Subject – Math AI(Higher Level)
Topic - Number and Algebra
Year - May 2021 – Nov 2024
Paper -1
Questions

Question 1

[Maximum mark: 6]

The rate, A, of a chemical reaction at a fixed temperature is related to the concentration of two compounds, B and C, by the equation

$$A = kB^x C^y, \text{ where } x,\ y,\ k \in \mathbb{R}.$$

A scientist measures the three variables three times during the reaction and obtains the following values.

Experiment	A (mol l^{-1} s^{-1})	B (mol l^{-1})	C (mol l^{-1})
1	5.74	2.1	3.4
2	2.88	1.5	2.4
3	0.980	0.8	1.9

Find x, y and k.

Question 2

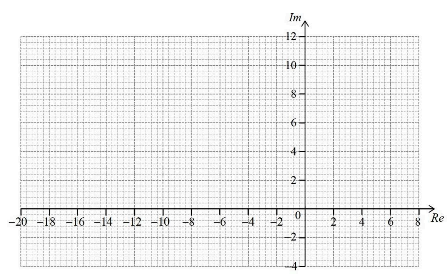
[Maximum mark: 7]

Let $w = ae^{\frac{\pi}{4}i}$, where $a \in \mathbb{R}^+$.

(a) For $a = 2$,

(i) find the values of w^2, w^3, and w^4;

(ii) draw w, w^2, w^3 and w^4 on the following Argand diagram. [5]



Let $z = \dfrac{w}{2 - i}$.

(b) Find the value of a for which successive powers of z lie on a circle. [2]

Question 3

[Maximum mark: 7]

Product research leads a company to believe that the revenue (R) made by selling its goods at a price (p) can be modelled by the equation.

$$R(p) = cpe^{dp},\ c,\ d \in \mathbb{R}$$

There are two competing models, A and B with different values for the parameters c and d.

Model A has $c = 3$, $d = -0.5$ and model B has $c = 2.5$, $d = -0.6$.

The company experiments by selling the goods at three different prices in three similar areas and the results are shown in the following table.

Area	**Price (p)**	**Revenue (R)**
1	1	1.5
2	2	1.8
3	3	1.5

The company will choose the model with the smallest value for the sum of square residuals.

Determine which model the company chose.

Question 4

[Maximum mark: 8]

Consider $w = iz + 1$, where $w, z \in \mathbb{C}$.

(a) Find w when

(i) $z = 2i$.

(ii) $z = 1 + i$. [3]

Point z on the Argand diagram can be transformed to point w by two transformations.

(b) Describe these two transformations and give the order in which they are applied. [3]

(c) Hence, or otherwise, find the value of z when $w = 2 - i$. [2]

Question 5

[Maximum mark: 8]

Charlie and Daniella each began a fitness programme. On day one, they both ran $500\,\mathrm{m}$. On each subsequent day, Charlie ran $100\,\mathrm{m}$ more than the previous day whereas Daniella increased her distance by 2% of the distance ran on the previous day.

(a) Calculate how far

(i) Charlie ran on day 20 of his fitness programme.

(ii) Daniella ran on day 20 of her fitness programme. [5]

On day n of the fitness programmes Daniella runs more than Charlie for the first time.

(b) Find the value of n. [3]

Question 6

[Maximum mark: 8]

It is given that $z_1 = 3\operatorname{cis}\left(\frac{3\pi}{4}\right)$ and $z_2 = 2\operatorname{cis}\left(\frac{n\pi}{16}\right)$, $n \in \mathbb{Z}^+$.

(a) In parts (a)(i) and (a)(ii), give your answers in the form $re^{i\theta}$, $r \geq 0$, $-\pi < \theta \leq \pi$.

(i) Find the value of $z_1^{\ 3}$.

(ii) Find the value of $\left(\frac{z_1}{z_2}\right)^4$ for $n = 2$. [5]

(b) Find the least value of n such that $z_1 z_2 \in \mathbb{R}^+$. [3]

Question 7

[Maximum mark: 6]

A meteorologist models the height of a hot air balloon launched from the ground. The model assumes the balloon travels vertically upwards and travels 450 metres in the first minute.

Due to the decrease in temperature as the balloon rises, the balloon will continually slow down. The model suggests that each minute the balloon will travel only 82% of the distance travelled in the previous minute.

(a) Find how high the balloon will travel in the first 10 minutes after it is launched. [3]

(b) The balloon is required to reach a height of at least 2520 metres.

Determine whether it will reach this height. [2]

(c) Suggest a limitation of the given model. [1]

Question 8

[Maximum mark: 5]

Roger buys a new laptop for himself at a cost of £495. At the same time, he buys his daughter Chloe a higher specification laptop at a cost of £2200.

It is anticipated that Roger's laptop will depreciate at a rate of 10% per year, whereas Chloe's laptop will depreciate at a rate of 15% per year.

(a) Estimate the value of Roger's laptop after 5 years. [2]

Roger and Chloe's laptops will have the same value k years after they were purchased.

(b) Find the value of k. [2]

(c) Comment on the validity of your answer to part (b). [1]

Question 9

[Maximum mark: 5]

The following table shows the time, in days, from December 1st and the percentage of Christmas trees in stock at a shop on the beginning of that day.

