horizontal athletic field hits a ball. The height of the ball above the field, in metres, after it is hit can be modelled using a quadratic function of the form $f(x) = ax^2 + bx + c$, where x represents the horizontal distance, in metres, that the ball has travelled from the player.

A specialized camera tracks the initial path of the ball after it is hit by the player. The camera records that the ball travels through the three points $(3, 2.36)$, $(10, 5)$ and $(17, 7.16)$, as shown in **Diagram 1**.
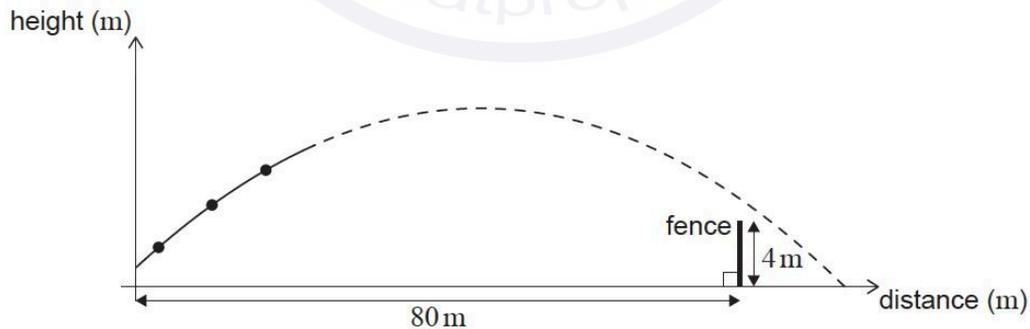
diagram not to scale

Diagram 1
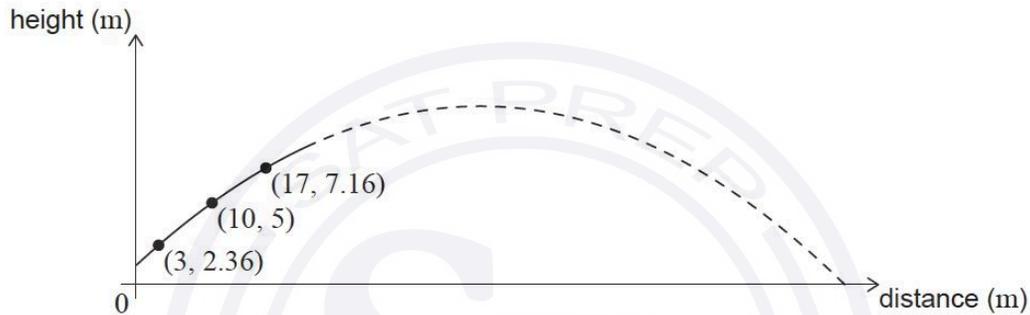
(a)   Use the coordinates $(3, 2.36)$ to write down an equation in terms of a, b, and c.   [1]

(b)   Use your answer to part (a) and two similar equations to find the equation of the quadratic model for the height of the ball.   [3]

A 4-metre-high fence is 80 metres from where the player hit the ball, as shown in **Diagram 2**.
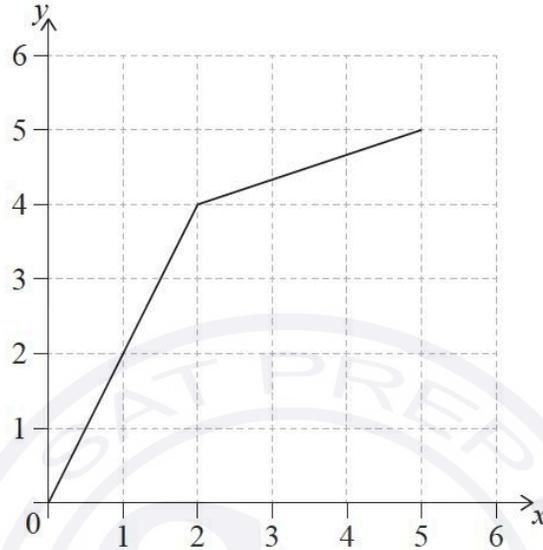
diagram not to scale

Diagram 2

(c)   Show that the model predicts that the ball will go over the fence.   [3]

(d)   Find the horizontal distance that the ball will travel, from the player until it first hits the ground, according to this model.   [2]

Question 17

[Maximum mark: 7]

The graph of the function f is given in the following diagram.

(a) Write down $f(2)$.   [1]

(b) On the axes, sketch $y = f^{-1}(x)$.   [2]

The function g is defined as $g(x) = 3x - 1$.

