Subject – Math AI(Higher Level)
Topic - Calculus
Year - May 2021 – Nov 2024
Paper -2
Answers

Question 1

(a) $\frac{dv}{dt} = 9.81 - 0.9v$ ***M1***

$\int \frac{1}{9.81 - 0.9v} dv = \int 1 \, dt$ ***M1***

$-\frac{1}{0.9}\ln(9.81 - 0.9v) = t + c$ ***A1***

$9.81 - 0.9v = Ae^{-0.9t}$ ***A1***

$v = \frac{9.81 - Ae^{-0.9t}}{0.9}$ ***A1***

when $t = 0$, $v = 0$ hence $A = 9.81$ ***A1***

$v = \frac{9.81(1 - e^{-0.9t})}{0.9}$

$v = 10.9(1 - e^{-0.9t})$ ***A1***

[7 marks]

(b) **either** let t tend to infinity, **or** $\frac{dv}{dt} = 0$ ***(M1)***

$v = 10.9$ ***A1***

[2 marks]

(c) $\frac{dx}{dt} = y$ ***M1***

$\frac{dy}{dt} = 9.81 - 0.9y^2$ ***A1***

[2 marks]

(d) $x_{n+1} = x_n + 0.2y_n$, $y_{n+1} = y_n + 0.2(9.81 - 0.9(y_n)^2)$ ***(M1)(A1)***

$x = 1.04$, $\frac{dx}{dt} = 3.31$ ***(M1)A1***

[4 marks]

(e) 3.3015 **A1**

[1 mark]

(f) $0 = 9.81 - 0.9(v)^2$ **M1**

$\Rightarrow v = \sqrt{\frac{9.81}{0.9}} = 3.301511\ldots\ (= 3.30)$ **A1**

[2 marks]

(g) the model found the terminal velocity very accurately, so good approximation **R1**

intermediate values had object exceeding terminal velocity so not good approximation **R1**

[2 marks]

Total [20 marks]

Question 2

(a) $2(8\times4+3\times4+3\times8)$ **_M1_**

$=136\ (\text{cm}^2)$ **_A1_**

[2 marks]

(b) $\sqrt{8^2+4^2+3^2}$ **_M1_**

$(\text{AG}=)\ 9.43\ (\text{cm})\ (9.4339\ldots,\ \sqrt{89})$ **_A1_**

[2 marks]

(c) $-2x+220=0$ **_M1_**

$x=110$ **_A1_**

$110\,000$ (boxes) **_A1_**

[3 marks]

(d) $P(x)=\int -2x+220\,\mathrm{d}x$ **_M1_**

Note: Award ***M1*** for evidence of integration.

$P(x)=-x^2+220x+c$ **_A1A1_**

Note: Award ***A1*** for either $-x^2$ or $220x$ award ***A1*** for both correct terms and constant of integration.

$1700=-(20)^2+220(20)+c$ **_M1_**

$c=-2300$

$P(x)=-x^2+220x-2300$ **_A1_**

[5 marks]

(e) $-x^2+220x-2300=0$ **_M1_**

$x=11.005$ **_A1_**

$11\,006$ (boxes) **_A1_**

Note: Award ***M1*** for their $P(x)=0$, award ***A1*** for their correct solution to x. Award the final ***A1*** for expressing their solution to the minimum number of boxes. Do not accept $11\,005$, the nearest integer, nor $11\,000$, the answer expressed to 3 significant figures, as these will not satisfy the demand of the question.

[3 marks]

Total [15 marks]

Question 3

(a) $\int \frac{1}{x}\mathrm{d}x = \int 2\mathrm{d}t$ ***(M1)***

$\ln x = 2t + c$

$x = A\mathrm{e}^{2t}$ ***(A1)***

$x(0) = 100 \Rightarrow A = 100$ ***(M1)***

$x = 100\mathrm{e}^{2t}$ ***(A1)***

$x(1) = 739$ ***A1***

Note: Accept 738 for the final ***A1***.

