Subject – Math AI(Higher Level)
Topic - Calculus
Year - May 2021 – Nov 2024
Paper -2
Questions

Question 1

[Maximum mark: 20]

An object is placed into the top of a long vertical tube, filled with a thick viscous fluid, at time $t = 0$.

Initially it is thought that the resistance of the fluid would be proportional to the velocity of the object. The following model was proposed, where the object's displacement, x, from the top of the tube, measured in metres, is given by the differential equation

$$\frac{d^2x}{dt^2} = 9.81 - 0.9\left(\frac{dx}{dt}\right).$$

(a) By substituting $v = \frac{dx}{dt}$ into the equation, find an expression for the velocity of the particle at time t. Give your answer in the form $v = f(t)$. [7]

The maximum velocity approached by the object as it falls is known as the terminal velocity.

(b) From your solution to part (a), or otherwise, find the terminal velocity of the object predicted by this model. [2]

An experiment is performed in which the object is placed in the fluid on a number of occasions and its terminal velocity recorded. It is found that the terminal velocity was consistently smaller than that predicted by the model used. It was suggested that the resistance to motion is actually proportional to the velocity squared and so the following model was set up.

$$\frac{d^2x}{dt^2} = 9.81 - 0.9\left(\frac{dx}{dt}\right)^2$$

(c) Write down the differential equation as a system of first order differential equations. [2]

(d) Use Euler's method, with a step length of 0.2, to find the displacement and velocity of the object when $t = 0.6$. [4]

(e) By repeated application of Euler's method, find an approximation for the terminal velocity, to five significant figures. [1]

At terminal velocity the acceleration of an object is equal to zero.

(f) Use the differential equation to find the terminal velocity for the object. [2]

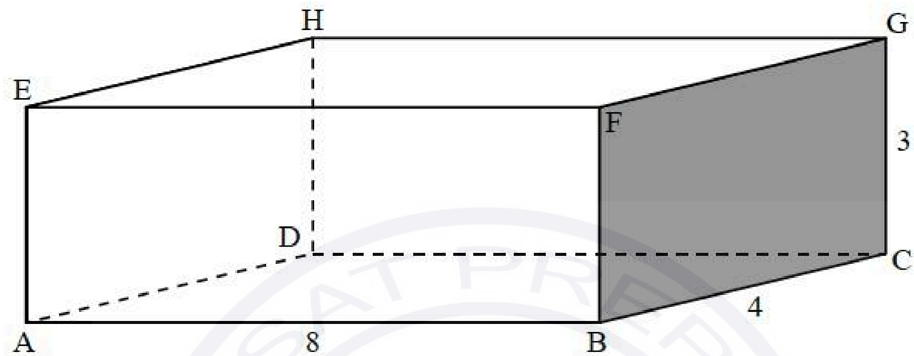
(g) Use your answers to parts (d), (e) and (f) to comment on the accuracy of the Euler approximation to this model. [2]

Question 2

[Maximum mark: 15]

The Happy Straw Company manufactures drinking straws.

The straws are packaged in small closed rectangular boxes, each with length $8\,\text{cm}$, width $4\,\text{cm}$ and height $3\,\text{cm}$. The information is shown in the diagram.



(a) Calculate the surface area of the box in cm^2. [2]

(b) Calculate the length AG. [2]

Each week, the Happy Straw Company sells x boxes of straws. It is known that $\frac{dP}{dx} = -2x + 220$, $x \geq 0$, where P is the weekly profit, in dollars, from the sale of x thousand boxes.

(c) Find the number of boxes that should be sold each week to maximize the profit. [3]

The profit from the sale of $20\,000$ boxes is $\$1700$.

(d) Find $P(x)$. [5]

(e) Find the least number of boxes which must be sold each week in order to make a profit. [3]

Question 3

[Maximum mark: 17]

Consider the following system of coupled differential equations.

$$\frac{dx}{dt} = -4x$$
$$\frac{dy}{dt} = 3x - 2y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix}$. [6]

(b) Hence, write down the general solution of the system. [2]

(c) Determine, with justification, whether the equilibrium point $(0, 0)$ is stable or unstable. [2]

(d) Find the value of $\frac{dy}{dx}$

(i) at $(4, 0)$.

(ii) at $(-4, 0)$. [3]

(e) Sketch a phase portrait for the general solution to the system of coupled differential equations for $-6 \le x \le 6$, $-6 \le y \le 6$. [4]

Question 4

[Maximum mark: 17]

A biologist introduces 100 rabbits to an island and records the size of their population (x) over a period of time. The population growth of the rabbits can be approximately modelled by the following differential equation, where t is time measured in years.

$$\frac{dx}{dt} = 2x$$

(a) Find the population of rabbits 1 year after they were introduced. [5]

A population of 100 foxes is introduced to the island when the population of rabbits has reached 1000. The subsequent population growth of rabbits and foxes, where y is the population of foxes at time t, can be approximately modelled by the coupled equations:

$$\frac{dx}{dt} = x(2 - 0.01y)$$

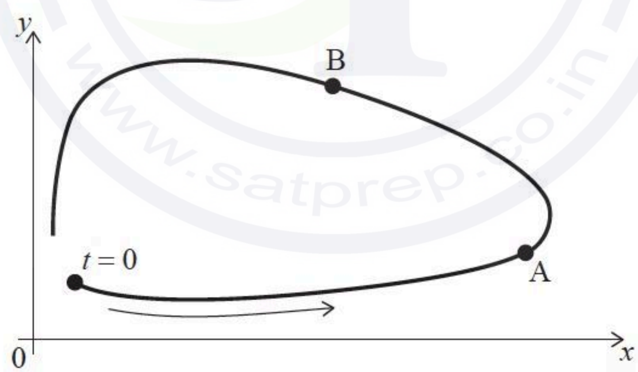
$$\frac{dy}{dt} = y(0.0002x - 0.8)$$

(b) Use Euler's method with a step size of 0.25, to find

(i) the population of rabbits 1 year after the foxes were introduced.

(ii) the population of foxes 1 year after the foxes were introduced. [6]

(c) The graph of the population sizes, according to this model, for the first 4 years after the foxes were introduced is shown below.



Describe the changes in the populations of rabbits and foxes for these 4 years at

(i) point A.