Days since December 1st (d)	1	3	6	9	12	15	18
Percentage of Christmas trees left in stock (x)	100	51	29	21	18	16	14

The following table shows the natural logarithm of both d and x on these days to 2 decimal places.

ln (d)	0	1.10	1.79	2.20	2.48	2.71	2.89
ln (x)	4.61	3.93	3.37	3.04	2.89	2.77	2.64

(a) Use the data in the second table to find the value of m and the value of b for the regression line, $\ln x = m(\ln d) + b$. [2]

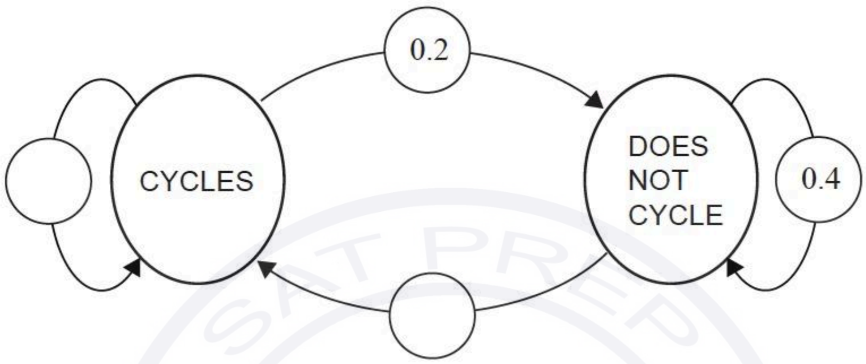
(b) Assuming that the model found in part (a) remains valid, estimate the percentage of trees in stock when $d = 25$. [3]

Question 10

[Maximum mark: 5]

Katie likes to cycle to work as much as possible. If Katie cycles to work one day then she has a probability of 0.2 of not cycling to work on the next work day. If she does not cycle to work one day then she has a probability of 0.4 of not cycling to work on the next work day.

(a) Complete the following transition diagram to represent this information. [2]



Katie works for 180 days in a year.

(b) Find the probability that Katie cycles to work on her final working day of the year. [3]

Question 11

[Maximum mark: 5]

An infinite geometric sequence, with terms u_n, is such that $u_1 = 2$ and $\sum_{k=1}^{\infty} u_k = 10$.

(a) Find the common ratio, r, for the sequence. [2]

(b) Find the least value of n such that $u_n < \frac{1}{2}$. [3]

Question 12

[Maximum mark: 9]

In this question, give all answers correct to 2 decimal places.

Raul and Rosy want to buy a new house and they need a loan of $170\,000$ Australian dollars (AUD) from a bank. The loan is for 30 years and the annual interest rate for the loan is 3.8%, compounded monthly. They will pay the loan in fixed monthly instalments at the end of each month.

(a) Find the amount they will pay the bank each month. [3]

(b) (i) Find the amount Raul and Rosy will still owe the bank at the end of the first 10 years.

(ii) Using your answers to parts (a) and (b)(i), calculate how much interest they will have paid in total during the first 10 years. [6]

Question 13

[Maximum mark: 5]

Inspectors are investigating the carbon dioxide emissions of a power plant. Let R be the rate, in tonnes per hour, at which carbon dioxide is being emitted and t be the time in hours since the inspection began.

When R is plotted against t, the total amount of carbon dioxide produced is represented by the area between the graph and the horizontal t-axis.

The rate, R, is measured over the course of two hours. The results are shown in the following table.

t	0	0.4	0.8	1.2	1.6	2
R	30	50	60	40	20	50

(a) Use the trapezoidal rule with an interval width of 0.4 to estimate the total amount of carbon dioxide emitted during these two hours. [3]

The real amount of carbon dioxide emitted during these two hours was 72 tonnes.

(b) Find the percentage error of the estimate found in part (a). [2]

Question 14

[Maximum Mark 5]

The strength of earthquakes is measured on the Richter magnitude scale, with values typically between 0 and 8 where 8 is the most severe.

The Gutenberg–Richter equation gives the average number of earthquakes per year, N, which have a magnitude of at least M. For a particular region the equation is

$$\log_{10} N = a - M, \text{ for some } a \in \mathbb{R}.$$

This region has an average of 100 earthquakes per year with a magnitude of at least 3.

(a) Find the value of a. [2]

The equation for this region can also be written as $N = \frac{b}{10^M}$.

(b) Find the value of b. [2]

Within this region the most severe earthquake recorded had a magnitude of 7.2.

(c) Find the average number of earthquakes in a year with a magnitude of at least 7.2. [1]

Question 15

[Maximum mark: 5]

The matrix $\boldsymbol{M} = \begin{pmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{pmatrix}$ has eigenvalues -0.5 and 1.