[5 marks]

(b) $t_{n+1} = t_n + 0.25$ ***(A1)***

Note: This may be inferred from a correct t column, where this is seen.

$x_{n+1} = x_n + 0.25x_n(2 - 0.01y_n)$ ***(A1)***

$y_{n+1} = y_n + 0.25y_n(0.0002x_n - 0.8)$ ***(A1)***

t	x	y
0	1000	100
0.25	1250	85
0.5	1609	73
0.75	2119	65
1	2836	58

(A1)

Note: Award ***A1*** for whole line correct when $t = 0.5$ or $t = 0.75$. The t column may be omitted and implied by the correct x and y values. The formulas are implied by the correct x and y columns.

(i) 2840 (2836 **OR** 2837) ***A1***

(ii) 58 **OR** 59 ***A1***

[6 marks]

(c) (i) both populations are increasing ***A1***

(ii) rabbits are decreasing and foxes are increasing ***A1A1***

[3 marks]

(d) setting at least one DE to zero ***(M1)***

$x = 4000$, $y = 200$ ***A1A1***

[3 marks]

Total [17 marks]

Question 4

(a) $\int \frac{1}{x}\,dx = \int 2\,dt$ *(M1)*

$\ln x = 2t + c$

$x = Ae^{2t}$ *(A1)*

$x(0) = 100 \Rightarrow A = 100$ *(M1)*

$x = 100e^{2t}$ *(A1)*

$x(1) = 739$ ***A1***

Note: Accept 738 for the final ***A1***.

[5 marks]

(b) $t_{n+1} = t_n + 0.25$ *(A1)*

Note: This may be inferred from a correct t column, where this is seen.

$x_{n+1} = x_n + 0.25x_n(2 - 0.01y_n)$ *(A1)*

$y_{n+1} = y_n + 0.25y_n(0.0002x_n - 0.8)$ *(A1)*

t	x	y
0	1000	100
0.25	1250	85
0.5	1609	73
0.75	2119	65
1	2836	58

(A1)

Note: Award ***A1*** for whole line correct when $t = 0.5$ or $t = 0.75$. The t column may be omitted and implied by the correct x and y values. The formulas are implied by the correct x and y columns.

(i) 2840 (2836 **OR** 2837) ***A1***

(ii) 58 **OR** 59 ***A1***

[6 marks]

(c) (i) both populations are increasing ***A1***

(ii) rabbits are decreasing and foxes are increasing ***A1A1***

[3 marks]

(d) setting at least one DE to zero *(M1)*

$x = 4000$, $y = 200$ ***A1A1***

[3 marks]

Total [17 marks]

Question 5

(a) (i) evidence of power rule (at least one correct term seen) ***(M1)***

$\frac{dy}{dx} = -0.3x^2 + 1.6x$ ***A1***

(ii) $-0.3x^2 + 1.6x = 0$ ***M1***

$x = 5.33 \left(5.33333\ldots, \frac{16}{3}\right)$ ***A1***

$y = -0.1 \times 5.33333\ldots^3 + 0.8 \times 5.33333\ldots^2$ ***(M1)***

Note: Award ***M1*** for substituting their zero for $\frac{dy}{dx}$ $(5.333\ldots)$ into y.

$7.59 \text{ m} \ (7.58519\ldots)$ ***A1***

Note: Award ***M0A0M0A0*** for an unsupported 7.59.
Award at most ***M0A0M1A0*** if only the last two lines in the solution are seen.
Award at most ***M1A0M1A1*** if their $x = 5.33$ is not seen.

[6 marks]

(b) One correct substitution seen ***(M1)***

(i) 6.4 m ***A1***

(ii) 7.2 m ***A1***

[3 marks]

(c) $A = \frac{1}{2} \times 2\left((2.4+0)+2(6.4+7.2)\right)$ ***(A1)(M1)***

> **Note:** Award ***A1*** for $h=2$ seen. Award ***M1*** for correct substitution into the trapezoidal rule (the zero can be omitted in working).