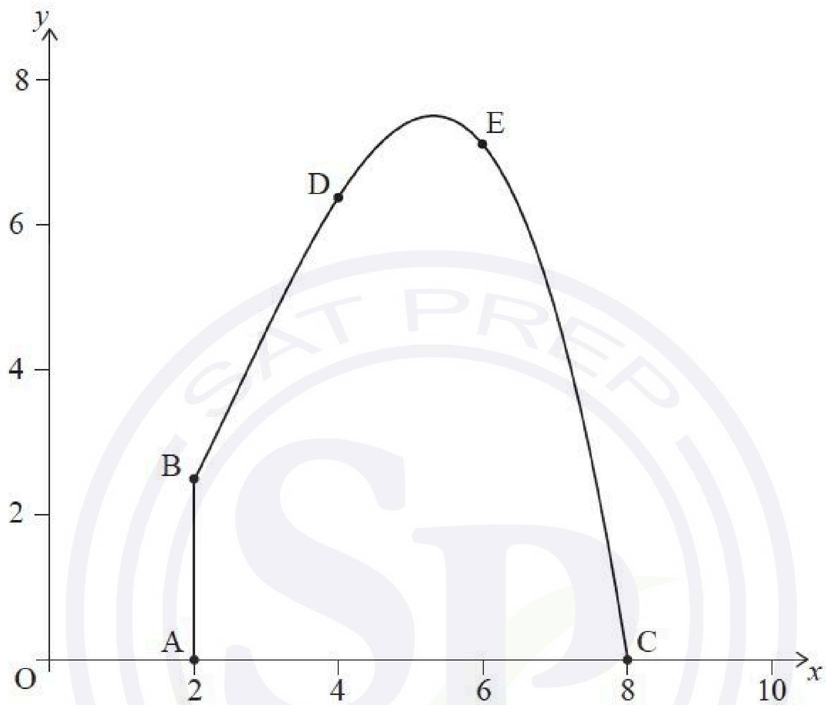
(ii) point B. [3]

(d) Find the non-zero equilibrium point for the populations of rabbits and foxes. [3]

Question 5

[Maximum mark: 16]

The cross-sectional view of a tunnel is shown on the axes below. The line [AB] represents a vertical wall located at the left side of the tunnel. The height, in metres, of the tunnel above the horizontal ground is modelled by $y = -0.1x^3 + 0.8x^2,\ 2 \le x \le 8$, relative to an origin O.



Point A has coordinates $(2, 0)$, point B has coordinates $(2, 2.4)$, and point C has coordinates $(8, 0)$.

(a) (i) Find $\frac{dy}{dx}$.

(ii) Hence find the maximum height of the tunnel. [6]

(b) Find the height of the tunnel when

(i) $x = 4$.

(ii) $x = 6$. [3]

(c) Use the trapezoidal rule, with three intervals, to estimate the cross-sectional area of the tunnel. [3]

(d) (i) Write down the integral which can be used to find the cross-sectional area of the tunnel.

(ii) Hence find the cross-sectional area of the tunnel. [4]

Question 6

[Maximum mark: 17]

Consider the following system of coupled differential equations.

$$\frac{dx}{dt} = -4x$$
$$\frac{dy}{dt} = 3x - 2y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix}$. [6]

(b) Hence, write down the general solution of the system. [2]

(c) Determine, with justification, whether the equilibrium point $(0, 0)$ is stable or unstable. [2]

(d) Find the value of $\frac{dy}{dx}$

(i) at $(4, 0)$.

(ii) at $(-4, 0)$. [3]

(e) Sketch a phase portrait for the general solution to the system of coupled differential equations for $-6 \le x \le 6$, $-6 \le y \le 6$. [4]

Question 7

[Maximum mark: 15]

A particle P moves along the x-axis. The velocity of P is $v\,\text{m s}^{-1}$ at time t seconds, where $v = -2t^2 + 16t - 24$ for $t \ge 0$.

(a) Find the times when P is at instantaneous rest. [2]

(b) Find the magnitude of the particle's acceleration at 6 seconds. [4]

(c) Find the greatest speed of P in the interval $0 \le t \le 6$. [2]

(d) The particle starts from the origin O. Find an expression for the displacement of P from O at time t seconds. [4]

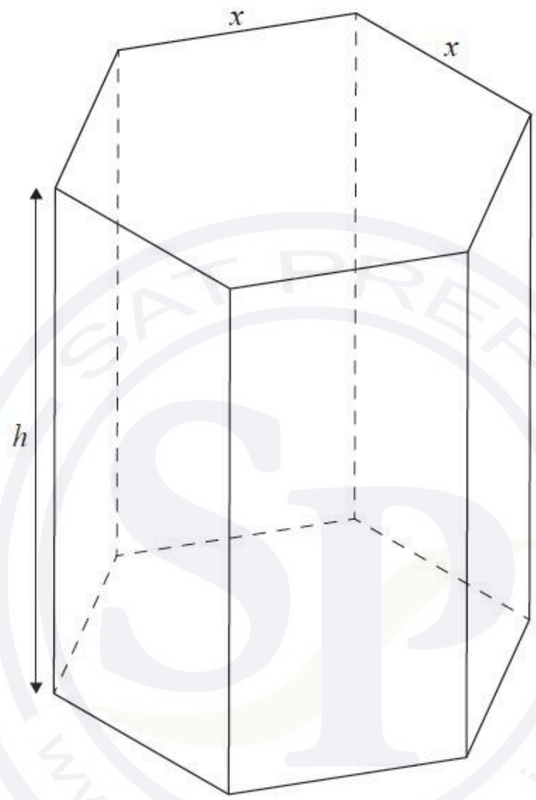
(e) Find the total distance travelled by P in the interval $0 \le t \le 4$. [3]

Question 8

[Maximum mark: 15]

A hollow chocolate box is manufactured in the form of a right prism with a regular hexagonal base. The height of the prism is h cm, and the top and base of the prism have sides of length x cm.

diagram not to scale



(a) Given that $\sin 60° = \frac{\sqrt{3}}{2}$, show that the area of the base of the box is equal to $\frac{3\sqrt{3}x^2}{2}$. [2]

(b) Given that the total external surface area of the box is $1200\,\text{cm}^2$, show that the volume of the box may be expressed as $V = 300\sqrt{3}x - \frac{9}{4}x^3$. [5]

(c) Sketch the graph of $V = 300\sqrt{3}x - \frac{9}{4}x^3$, for $0 \le x \le 16$. [2]

(d) Find an expression for $\frac{dV}{dx}$. [2]

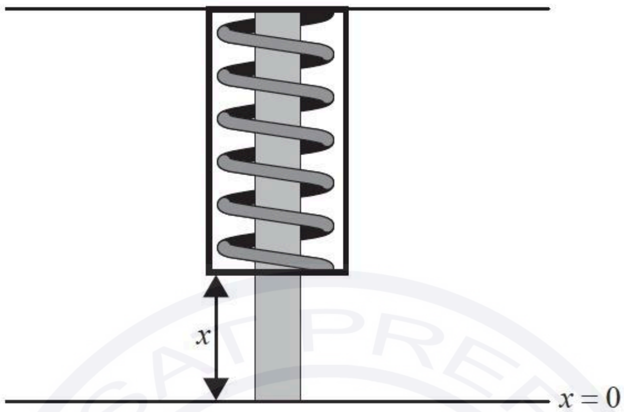
(e) Find the value of x which maximizes the volume of the box. [2]

(f) Hence, or otherwise, find the maximum possible volume of the box. [2]

Question 9

[Maximum mark: 15]

A shock absorber on a car contains a spring surrounded by a fluid. When the car travels over uneven ground the spring is compressed and then returns to an equilibrium position.