(a) Find an eigenvector corresponding to the eigenvalue of 1. Give your answer in the form $\begin{pmatrix} a \\ b \end{pmatrix}$, where $a,\ b \in \mathbb{Z}$. [3]

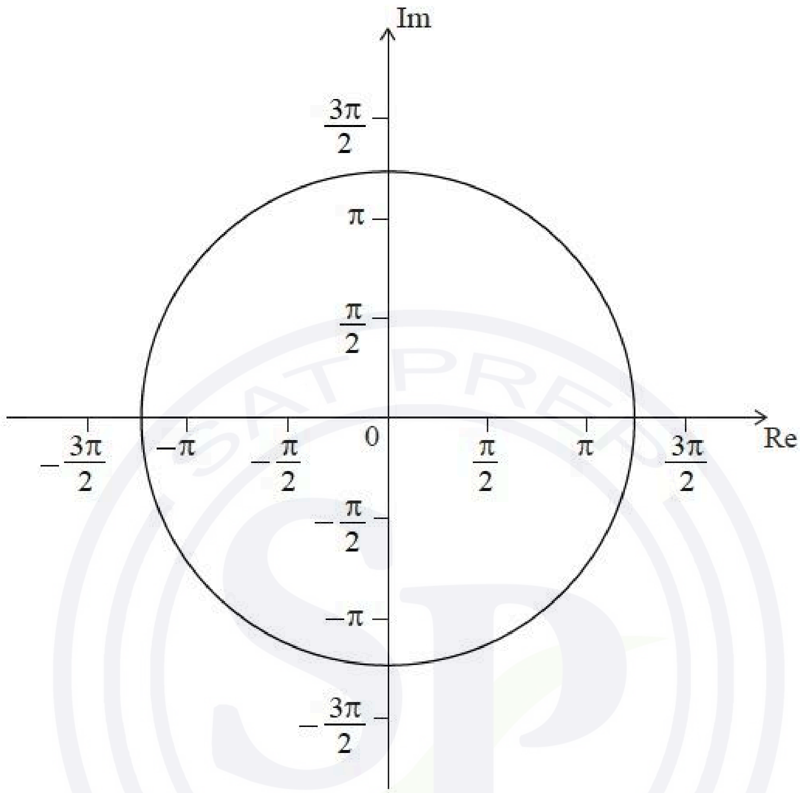
A switch has two states, A and B. Each second it either remains in the same state or moves according to the following rule: If it is in state A it will move to state B with a probability of 0.8 and if it is in state B it will move to state A with a probability of 0.7.

(b) Using your answer to (a), or otherwise, find the long-term probability of the switch being in state A. Give your answer in the form $\frac{c}{d}$, where $c,\ d \in \mathbb{Z}^+$. [2]

Question 16

[Maximum mark: 7]

The following Argand diagram shows a circle centre 0 with a radius of 4 units.



A set of points, $\{z_\theta\}$, on the Argand plane are defined by the equation

$$z_\theta = \frac{1}{2}\theta e^{\theta i}, \text{ where } \theta \geq 0.$$

(a) Plot on the Argand diagram the points corresponding to

(i) $\theta = \frac{\pi}{2}$.

(ii) $\theta = \pi$.

(iii) $\theta = \frac{3\pi}{2}$. [3]

Consider the case where $|z_\theta| = 4$.

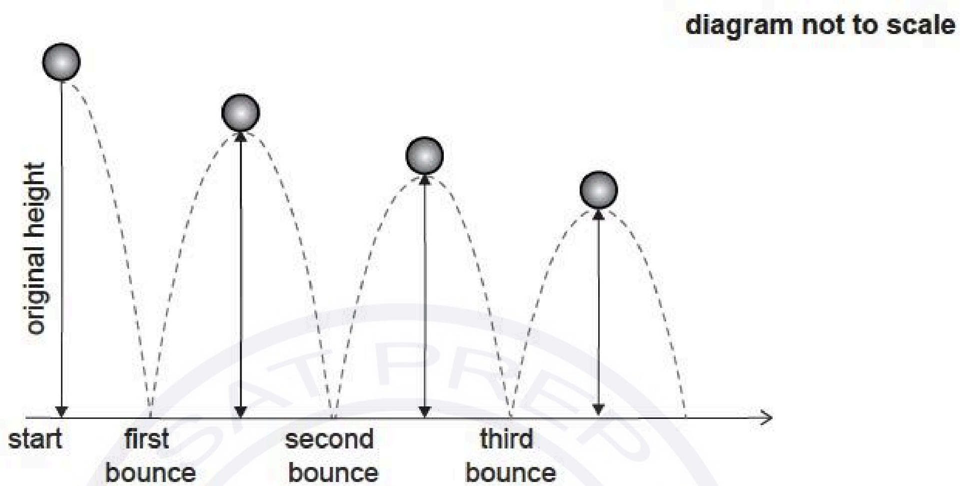
(b) (i) Find this value of θ.

(ii) For this value of θ, plot the approximate position of z_θ on the Argand diagram. [4]

Question 17

[Maximum mark: 7]

A ball is dropped from a height of 1.8 metres and bounces on the ground. The maximum height reached by the ball, after each bounce, is 85% of the previous maximum height.