$= 29.6 \text{ m}^2$ ***A1***

[3 marks]

(d) (i) $A = \int_2^8 -0.1x^3 + 0.8x^2 \, dx$ **OR** $A = \int_2^8 y \, dx$ ***A1A1***

> **Note:** Award ***A1*** for a correct integral, ***A1*** for correct limits in the correct location. Award at most ***A0A1*** if dx is omitted.

(ii) $A = 32.4 \text{ m}^2$ ***A2***

> **Note:** As per the marking instructions, ***FT*** from their integral in part (d)(i). Award at most ***A1FTA0*** if their area is >48, this is outside the constraints of the question (a 6x8 rectangle).

[4 marks]

Total [16 marks]

Question 6

(a) $\begin{vmatrix} -4-\lambda & 0 \\ 3 & -2-\lambda \end{vmatrix} = 0$ ***(M1)***

$(-4-\lambda)(-2-\lambda) = 0$ ***(A1)***

$\lambda = -4$ **OR** $\lambda = -2$ ***A1***

$\lambda = -4$

$\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4x \\ -4y \end{pmatrix}$ ***(M1)***

Note: This ***M1*** can be awarded for attempting to find either eigenvector.

$3x - 2y = -4y$

$3x = -2y$

possible eigenvector is $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ (or any real multiple) ***A1***

$\lambda = -2$

$\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$

$x = 0,\ y = 1$

possible eigenvector is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (or any real multiple) ***A1***

[6 marks]

(b) $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-4t}\begin{pmatrix} -2 \\ 3 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ***(M1)A1***

Note: Award ***M1A1*** for $x = -2Ae^{-4t}$, $y = 3Ae^{-4t} + Be^{-2t}$, ***M1A0*** if LHS is missing or incorrect.

[2 marks]

(c) two (distinct) real negative eigenvalues ***R1***

(or equivalent (*eg* both $e^{-4t} \to 0, e^{-2t} \to 0$ as $t \to \infty$))

$\Rightarrow$ stable equilibrium point ***A1***

Note: Do not award ***R0A1***.

[2 marks]

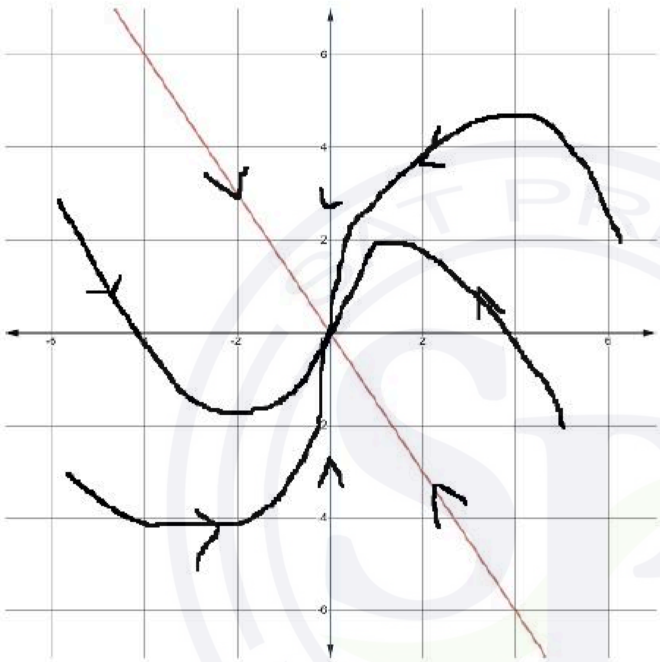
(d) $\frac{dy}{dx} = \frac{3x - 2y}{-4x}$ ***(M1)***