The displacement, x, of the spring is measured, in centimetres, from the equilibrium position of $x = 0$. The value of x can be modelled by the following second order differential equation, where t is the time, measured in seconds, after the initial displacement.

$$\ddot{x} + 3\dot{x} + 1.25x = 0$$

(a) Given that $y = \dot{x}$, show that $\dot{y} = -1.25x - 3y$. [2]

The differential equation can be expressed in the form $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} x \\ y \end{pmatrix}$, where $\boldsymbol{A}$ is a 2×2 matrix.

(b) Write down the matrix $\boldsymbol{A}$. [1]

(c) (i) Find the eigenvalues of matrix $\boldsymbol{A}$.

(ii) Find the eigenvectors of matrix $\boldsymbol{A}$. [6]

(d) Given that when $t = 0$ the shock absorber is displaced $8\,\text{cm}$ and its velocity is zero, find an expression for x in terms of t. [6]

Question 10

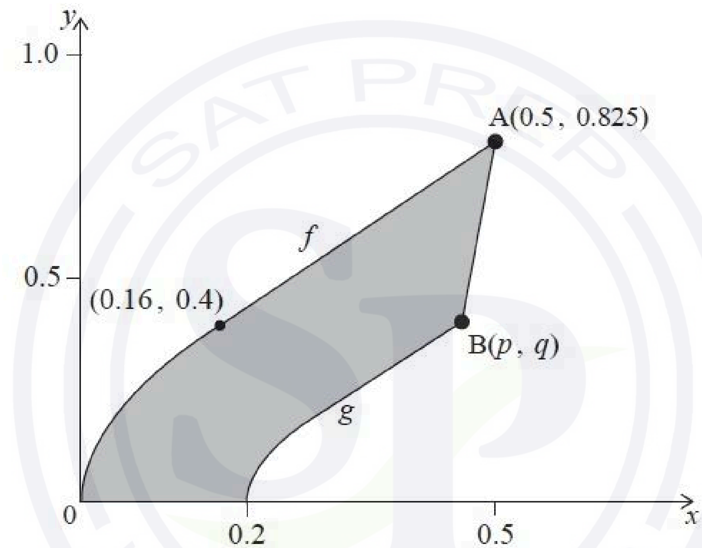
[Maximum mark: 18]

Consider the curve $y = \sqrt{x}$.

(a) (i) Find $\frac{dy}{dx}$.

(ii) Hence show that the equation of the tangent to the curve at the point $(0.16, 0.4)$ is $y = 1.25x + 0.2$. [4]

The shape of a piece of metal can be modelled by the region bounded by the functions f, g, the x-axis and the line segment [AB], as shown in the following diagram. The units on the x and y axes are measured in metres.



The piecewise function f is defined by

$$f(x) = \begin{cases} \sqrt{x} & 0 \leq x \leq 0.16 \\ 1.25x + 0.2 & 0.16 < x \leq 0.5 \end{cases}$$

The graph of g is obtained from the graph of f by:

- a stretch scale factor of $\frac{1}{2}$ in the x direction,
- followed by a stretch scale factor $\frac{1}{2}$ in the y direction,
- followed by a translation of 0.2 units to the right.

Point A lies on the graph of f and has coordinates $(0.5, 0.825)$. Point B is the image of A under the given transformations and has coordinates (p, q).

(b) Find the value of p and the value of q. [2]

The piecewise function g is given by

$$g(x) = \begin{cases} h(x) & 0.2 \le x \le a \\ 1.25x + b & a < x \le p \end{cases}$$

(c) Find

(i) an expression for $h(x)$.

(ii) the value of a.

(iii) the value of b. [5]

(d) (i) Find the area enclosed by $y = f(x)$, the x-axis and the line $x = 0.5$.

The area enclosed by $y = g(x)$, the x-axis and the line $x = p$ is $0.0627292\,\mathrm{m}^2$ correct to six significant figures.

(ii) Find the area of the shaded region on the diagram. [7]

Question 11

[Maximum mark: 13]

A particle moves such that its displacement, x metres, from a point O at time t seconds is given by the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$

(a) (i) Use the substitution $y = \frac{dx}{dt}$ to show that this equation can be written as

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(ii) Find the eigenvalues for the matrix $\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$.

(iii) Hence state the long-term velocity of the particle. [5]

The equation for the motion of the particle is amended to

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 3t + 4.$$

(b) (i) Use the substitution $y = \frac{dx}{dt}$ to write the differential equation as a system of coupled, first order differential equations.

When $t = 0$ the particle is stationary at O.

(ii) Use Euler's method with a step length of 0.1 to find the displacement of the particle when $t = 1$.

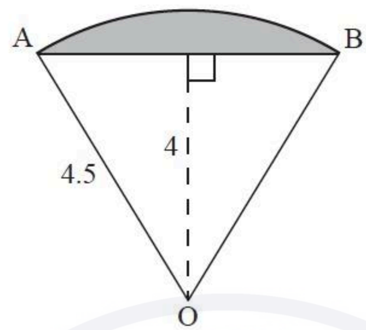
(iii) Find the long-term velocity of the particle. [8]

Question 12

[Maximum mark: 17]

A sector of a circle, centre O and radius 4.5 m, is shown in the following diagram.

diagram not to scale

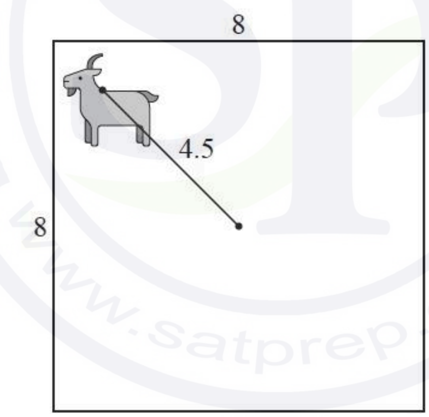


(a) (i) Find the angle $\text{A}\hat{\text{O}}\text{B}$.

(ii) Find the area of the shaded segment. [8]

A square field with side 8 m has a goat tied to a post in the centre by a rope such that the goat can reach all parts of the field up to 4.5 m from the post.

diagram not to scale



(b) Find the area of the field that can be reached by the goat. [4]

Let V be the volume of grass eaten by the goat, in cubic metres, and t be the length of time, in hours, that the goat has been in the field.

The goat eats grass at the rate of $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.3\,t\mathrm{e}^{-t}$.

(c) Find the value of t at which the goat is eating grass at the greatest rate. [2]

The goat is tied in the field for 8 hours.

(d) Find the total volume of grass eaten by the goat during this time. [3]

Question 13

[Maximum mark: 16]

An environmental scientist is asked by a river authority to model the effect of a leak from a power plant on the mercury levels in a local river. The variable x measures the concentration of mercury in micrograms per litre.