(a) Show that the maximum height reached by the ball after it has bounced for the sixth time is $68\,\text{cm}$, to the nearest cm. [2]

(b) Find the number of times, after the first bounce, that the maximum height reached is greater than $10\,\text{cm}$. [2]

(c) Find the total **vertical** distance travelled by the ball from the point at which it is dropped until the fourth bounce. [3]

Question 18

[Maximum mark: 5]

The ticket prices for a concert are shown in the following table.

Ticket Type	Price (in Australian dollars, \$)
Adult	15
Child	10
Student	12

- A total of 600 tickets were sold.
- The total amount of money from ticket sales was \$7816.
- There were twice as many adult tickets sold as child tickets.

Let the number of adult tickets sold be x, the number of child tickets sold be y, and the number of student tickets sold be z.

(a) Write down three equations that express the information given above. [3]

(b) Find the number of each type of ticket sold. [2]

Question 19

[Maximum mark: 7]

The equation of the line $y = mx + c$ can be expressed in vector form $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$.

(a) Find the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in terms of m and/or c. [2]

The matrix $\boldsymbol{M}$ is defined by $\begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$.

(b) Find the value of $\det \boldsymbol{M}$. [1]

The line $y = mx + c$ (where $m \neq -2$) is transformed into a new line using the transformation described by matrix $\boldsymbol{M}$.

(c) Show that the equation of the resulting line does not depend on m or c. [4]

Question 20

[Maximum mark: 5]

An electric circuit has two power sources. The voltage, V_1, provided by the first power source, at time t, is modelled by

$$V_1 = \mathrm{Re}\left(2e^{3t\mathrm{i}}\right).$$

The voltage, V_2, provided by the second power source is modelled by

$$V_2 = \mathrm{Re}\left(5e^{(3t+4)\mathrm{i}}\right).$$

The total voltage in the circuit, V_T, is given by

$$V_T = V_1 + V_2.$$

(a) Find an expression for V_T in the form $A\cos(Bt + C)$, where A, B and C are real constants. [4]

(b) Hence write down the maximum voltage in the circuit. [1]

Question 21

[Maximum mark: 6]

Juliana plans to invest money for 10 years in an account paying 3.5% interest, compounded annually. She expects the annual inflation rate to be 2% per year throughout the 10-year period.

Juliana would like her investment to be worth a real value of $\$4000$, compared to current values, at the end of the 10-year period. She is considering two options.

Option 1: Make a one-time investment at the start of the 10-year period.

Option 2: Invest $\$1000$ at the start of the 10-year period and then invest $\$x$ into the account at the end of each year (including the first and last years).

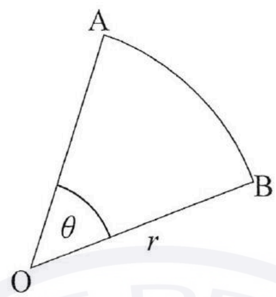
(a) For option 1, determine the minimum amount Juliana would need to invest. Give your answer to the nearest dollar. [3]

(b) For option 2, find the minimum value of x that Juliana would need to invest each year. Give your answer to the nearest dollar. [3]

Question 22

[Maximum mark: 8]

The diagram shows a sector, OAB, of a circle with centre O and radius r, such that $\mathrm{A\hat{O}B} = \theta$.



Sam measured the value of r to be $2\,\mathrm{cm}$ and the value of θ to be 30°.

(a) Use Sam's measurements to calculate the area of the sector. Give your answer to four significant figures. [2]

It is found that Sam's measurements are accurate to only one significant figure.

(b) Find the upper bound and lower bound of the area of the sector. [3]

(c) Find, with justification, the largest possible percentage error if the answer to part (a) is recorded as the area of the sector. [3]

Question 23

[Maximum mark: 5]

The sum of an infinite geometric sequence is 9.

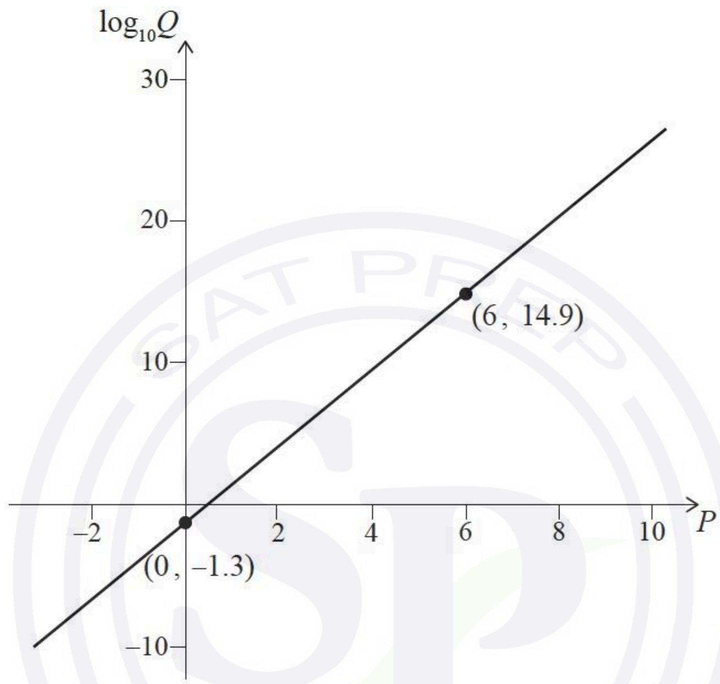
The first term is 4 more than the second term.