(c) Find an expression for $g^{-1}(x)$.   [2]

(d) Find a value of x where $f^{-1}(x) = g^{-1}(x)$.   [2]

Question 18

[Maximum mark: 7]

The pH scale is a measure of the acidity of a solution. Its value is given by the formula

$$pH = -\log_{10}[H^+],$$

where $[H^+]$ is the concentration of hydrogen ions in the solution (measured in moles per litre).

(a) Calculate the pH value if the concentration of hydrogen ions is 0.0003.   [2]

The pH of milk is 6.6.

(b) Calculate the concentration of hydrogen ions in milk.   [2]

The strength of an acid is measured by its concentration of hydrogen ions.

A lemon has a pH value of 2 and a tomato has a pH value of 4.5.

(c) Calculate how many times stronger the acid in a lemon is when compared to the acid in a tomato.   [3]

Question 19

[Maximum mark: 4]

Two model cars are driving along the same straight track. Car A leaves point O at $t = 0$ and its displacement (x_A metres) from O at t seconds is given by the equation

$$x_A = 4\sqrt{t} \text{ , for } 0 \leq t \leq 9.$$

A faster car, B, leaves O at $t = 3$.

After starting its motion, the time taken by car B to reach any point on the track is half the time that was taken by A.

(a) Write down the equation for the displacement of B, x_B, in terms of t.   [2]

(b) Find the value of t at which B catches up with A.   [2]

Question 20

In 2008, there were no beavers in Scotland, as they had become extinct. In 2009, beavers were introduced from Norway. Surveys were carried out in the region of Tayside to monitor the beaver population.

Figure 1: A beaver

It is believed that the population, P, grows according to a logistic model

$$P = \frac{L}{1+Ce^{-0.28t}},$$

where t is the number of years after 2008, $t \geq 1$.

A survey in 2017 estimated the population to be 433.

A survey in 2020 estimated the population to be 954.

Find the carrying capacity, L, for the number of beavers in Tayside, according to this model.
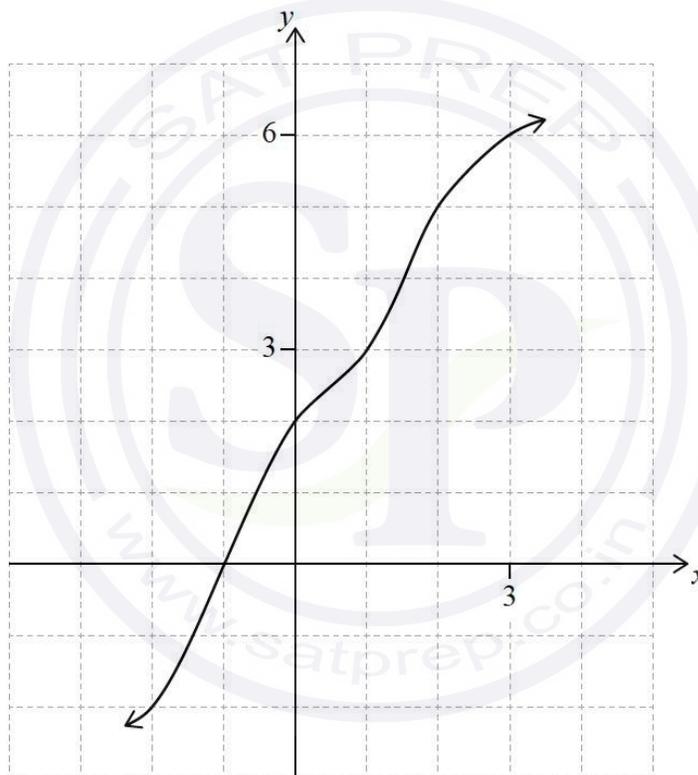
Question 21

[Maximum mark: 8]

The function f is defined by $f(x) = 4\ln(2x - 3)$, where $x > \dfrac{3}{2}$.

The graph of $y = f(x)$ is obtained from the graph of $y = \ln x$ by a sequence of three transformations.

(a) Describe fully these transformations, including the order in which they occur. [3]

The graph of $y = g(x)$ is shown.

The graph passes through the points $(-2, -2)$, $(-1, 0)$, $(0, 2)$, $(1, 3)$, $(2, 5)$ and $(3, 6)$.

(b) Find $(f \circ g)(2)$. [2]

(c) Solve $(f \circ g)(x) = 2\ln 9$. [3]

Question 22

Radioactive carbon is a material that decays over time.

The mass, $m(t)$ (in nanograms), of radioactive carbon in a fossil of a plant, after t years, can be modelled by the function

$$m(t) = 120e^{-0.000121t}$$

where t is the time since the plant died.

(a) Write down the initial mass of the radioactive carbon. [1]

(b) Find the mass of the radioactive carbon after $20\,000$ years. [2]

(c) Calculate the smallest number of complete years it takes for more than half the sample to decay. [3]