The situation is modelled using the second order differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$$

where $t \geq 0$ is the time measured in days since the leak started. It is known that when $t = 0$, $x = 0$ and $\frac{dx}{dt} = 1$.

(a) Show that the system of coupled first order equations:

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -2x - 3y$$

can be written as the given second order differential equation. [2]

(b) Find the eigenvalues of the system of coupled first order equations given in part (a). [3]

(c) Hence find the exact solution of the second order differential equation. [5]

(d) Sketch the graph of x against t, labelling the maximum point of the graph with its coordinates. [2]

If the mercury levels are greater than 0.1 micrograms per litre, fishing in the river is considered unsafe and is stopped.

(e) Use the model to calculate the total amount of time when fishing should be stopped. [3]

The river authority decides to stop people from fishing in the river for 10% longer than the time found from the model.

(f) Write down one reason, with reference to the context, to support this decision. [1]

Question 14

[Maximum mark: 15]

A student investigating the relationship between chemical reactions and temperature finds the Arrhenius equation on the internet.

$$k = Ae^{-\frac{c}{T}}$$

This equation links a variable k with the temperature T, where A and c are positive constants and $T > 0$.

(a) Show that $\frac{dk}{dT}$ is always positive. [3]

(b) Given that $\lim_{T \to \infty} k = A$ and $\lim_{T \to 0} k = 0$, sketch the graph of k against T. [3]

The Arrhenius equation predicts that the graph of $\ln k$ against $\frac{1}{T}$ is a straight line.

(c) Write down

(i) the gradient of this line in terms of c;

(ii) the y-intercept of this line in terms of A. [4]

The following data are found for a particular reaction, where T is measured in Kelvin and k is measured in $cm^3\ mol^{-1}\ s^{-1}$:

T	k
590	5×10^{-4}
600	6×10^{-4}
610	10×10^{-4}
620	14×10^{-4}
630	20×10^{-4}
640	29×10^{-4}
650	36×10^{-4}

(d) Find the equation of the regression line for $\ln k$ on $\frac{1}{T}$. [2]

(e) Find an estimate of

(i) c;

(ii) A.

It is not required to state units for these values. [3]

Question 15

[Maximum mark: 15]

A cafe makes x litres of coffee each morning. The cafe's profit each morning, C, measured in dollars, is modelled by the following equation

$$C = \frac{x}{10}\left(k^2 - \frac{3}{100}x^2\right)$$

where k is a positive constant.

(a) Find an expression for $\frac{\mathrm{d}C}{\mathrm{d}x}$ in terms of k and x. [3]

(b) Hence find the maximum value of C in terms of k. Give your answer in the form pk^3, where p is a constant. [4]

The cafe's manager knows that the cafe makes a profit of $\$426$ when 20 litres of coffee are made in a morning.

(c) (i) Find the value of k.

(ii) Use the model to find how much coffee the cafe should make each morning to maximize its profit. [3]

(d) Sketch the graph of C against x, labelling the maximum point and the x-intercepts with their coordinates. [3]

The manager of the cafe wishes to serve as many customers as possible.

(e) Determine the maximum amount of coffee the cafe can make that will not result in a loss of money for the morning. [2]

Question 16

[Maximum mark: 20]

The position vector of a particle at time t is given by $\boldsymbol{r} = 3\cos(3t)\boldsymbol{i} + 4\sin(3t)\boldsymbol{j}$.
Displacement is measured in metres and time is measured in seconds.

(a) (i) Find an expression for the velocity of the particle at time t.

(ii) Hence find the speed when $t = 3$. [4]

(b) (i) Find an expression for the acceleration of the particle at time t.

(ii) Hence show that the acceleration is always directed towards the origin. [4]

The position vector of a second particle is given by $\boldsymbol{r} = -4\sin(4t)\boldsymbol{i} + 3\cos(4t)\boldsymbol{j}$.

(c) For $0 \le t \le 10$, find the time when the two particles are closest to each other. [5]

At time k, where $0 < k < 1.5$, the second particle is moving parallel to the first particle.

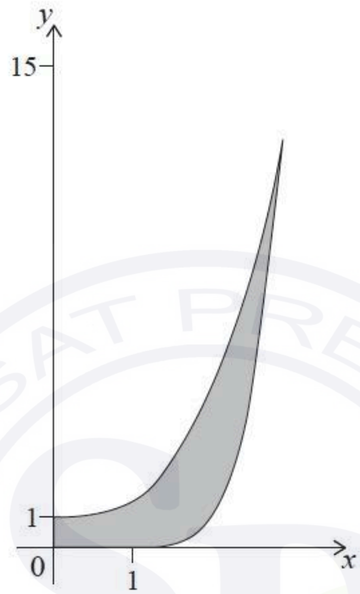
(d) (i) Find the value of k.

(ii) At time k, show that the two particles are moving in the opposite direction. [7]

Question 17

[Maximum mark: 13]

Adesh is designing a glass. The glass has an inner surface and an outer surface. Part of the cross section of his design is shown in the following graph, where the shaded region represents the glass. The two surfaces meet at the top of the glass. 1 unit represents 1 cm.



The inner surface is modelled by $f(x)=\frac{1}{2}x^3+1$ for $0 \le x \le p$.

The outer surface is modelled by $g(x)=\begin{cases} 0 & \text{for } 0 \le x < 1 \\ (x-1)^4 & \text{for } 1 \le x \le p \end{cases}$.

(a) Find the value of p. [2]

The glass design is finished by rotating the shaded region in the diagram through 360° about the y-axis.

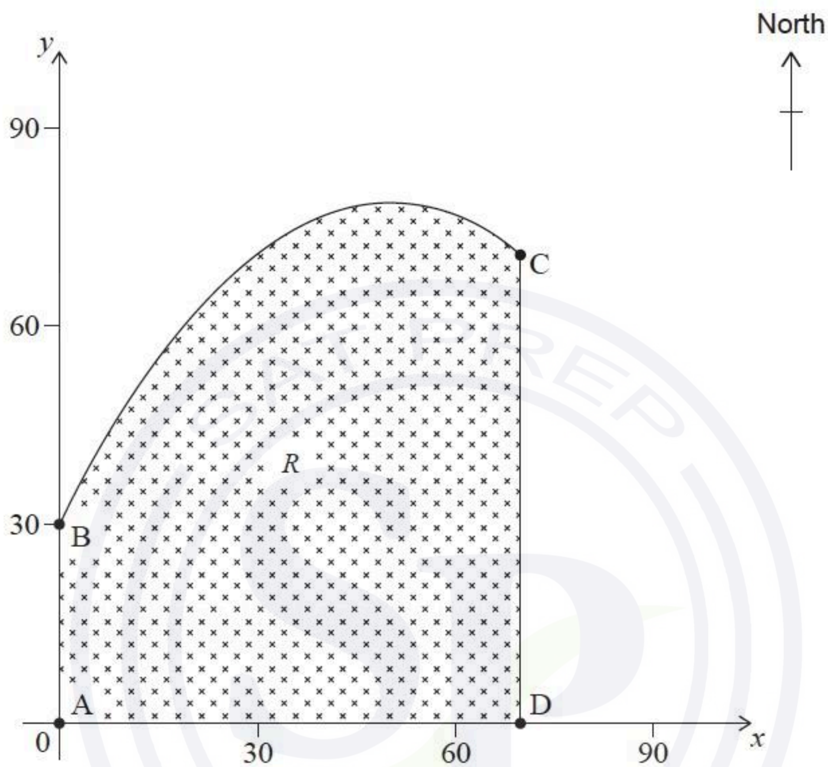
(b) Find the volume of liquid that can be contained inside the finished glass. [5]