Find the third term. Justify your answer.

Question 24

[Maximum mark: 6]

Gen is investigating the relationship between two sets of data, labelled P and Q, that she collected. She created a scatter plot with P on the x-axis and $\log_{10} Q$ on the y-axis. Gen noticed that the points had a strong linear correlation, so she drew a line of best fit, as shown in the diagram. The line passes through the points $(0, -1.3)$ and $(6, 14.9)$.



(a) Find an equation for Q in terms of P. [3]

Gen also investigates the relationship between the same data, Q, and some new data, R. She believes that the data can be modelled by $Q = a \ln R + b$ and she decides to create a scatter plot to verify her belief.

(b) State what expression Gen should plot on each axis to verify her belief. [1]

The scatter plot has a linear relationship and Gen finds $a = 4.3$ and $b = 12.1$.

(c) Find an equation for P in terms of R. [2]

Question 25

[Maximum mark: 7]

Stars are classified by their brightness. The brightest stars in the sky have a magnitude of 1. The magnitude, m, of another star can be modelled as a function of its brightness, b, relative to a star of magnitude 1, as shown by the following equation.

$$m = 1 - 2.5\log_{10}(b)$$

The star called Acubens has a brightness of 0.0525.

(a) Find the magnitude of Acubens. [2]

Ceres has a magnitude of 7 and is the least bright star visible without magnification.

(b) Find the brightness of Ceres. [2]

The star Proxima Centauri has a greater magnitude than the planet Neptune. The difference in their magnitudes is 3.2.

(c) Find how many times brighter Neptune is compared to Proxima Centauri. [3]

Question 26

[Maximum mark: 8]

The transformation T is represented by the matrix $\boldsymbol{M} = \begin{pmatrix} 2 & -4 \\ 3 & 1 \end{pmatrix}$.

A pentagon with an area of $12\,\text{cm}^2$ is transformed by T.

(a) Find the area of the image of the pentagon. [2]

Under the transformation T, the image of point X has coordinates $(2t-3\,,\ 6-5t)$, where $t \in \mathbb{R}$.

(b) Find, in terms of t, the coordinates of X. [6]

Question 27

[Maximum mark: 7]

On 1 December 2022, Laviola invests 800 euros (EUR) into a savings account which pays a nominal annual interest rate of 7.5% compounded monthly. At the end of each month, Laviola deposits an additional $\text{EUR } 500$ into the savings account.

At the end of k months, Laviola will have saved enough money to withdraw $\text{EUR } 10\,000$.

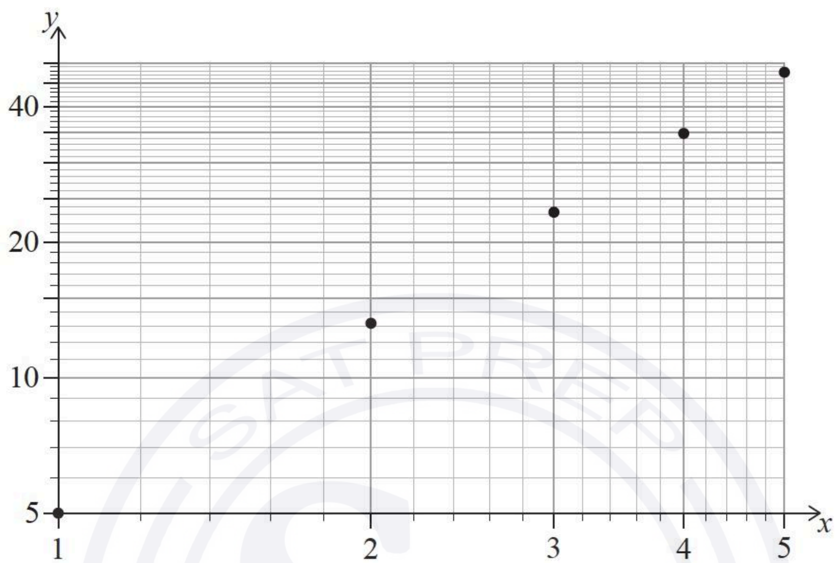
(a) Find the smallest possible value of k, for $k \in \mathbb{Z}^+$. [4]

(b) For this value of k, find the interest earned in the savings account.
Express your answer correct to the nearest EUR. [3]

Question 28

[Maximum mark: 6]

Petra examines two quantities, x and y, and plots data points on a log-log graph.



She observes that on this graph the data points follow a perfect straight line. Given that the line passes through the points $(2, 13.1951)$ and $(4, 34.822)$, find the equation of the relationship connecting x and y. Your final answer should not include logarithms.

Question 29

[Maximum mark: 5]

Two AC (alternating current) electrical sources with the same frequencies are combined. The voltages from these sources can be expressed as $V_1 = 6\sin(at + 30°)$ and $V_2 = 6\sin(at + 90°)$.

The combined total voltage can be expressed in the form $V_1 + V_2 = V\sin(at + \theta°)$.