(i) $(4,\ 0) \Rightarrow \frac{dy}{dx} = -\frac{3}{4}$ ***A1***

(ii) $(-4,\ 0) \Rightarrow \frac{dy}{dx} = -\frac{3}{4}$ ***A1***

[3 marks]

(e)



A1A1A1A1

Note: Award ***A1*** for a phase plane, with correct axes (condone omission of labels) and at least three non-overlapping trajectories. Award ***A1*** for all trajectories leading to a stable node at $(0,\ 0)$. Award ***A1*** for showing gradient is negative at $x = 4$ and -4. Award ***A1*** for both eigenvectors on diagram.

[4 marks]

Total [17 marks]

Question 7

(a) solving $v = 0$ **M1**

$t = 2,\ t = 6$ **A1**

[2 marks]

(b) use of power rule ***(M1)***

$\frac{dv}{dt} = -4t + 16$ ***(A1)***

$(t = 6)$

$\Rightarrow a = -8$ ***(A1)***

magnitude $= 8\ \text{m s}^{-2}$ ***A1***

[4 marks]

(c) using a sketch graph of v ***(M1)***

$24\ \text{m s}^{-1}$ ***A1***

[2 marks]

(d) **METHOD ONE**

$x = \int v\ dt$

attempt at integration of v ***(M1)***

$-\frac{2t^3}{3} + 8t^2 - 24t\ (+c)$ ***A1***

attempt to find c (use of $t = 0,\ x = 0$) ***(M1)***

$c = 0$ ***A1***

$\left(x = -\frac{2t^3}{3} + 8t^2 - 24t\right)$

METHOD TWO

$x = \int_0^t v\ dt$

attempt at integration of v ***(M1)***

$\left[-\frac{2t^3}{3} + 8t^2 - 24t\right]_0^t$ ***A1***

attempt to substituted limits into their integral ***(M1)***

$x = -\frac{2t^3}{3} + 8t^2 - 24t$ ***A1***

[4 marks]

(e) $\int_0^4 |v|\ dt$ ***(M1)(A1)***

Note: Award ***M1*** for using the absolute value of v, or separating into two integrals, ***A1*** for the correct expression.

$= 32\ \text{m}$ ***A1***

[3 marks]

Total [15 marks]

Question 8

(a) evidence of splitting diagram into equilateral triangles **M1**

area $= 6\left(\frac{1}{2}x^2 \sin 60^\circ\right)$ **A1**

$= \frac{3\sqrt{3}x^2}{2}$ **AG**

Note: The ***AG*** line must be seen for the final ***A1*** to be awarded.

[2 marks]

(b) total surface area of prism $1200 = 2\left(3x^2\frac{\sqrt{3}}{2}\right) + 6xh$ **M1A1**

Note: Award ***M1*** for expressing total surface areas as a sum of areas of rectangles and hexagon(s), and ***A1*** for a correctly substituted formula, equated to 1200.

[5 marks]

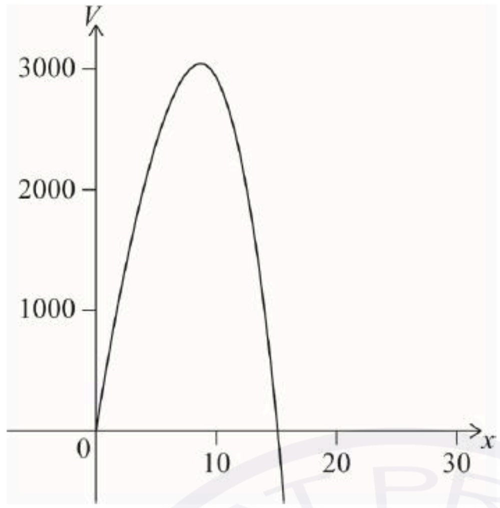
$h = \frac{400 - \sqrt{3}x^2}{2x}$ **A1**

volume of prism $= \frac{3\sqrt{3}}{2}x^2 h$ ***(A1)***

$= \frac{3\sqrt{3}}{2}x^2\left(\frac{400 - \sqrt{3}x^2}{2x}\right)$ **A1**

$= 300\sqrt{3}x - \frac{9}{4}x^3$ ***(AG)***

(c)



A1A1

Note: Award ***A1*** for correct shape, ***A1*** for roots in correct place with some indication of scale (indicated by a labelled point).