(c) Find the volume of the region between the two surfaces of the finished glass. [6]

Question 18

[Maximum mark: 17]

Linda owns a field, represented by the shaded region R. The plan view of the field is shown in the following diagram, where both axes represent distance and are measured in metres.



The segments [AB], [CD] and [AD] respectively represent the western, eastern and southern boundaries of the field. The function, $f(x)$, models the northern boundary of the field between points B and C and is given by

$$f(x) = \frac{-x^2}{50} + 2x + 30, \text{ for } 0 \le x \le 70.$$

(a) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the point on the field that is furthest north. [5]

Point A has coordinates $(0,\ 0)$, point B has coordinates $(0,\ 30)$, point C has coordinates $(70,\ 72)$ and point D has coordinates $(70,\ 0)$.

(b) (i) Write down the integral which can be used to find the area of the shaded region R.

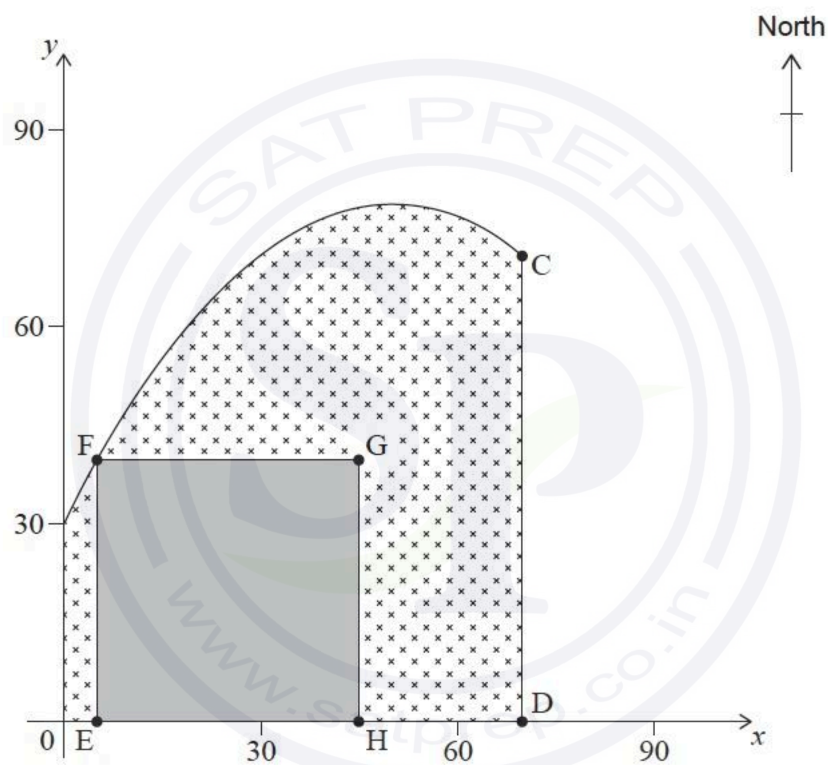
(ii) Find the area of Linda's field. [4]

Linda used the trapezoidal rule with ten intervals to estimate the area. This calculation underestimated the area by $11.4\,\text{m}^2$.

(c) (i) Calculate the percentage error in Linda's estimate.

(ii) Suggest how Linda might be able to reduce the error whilst still using the trapezoidal rule. [3]

Linda would like to construct a building on her field. The **square** foundation of the building, EFGH, will be located such that [EH] is on the southern boundary and point F is on the northern boundary of the property. A possible location of the foundation of the building is shown in the following diagram.



The area of the square foundation will be largest when [GH] lies on [CD].

(d) (i) Find the x-coordinate of point E for the largest area of the square foundation of building EFGH.

(ii) Find the largest area of the foundation. [5]

Question 19

[Maximum mark: 18]

A biologist suggests that the rates of change of the population of fruit flies (after time $t \geq 0$) in a particular ecosystem are given by the following equations, where x is the population of male fruit flies and y is the population of female fruit flies.

$$\frac{dx}{dt} = -4x + 6y$$

$$\frac{dy}{dt} = 9x - y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -4 & 6 \\ 9 & -1 \end{pmatrix}$. [6]

(b) Hence write down the general solution of the system, giving your answer in the form $\begin{pmatrix} x \\ y \end{pmatrix} = A\boldsymbol{p}_1 e^{\lambda_1 t} + B\boldsymbol{p}_2 e^{\lambda_2 t}$, where A, B, λ_1, λ_2 $(\lambda_2 > \lambda_1)$ are scalar constants and $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ are vector constants. [2]

Initially $x = 500$ and $y = 125$.

(c) Determine the value of A and the value of B. [2]

(d) State the long-term ratio of male fruit flies to female fruit flies. [1]

(e) Find the value of $\frac{dy}{dx}$ at time $t = 0$. [3]

(f) Sketch the trajectory, on the phase portrait, for the population growth of the fruit flies. [4]

Question 20

[Maximum mark: 16]

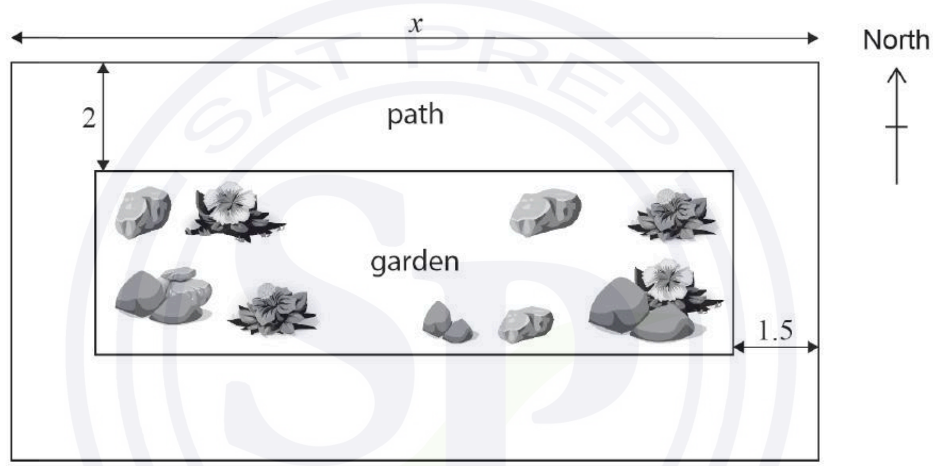
A particular park consists of a rectangular garden, of area $A\,\mathrm{m}^2$, and a concrete path surrounding it. The park has a total area of $1200\,\mathrm{m}^2$.