Determine the value of V and the value of θ.

Question 30

[Maximum mark: 5]

On 1 January 2022, Mina deposited $\$1000$ into a bank account with an annual interest rate of 4%, compounded monthly. At the end of January, and the end of every month after that, she deposits $\$100$ into the same account.

(a) Calculate the amount of money in her account at the start of 2024. Give your answer to two decimal places. [3]

(b) Find how many complete months, counted from 1 January 2022, it will take for Mina to have more than $\$5000$ in her account. [2]

Question 31

[Maximum mark: 7]

The matrices $\boldsymbol{P}=\begin{pmatrix}3 & 1\\ 0 & 2\end{pmatrix}$ and $\boldsymbol{Q}=\begin{pmatrix}-4 & 1\\ 1 & 3\end{pmatrix}$ represent two transformations.

A triangle T is transformed by $\boldsymbol{P}$, and this image is then transformed by $\boldsymbol{Q}$ to form a new triangle, T'.

(a) Find the single matrix that represents the transformation $T' \to T$, which will undo the transformation described above. [4]

The area of T' is $273\,\text{cm}^2$.

(b) Using your answer to part (a), or otherwise, determine the area of T. [3]

Question 32

[Maximum mark: 6]

Angel has $\$520$ in his savings account. Angel considers investing the money for 5 years with a bank. The bank offers an annual interest rate of 1.2% compounded quarterly.

(a) Calculate the amount of money Angel would have at the end of 5 years with the bank. Give your answer correct to two decimal places. [3]

Instead of investing the money, Angel decides to buy a phone that costs $\$520$. At the end of 5 years, the phone will have a value of $\$30$. It may be assumed that the depreciation rate per year is constant.

(b) Calculate the annual depreciation rate of the phone. [3]

Question 33

[Maximum mark: 7]

The eating habits of students in a school are studied over a number of months. The focus of the study is whether non-vegetarians become vegetarians, and whether vegetarians remain vegetarians.

Each month, students choose between the vegetarian or non-vegetarian lunch options. Once they have chosen for the month, they cannot change the option until the next month.

In any month during the study, it is noticed that the probability of a non-vegetarian becoming vegetarian the following month is 0.1, and that the probability of a vegetarian remaining a vegetarian the following month is 0.8.

This situation can be represented by the transition matrix

$$\boldsymbol{T} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}.$$

(a) Interpret the value 0.9 in $\boldsymbol{T}$ in terms of the changes in the eating habits of the students in the school. [1]

(b) Find the eigenvalues of matrix $\boldsymbol{T}$. [3]

One of the eigenvectors of $\boldsymbol{T}$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(c) Find another, non-parallel, eigenvector and interpret it in context. [3]

Question 34

[Maximum mark: 6]

Given $z = \sqrt{3} - \mathrm{i}$.

(a) Write z in the form $z = r\mathrm{e}^{\theta \mathrm{i}}$, where $r \in \mathbb{R}^{+}$, $-\pi < \theta \le \pi$. [2]

Let $z_1 = \mathrm{e}^{2t\mathrm{i}}$ and $z_2 = 2\mathrm{e}^{\left(2t - \frac{\pi}{6}\right)\mathrm{i}}$.

(b) Find $\mathrm{Im}(z_1 + z_2)$ in the form $p\sin(2t + q)$, where $p > 0$, $t \in \mathbb{R}$ and $-\pi \le q \le \pi$. [4]

Question 35

[Maximum mark: 7]

The growth of a particular type of seashell is being studied by Manon. At the end of each month Manon records the increase in the width of a seashell since the end of the previous month.

She models the monthly increase in the width of the seashell by a geometric sequence with common ratio 0.8. In the first month, the width of the seashell increases by $4\,\text{mm}$.

(a) Find by how much the width of the seashell will increase during the third month, according to her model. [2]

(b) Find the total increase in the width of the seashell, predicted by Manon's model, during the first year. [2]

Manon's seashell had a width of $25\,\text{mm}$ at the beginning of the first month.

(c) Find the maximum possible width of the seashell, predicted by Manon's model. [3]

Question 36

[Maximum mark: 8]

Consider the complex number $z = -1 + \mathrm{i}$.

(a) Express z in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $-\pi < \theta \le \pi$. [2]

A and B are the points on the Argand diagram that represent the complex numbers z and z^2, respectively.

A is mapped onto B by the composition of a rotation and an enlargement.

(b) (i) Describe fully this mapping of A onto B, stating the scale factor of the enlargement and the angle of rotation.

(ii) Find and simplify a matrix that maps A onto B. [5]

(c) Find the smallest positive integer, n, for which z^n is real and positive. [1]

Question 37

[Maximum mark: 8]

The annual growth of a tree is 80% of its growth during the previous year.

This year the tree is 42 m in height and one year ago its height was 37 m.

(a) Calculate the annual growth of the tree in the coming year. [2]

(b) Calculate the height of the tree 6 years from now. Give your answer correct to the nearest cm. [4]

If the tree continues to follow this pattern of growth, its height will never exceed k metres.