[2 marks]

(d) $\frac{dV}{dx} = 300\sqrt{3} - \frac{27}{4}x^2$ ***A1A1***

Note: Award ***A1*** for a correct term.

[2 marks]

(e) from the graph of V or $\frac{dV}{dx}$ **OR** solving $\frac{dV}{dx} = 0$ ***(M1)***

$x = 8.77$ (8.77382...) ***A1***

[2 marks]

(f) from the graph of V **OR** substituting their value for x into V ***(M1)***

$V_{max} = 3040\ cm^3$ (3039.34...) ***A1***

[2 marks]

Total [15 marks]

Question 9

(a) $y = \dot{x} \Rightarrow \dot{y} = \ddot{x}$ **A1**

$\dot{y} + 3(y) + 1.25x = 0$ **R1**

Note: If no explicit reference is made to $\dot{y} = \ddot{x}$, or equivalent, award ***A0R1*** if second line is seen. If $\frac{\mathrm{d}y}{\mathrm{d}x}$ used instead of $\frac{\mathrm{d}y}{\mathrm{d}t}$, award ***A0R0***.

$\dot{y} = -3y - 1.25x$ **AG**

[2 marks]

(b) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -1.25 & -3 \end{pmatrix}$ **A1**

[1 mark]

(c) (i) $\begin{vmatrix} -\lambda & 1 \\ -1.25 & -3-\lambda \end{vmatrix} = 0$ ***(M1)***

$\lambda(\lambda + 3) + 1.25 = 0$ ***(A1)***

$\lambda = -2.5$; $\lambda = -0.5$ **A1**

(ii) $\begin{pmatrix} 2.5 & 1 \\ -1.25 & -0.5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ***(M1)***

$2.5a + b = 0$

$\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ **A1**

$\begin{pmatrix} 0.5 & 1 \\ -1.25 & -2.5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$0.5a + b = 0$

$\boldsymbol{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ **A1**

Note: Award ***M1*** for a valid attempt to find either eigenvector. Accept equivalent forms of the eigenvectors. Do not award ***FT*** for eigenvectors that do not satisy both rows of the matrix.

[6 marks]

(d) $\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{-2.5t}\begin{pmatrix} -2 \\ 5 \end{pmatrix} + B\mathrm{e}^{-0.5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ ***M1A1***

$t = 0 \;\Rightarrow\; x = 8,\; \dot{x} = y = 0$ ***(M1)***

$-2A - 2B = 8$
$5A + B = 0$ ***(M1)***

$A = 1;\; B = -5$ ***A1***

$x = -2\mathrm{e}^{-2.5t} + 10\mathrm{e}^{-0.5t}$ ***A1***

Note: Do not award the final ***A1*** if the answer is given in the form $\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{-2.5t}\begin{pmatrix} -2 \\ 5 \end{pmatrix} + B\mathrm{e}^{-0.5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

[6 marks]
Total: [15 marks]

Question 10

(a) (i) $y = x^{\frac{1}{2}}$ *(M1)*

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ ***A1***

(ii) gradient at $x = 0.16$ is $\frac{1}{2} \times \frac{1}{\sqrt{0.16}}$ ***M1***

$= 1.25$

EITHER
$y - 0.4 = 1.25(x - 0.16)$ ***M1***

OR
$0.4 = 1.25(0.16) + b$ ***M1***

Note: Do not allow working backwards from the given answer.