The width of the path at the north and south side of the park is $2\,\mathrm{m}$.

The width of the path at the west and east side of the park is $1.5\,\mathrm{m}$.

The length of the park (along the north and south sides) is x metres, $3 < x < 300$.

diagram not to scale



(a) Show that $A = 1212 - 4x - \dfrac{3600}{x}$. [5]

(b) Find the possible dimensions of the park if the area of the garden is $800\,\mathrm{m}^2$. [4]

(c) Find an expression for $\dfrac{\mathrm{d}A}{\mathrm{d}x}$. [3]

(d) Use your answer from part (c) to find the value of x that will maximize the area of the garden. [2]

(e) Find the maximum possible area of the garden. [2]

Question 21

[Maximum mark: 15]

A model speedboat has its position, at time t seconds $t \geq 0$, defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5y - 0.05x\,,\ \frac{\mathrm{d}y}{\mathrm{d}t} = -5x - 0.05y\,,$$

where x metres is the distance east and y metres is the distance north of a fixed point O.

(a) Find the eigenvalues of $A = \begin{pmatrix} -0.05 & 5 \\ -5 & -0.05 \end{pmatrix}$, giving your answers in the form $a + b\mathrm{i}$, where $a \neq 0,\ b \neq 0$. [4]

(b) (i) State what $a \neq 0$ indicates about the path of the speedboat.

(ii) State what the sign of a indicates about the path of the speedboat. [2]

At time $t = 0$, the speedboat has position $(20, 0)$.

(c) At time $t = 0$, find the value of

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}t}$.

(ii) $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. [5]

(d) Use your answers to parts (b) and (c) to sketch the path of the model speedboat. [4]

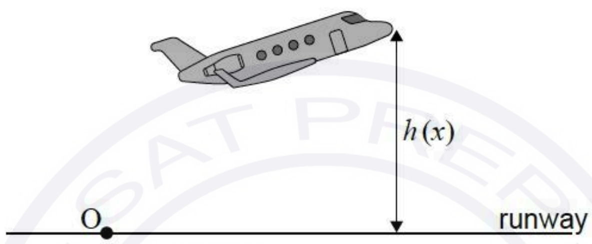
Question 22

[Maximum mark: 12]

A plane takes off from a horizontal runway. Let point O be the point where the plane begins to leave the runway and x be the horizontal distance, in km, of the plane from O. The function h models the vertical height, in km, of the nose of the plane from the horizontal runway, and is defined by

$$h(x)=\frac{10}{1+150e^{-0.07x}}-0.06,\ x\geq 0.$$

diagram not to scale



(a) (i) Find $h(0)$.

(ii) Interpret this value in terms of the context. [2]

(b) (i) Find the horizontal asymptote of the graph of $y=h(x)$.

(ii) Interpret this value in terms of the context. [2]

(c) Find $h'(x)$ in terms of x. [4]

A safety regulation recommends that $h'(x)$ never exceed 0.2.

(d) Given that this plane flies a distance of at least 200 km horizontally from point O, determine whether the plane is following this safety regulation. [4]

Question 22

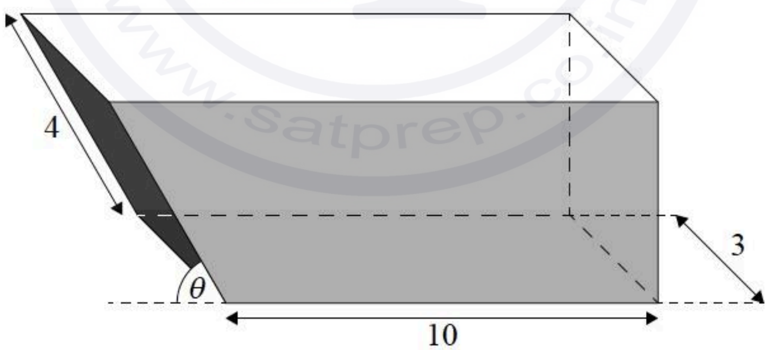
[Maximum mark: 22]

A skip is a container used to carry garbage away from a construction site. For safety reasons the garbage must not extend beyond the top of the skip. The maximum volume of garbage to be removed is therefore equal to the volume of the skip.

A particular design of skip can be modelled as a prism with a trapezoidal cross section. For the skip to be transported, it must have a rectangular base of length $10\,\text{m}$ and width $3\,\text{m}$. The length of the sloping edge is fixed at $4\,\text{m}$, and makes an angle of θ with the horizontal.

The following diagram shows such a skip.

diagram not to scale



(a) Find the volume of this skip,

(i) if the length of the top edge of the skip is $11\,\text{m}$.

(ii) if the height of the skip is $3.2\,\text{m}$.

(iii) if θ is $\frac{\pi}{3}$. [9]

(b) Show that the volume, $V\,\text{m}^3$, of the skip is given by

$$24\sin(\theta)(5+\cos(\theta)).$$ [2]

(c) Explain, in context, why $\theta \neq 0$. [1]

(d) (i) Sketch the graph of $V = 24\sin(\theta)\big(5+\cos(\theta)\big),\ 0 < \theta < \frac{\pi}{2}$.

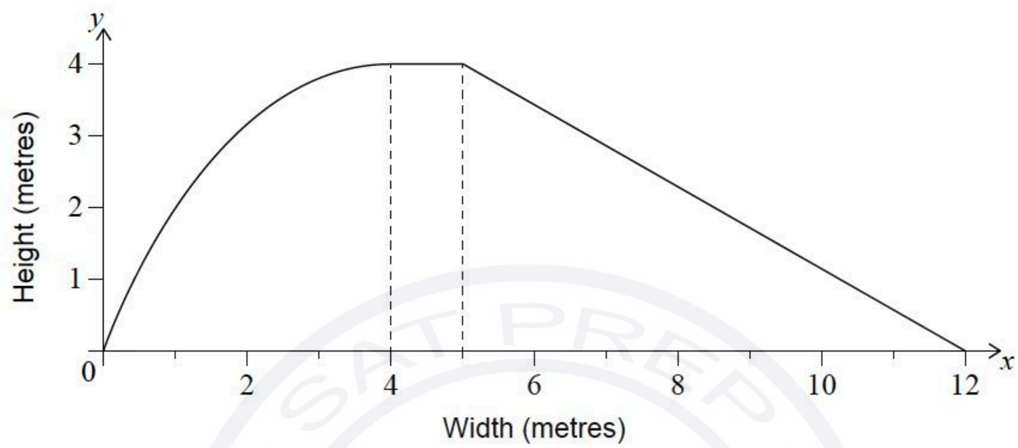
(ii) Find the maximum volume of the skip and the value of θ for which this maximum volume occurs. [4]

(e) Show, by differentiation, that the maximum volume occurs at a value of θ that satisfies the equation $2\cos^2\theta + 5\cos\theta - 1 = 0$. [6]

Question 23

[Maximum mark: 15]

The following diagram shows a model of the side view of a water slide. All lengths are measured in metres.