(c) Find the smallest possible value of k. [2]

Question 38

[Maximum mark: 9]

Imani invests $\$3000$ in a bank that pays a nominal annual interest rate of 1.25% compounded monthly.

(a) Calculate the amount of money Imani will have in the bank at the end of 6 years. Give your answer correct to two decimal places. [3]

(b) Calculate the number of months it takes until Imani has at least $\$3550$ in the bank. [2]

Imani uses the $\$3550$ as a partial payment for a used car costing $\$22\,000$. For the remainder she takes out a loan from a bank.

(c) Write down the amount of money that Imani takes out as a loan. [1]

The loan is for 8 years and the nominal annual interest rate is 12.6% compounded monthly. Imani will pay the loan in fixed monthly instalments at the end of each month.

(d) Calculate the amount, correct to the nearest dollar, that Imani will have to pay the bank each month. [3]

Question 39

[Maximum mark: 9]

Phoebe opens a coffee shop, near to a well-established Apollo coffee shop.

After being open for a few months, Phoebe notices that

- 10% of customers who preferred the Apollo coffee shop in one month preferred her coffee shop the following month.
- 25% of customers who preferred her coffee shop in one month preferred the Apollo coffee shop the following month.

She decides to show these changes in the following transition matrix.

$$\begin{pmatrix} 0.9 & 0.25 \\ 0.1 & 0.75 \end{pmatrix}$$

The two eigenvalues for this matrix are 1 and 0.65. An eigenvector corresponding to the eigenvalue of 1 is $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

(a) Find an eigenvector corresponding to the eigenvalue of 0.65. [2]

A diagonal matrix of eigenvalues is $\boldsymbol{D} = \begin{pmatrix} 0.65 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) Write down an expression for $\boldsymbol{D}^n$, giving your answer as a 2×2 matrix in terms of n. [1]

When Phoebe's coffee shop first opened, the Apollo shop had 7000 customers the previous month.

(c) Assuming all 7000 customers continue to go to one of these coffee shops, find an expression for the number that will favour Phoebe's coffee shop after n months. [6]

Question 40

[Maximum mark: 6]

Let $z_1 = 4 + 5\text{i}$.

(a) (i) Find $|z_1|$.

(ii) Find $\arg(z_1)$. [2]

Let $z_2 = 3\text{e}^{2\text{i}}$.

(b) Find the area of the triangle on an Argand diagram with vertices 0, z_1 and z_2. [4]

Question 41

[Maximum mark: 8]

Let $\boldsymbol{R}(\alpha)$ be the matrix representing a rotation, counter-clockwise (anticlockwise) about the origin, through an angle of α.

(a) Write down $\boldsymbol{R}(2\alpha)$ as a 2×2 matrix. [2]

(b) Calculate $\boldsymbol{R}(\alpha) \times \boldsymbol{R}(\alpha)$. [2]

(c) Use your answers from part (a) and part (b) to

(i) explain why $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$.

(ii) show that $\cos(2\alpha) = 1 - 2\sin^2(\alpha)$. [4]

Question 42

[Maximum mark: 8]

When studying big cats, researchers use a model in which the mass (m kilograms) of an animal is directly proportional to the cube of its shoulder height (h metres).

Cheetah

Lion

A cheetah has a mass of $64\,\text{kg}$ and shoulder height of 0.8 metres.

(a) (i) Use the model to find an expression for m in terms of h.

(ii) Hence find the mass of a different cheetah, with a shoulder height of 0.75 metres. [4]

'Rubner's law' states that the energy needs of an animal (E) are directly proportional to the square of h.

The energy needs of a lion of mass $220\,\text{kg}$ are k times the energy needs of a cheetah of mass $64\,\text{kg}$.

(b) Find the value of k. [4]

Question 43

[Maximum mark: 7]

On 1 January 2025, the Faber Car Company will release a new car to global markets. The company expects to sell 40 cars in January 2025. The number of cars sold each month can be modelled by a geometric sequence where $r = 1.1$.

(a) Use this model to find the number of cars that will be sold in December 2025. [2]

(b) Use this model to find the total number of cars that will be sold in the year

(i) 2025.

(ii) 2026. [5]

Question 44

[Maximum mark: 6]

Maan deposited $\$100\,000$ into a savings account with a nominal annual interest rate of $I\,\%$ **compounded monthly**. At the end of the eighth year, the amount in the account had increased to $\$150\,000$.

(a) Find the value of I. [3]

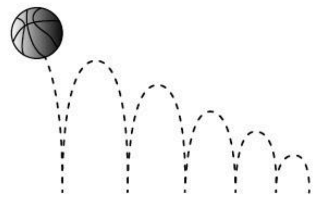
Maan withdraws the $\$150\,000$ and places it in an annuity, earning a nominal annual interest rate of $6.1\,\%$ **compounded monthly**. At the end of each month, Maan will receive a payment of $\$1000$.

(b) Find the amount of money remaining in the annuity at the end of 10 years. Express your answer to the nearest dollar. [3]

Question 45

[Maximum mark: 7]

Andreas drops a ball and records a video of the ball bouncing.



He uses the video to find the maximum height, in metres, after each of the first four bounces.
His results are shown in the table.