The curved edge of the slide is modelled by

$$f(x) = -\frac{1}{4}x^2 + 2x \text{ for } 0 \le x \le 4.$$

The remainder of the slide is modelled by

$$g(x) = \begin{cases} 4, \text{ for } 4 \le x \le 5 \\ \frac{48}{7} - \frac{4x}{7}, \text{ for } 5 \le x \le 12 \end{cases}$$

(a) Use the trapezoidal rule with an interval width of 1 to calculate the approximate area under the model of the slide in the interval $0 \le x \le 4$. [5]

(b) Find $\int \left(-\frac{1}{4}x^2 + 2x \right) \mathrm{d}x$. [3]

(c) Calculate the exact area under the entire model of the slide, for $0 \le x \le 12$. [4]

(d) Find the percentage error in the **total** area under the entire model of the slide when using the approximate value from part (a). [3]

Question 24

[Maximum mark: 14]

The interior of a vase is modelled by rotating the region bounded by the curve $y = \frac{1}{2}x^2 - 1$, and the lines $x = 0$, $y = 0$ and $y = 15$, through 2π radians about the y-axis. The values of x and y are measured in centimetres.

The vase is filled with water to a height of $h\,\text{cm}$.

(a) Find an explicit expression for the volume of water in terms of h. [5]

The vase is filled at a rate of $20\,\text{cm}^3\,\text{s}^{-1}$.

(b) Find the time taken to completely fill the vase. [2]

(c) Find the rate at which the height is changing when $h = 10\,\text{cm}$. [7]

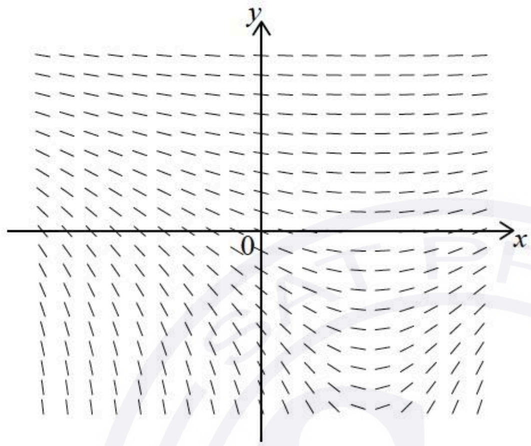
Question 25

[Maximum mark: 13]

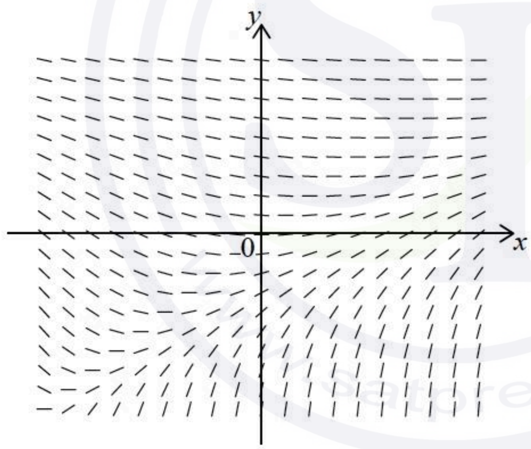
Consider the differential equation $\frac{dy}{dx} = \frac{x}{e^{2y}}$.

(a) Identify which of the following diagrams, **A**, **B** or **C**, represents the slope field for the differential equation. Give a reason for your answer. [2]

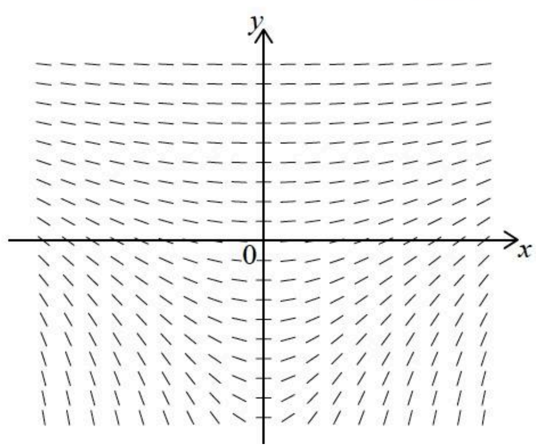
A.



B.



C.



It is given that, for a particular solution, $x = 0$ and $y = 0$.

(b) Find an expression for y, in terms of x, for this solution. [7]

(c) Find $\frac{dy}{dx}$, in terms of x, by differentiating your answer from part (b). [2]

(d) Hence verify that your answer to part (b) is a solution to $\frac{dy}{dx} = \frac{x}{e^{2y}}$. [2]

Question 26

[Maximum mark: 13]

A shop uses the following model to estimate n, the number of smoothies sold per day, in terms of x, the price of a single smoothie in pesos.

$$n = \frac{40\,000}{x^2}$$

The maximum number of smoothies the shop can make in a day is 400.

(a) Find the maximum price they could charge per smoothie for the shop to sell 400 in one day. [2]

(b) On a day when the shop sells smoothies at 50 pesos each, use the model to find

(i) the number of smoothies sold.

(ii) the total income from the smoothies sold. [2]

The cost of making each smoothie is 20 pesos. The profit per day (P) is the total income from the sale of smoothies that day minus the cost of making them.

(c) (i) Show that, according to the model, $P = \frac{40\,000}{x} - \frac{800\,000}{x^2}$.

(ii) Find $\frac{dP}{dx}$.

(iii) Find the value of x for which $\frac{dP}{dx} = 0$.