Bounce number, n	Maximum height, h
1	0.613
2	0.514
3	0.439
4	0.377

Andreas thinks the maximum height can be modelled by the function

$$h(n) = 0.613\left(\frac{0.514}{0.613}\right)^{n-1}, \text{ where } n \in \mathbb{Z}^+.$$

(a) Complete the following table. [2]

Bounce number, n	Maximum height, h, according to the model
1	0.613
2	0.514
3	
4	

(b) Hence, calculate the sum of square residuals (SS_{res}). [2]

Andreas' friend thinks a better model for h could be found using an exponential least squares regression curve.

(c) (i) Find the equation of this model.

(ii) Use this model to estimate the height from which the ball was originally dropped. [3]

Question 46

[Maximum mark: 8]

Let $\boldsymbol{M} = \begin{pmatrix} -4 & 2 \\ -3 & 3 \end{pmatrix}$.

(a) Find the eigenvalues of $\boldsymbol{M}$. [3]

$\boldsymbol{M}$ can be written in the form $\boldsymbol{M} = \boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix.

(b) (i) Write down $\boldsymbol{D}$.

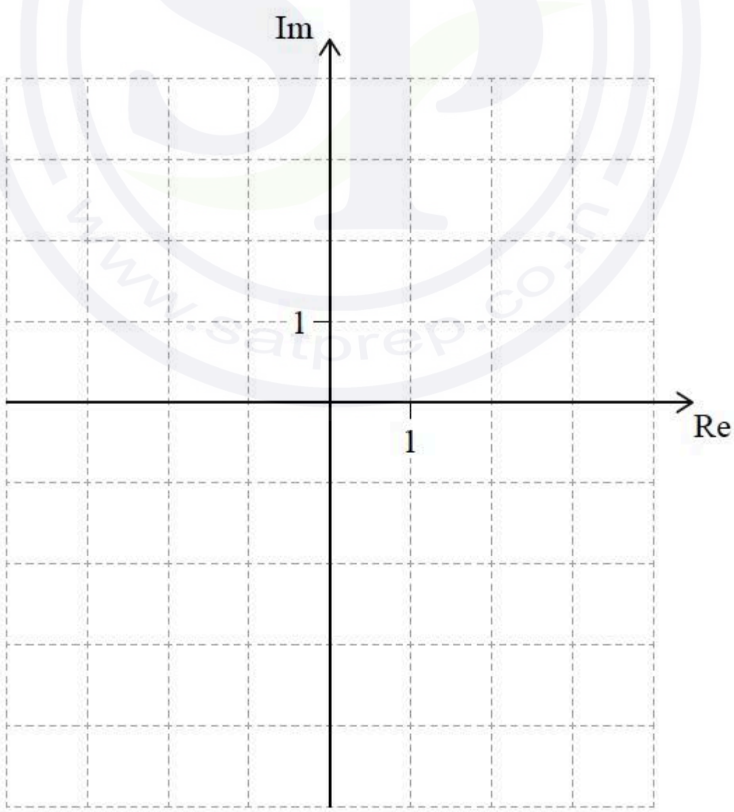
(ii) Find $\boldsymbol{P}$. [5]

Question 47

[Maximum mark: 5]

Let $z = 2 - 3\mathrm{i}$.

(a) Plot z on the Argand diagram. [1]



z can be written in the form $r \operatorname{cis} \theta$, where $r > 0$ and $-\pi < \theta \leq \pi$.

(b) Find the value of

(i) r

(ii) θ. [2]

(c) Find the value of $z\mathrm{i}$ in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{Z}$. [1]

$z\mathrm{i}$ can be obtained from z by a geometric transformation.

(d) By plotting $z\mathrm{i}$ on the Argand diagram, or otherwise, describe fully this transformation. [1]

Question 48

[Maximum mark: 6]

When Daniel retires, he invests $\$400\,000$ in an annuity fund that earns interest at a nominal rate of 4.5% per year, compounded monthly.

Daniel then withdraws $\$3600$ at the end of every month to pay for his living expenses.

(a) Find how much is in the annuity fund after 5 years. [3]

(b) Calculate how many times Daniel is able to make these withdrawals. [3]

Question 49

[Maximum mark: 4]

A museum has an annual membership fee of $\$200$, which includes 10 free visits. Any additional visits are charged at $\$30$ each. The total cost, $\$C$, of n visits during the year can be modelled by

$$C(n) = \begin{cases} 200, & n < p \\ an + b, & n \geq p \end{cases}, \text{ where } a, b, p, n \in \mathbb{Z}.$$

(a) Write down the value of

(i) a

(ii) p. [2]

(b) Find the value of b. [2]

Question 50

[Maximum mark: 5]

On 1 January in a particular year, Eva invests $\$25\,000$ in a new bank account. The account earns 5% simple interest, on the original $\$25\,000$, at the start of each subsequent year.

The amounts in the account at the start of each year form an arithmetic sequence.

(a) Find the common difference of this sequence. [2]

After k complete years, the amount in Eva's account will be greater than $\$44\,000$ for the first time.

(b) Find the value of k. [3]