(iv) Find the number of smoothies sold when the profit is maximized. [9]

Question 27

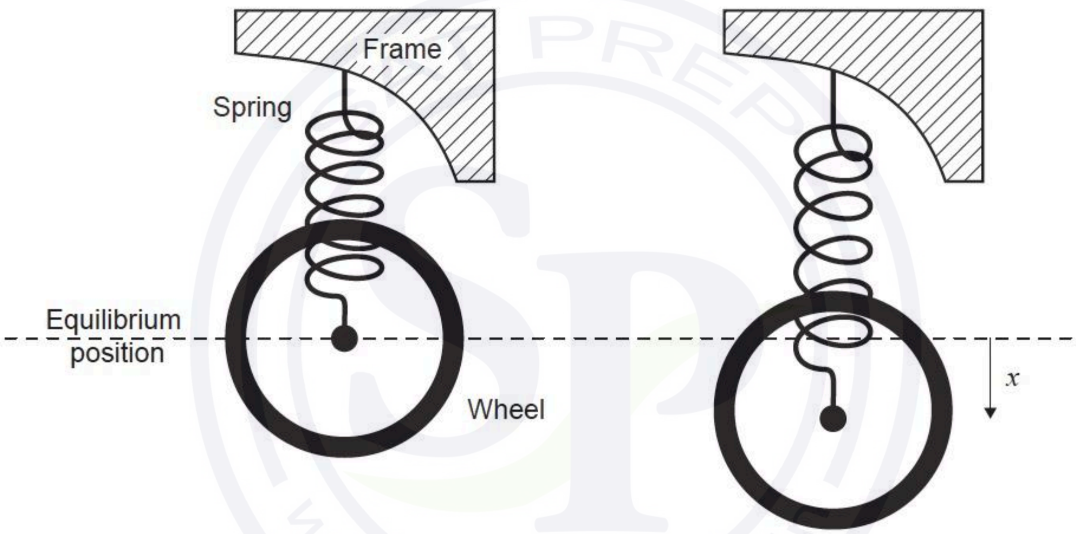
[Maximum mark: 18]

The wheel on a motorbike is attached to the frame by a spring. The movement of the spring acts as a **shock absorber**. When the rider sits on the motorbike, the spring compresses, and this position is called the equilibrium position.

When the wheel goes into holes or over bumps in the road, the spring will extend or compress to ensure a smooth ride.

Let x denote the vertical displacement, in centimetres, of the centre of the wheel below the equilibrium position, as shown in the diagram.

diagram not to scale



The vertical displacement of the centre of the wheel, at time t seconds, can be modelled by the differential equation

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0 \text{, where } a, b \in \mathbb{R}.$$

Let $y = \frac{dx}{dt}$.

(a) Show that $\frac{dy}{dt} = -bx - ay$. [2]

The equations $y = \frac{dx}{dt}$ and $\frac{dy}{dt} = -bx - ay$ can be written in matrix form as

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \boldsymbol{M}\begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } \boldsymbol{M} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}.$$

A manufacturer wants to compare two springs, Spring 1 and Spring 2, that could be used as shock absorbers.

The differential equation for Spring 1 has $a = 18$ and $b = 77$.

For these values, the eigenvalues of $\boldsymbol{M}$ are -7 and -11.

(b) Find the corresponding eigenvectors. [3]

The manufacturer models **both** springs using the same initial conditions:

$$t = 0,\ x = 5\,\text{cm and } \frac{dx}{dt} = 2\,\text{cm s}^{-1}.$$

(c) Hence, for Spring 1

(i) find the exact solution for $x(t)$

(ii) sketch the graph of $x(t)$, for $0 \le t \le 1$. [7]

The differential equation for Spring 2 has $a = 18$ and $b = 85$.

For these values, the eigenvalues of $\boldsymbol{M}$ are $-9 \pm 2\text{i}$.

(d) (i) Sketch the phase portrait for Spring 2, indicating the direction of the trajectory.

(ii) Hence, sketch the graph of x against t. [5]

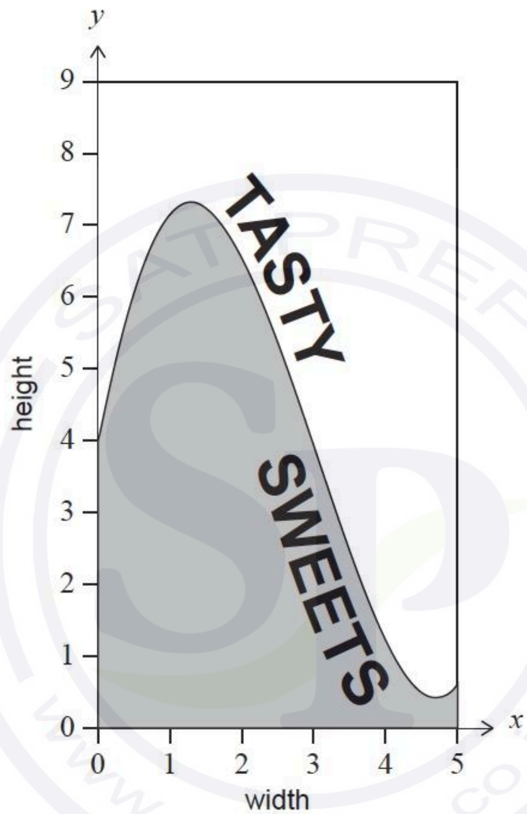
(e) Using your answers to parts (c)(ii) and (d)(ii), give a reason why Spring 1 might make a better shock absorber than Spring 2. [1]

Question 28

[Maximum mark: 18]

Sweets are sold in cylindrical containers. A new label for the container is being considered. The label will be a rectangle that is 5 cm wide and 9 cm high.

The design on the label is a curve, as shown on the following axes, where one unit represents 1 cm for both axes.



The values in the table approximate points on the curve, correct to one decimal place.

Width, x	0	1	2	3	4	5
Height, y	4	7.3	6.7	4.0	1.3	0.7

(a) Use the trapezoidal rule with five intervals, and the values given in the table, to estimate the shaded area below the curve. [3]

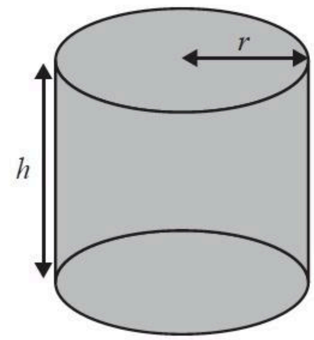
The curve used in the label design can be modelled by:

$$y = \frac{x^3}{3} - 3x^2 + 6x + 4, \text{ for } 0 \le x \le 5.$$

(b) Use this equation to find the area of the shaded region. [2]

The sweets are sold in closed cylindrical containers, with radius r and height h.

diagram not to scale



The whole container is made from one type of material, and it is assumed that the thickness of the material is negligible.

Each container has a volume of $600\,\text{cm}^3$.

(c) Write down an equation, in terms of r and h, that shows this information. [1]

The amount of material used for each container can be modelled by the external surface area of the container.

The external surface area, A, of the container can be expressed as

$$A = 2\pi r^2 + \frac{k}{r}, \text{ where } r > 0.$$

(d) Find the value of k. [4]

(e) (i) Find $\frac{\mathrm{d}A}{\mathrm{d}r}$.

(ii) Given that A has a minimum value, find the value of r that will minimize the material used. [5]

The containers are made so that the surface area is minimized. The $5\,\text{cm}$ by $9\,\text{cm}$ rectangular label is to be glued to the curved surface of the container.

(f) Show that the label will fit on the container. [3]