THEN
hence $y = 1.25x + 0.2$ ***AG***

[4 marks]

(b) $p = 0.45$, $q = 0.4125$ (or 0.413) (accept "$(0.45, 0.4125)$") ***A1A1***

[2 marks]

(c) (i) $(h(x) =)\ \frac{1}{2}\sqrt{2(x - 0.2)}$ ***A2***

Note: Award ***A1*** if only two correct transformations are seen.

(ii) $(a =)\ 0.28$ ***A1***

(iii) **EITHER**
Correct substitution of their part (b) (or $(0.28, 0.2)$) into the given expression *(M1)*

OR
$\frac{1}{2}(1.25 \times 2(x - 0.2) + 0.2)$ *(M1)*

Note: Award ***M1*** for transforming the equivalent expression for *f* correctly.

THEN
$(b =)\ -0.15$ ***A1***

[5 marks]

(d) (i) recognizing need to add two integrals ***(M1)***

$\int_0^{0.16} \sqrt{x}\ dx + \int_{0.16}^{0.5} (1.25x + 0.2)\ dx$ ***(A1)***

Note: The second integral could be replaced by the formula for the area of a trapezoid $\frac{1}{2} \times 0.34(0.4 + 0.825)$.

$0.251\ m^2\ (0.250916\ldots)$ ***A1***

(ii) **EITHER**

area of trapezoid $\frac{1}{2} \times 0.05(0.4125 + 0.825) = 0.0309375$ ***(M1)(A1)***

OR

$\int_{0.45}^{0.5} (8.25x - 3.3)\,dx = 0.0309375$ ***(M1)(A1)***

Note: If the rounded answer of 0.413 from part (b) is used, the integral is $\int_{0.45}^{0.5} (8.24x - 3.295)\,dx = 0.03095$ which would be awarded ***(M1)(A1)***.

THEN

shaded area $= 0.250916\ldots - 0.0627292 - 0.0309375$ ***(M1)***

Note: Award ***(M1)*** for the subtraction of both $0.0627292\ldots$ and their area for the trapezoid from their answer to (a)(i).

$= 0.157\ m^2\ (0.15725)$ ***A1***

[7 marks]

[Total 18 marks]

Question 11

(a) (i) $y=\frac{\mathrm{d}x}{\mathrm{d}t}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}t}+5\frac{\mathrm{d}x}{\mathrm{d}t}+6x=0$ **OR** $\frac{\mathrm{d}y}{\mathrm{d}t}+5y+6x=0$ ***M1***

Note: Award ***M1*** for substituting $\frac{\mathrm{d}y}{\mathrm{d}t}$ for $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$.

$$\begin{pmatrix}\frac{\mathrm{d}x}{\mathrm{d}t}\\ \frac{\mathrm{d}y}{\mathrm{d}t}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -6 & -5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$ ***AG***

(ii) $\det\begin{pmatrix}-\lambda & 1\\ -6 & -5-\lambda\end{pmatrix}=0$ ***(M1)***

Note: Award ***M1*** for an attempt to find eigenvalues. Any indication that $\det(\boldsymbol{M}-\lambda\boldsymbol{I})=0$ has been used is sufficient for the ***(M1)***.

$-\lambda(-5-\lambda)+6=0$ **OR** $\lambda^2+5\lambda+6=0$ ***(A1)***

$\lambda=-2,\ -3$ ***A1***

(iii) (on a phase portrait the particle approaches $(0,\ 0)$ as t increases so long term velocity (v) is)

0 ***A1***

Note: Only award ***A1*** for 0 if both eigenvalues in part (a)(ii) are negative. If at least one is positive accept an answer of '*no limit*' or '*infinity*', or in the case of one positive and one negative also accept '*no limit or 0 (depending on initial conditions)*'.

[5 marks]

(b) (i) $y=\frac{\mathrm{d}x}{\mathrm{d}t}$

$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\frac{\mathrm{d}y}{\mathrm{d}t}$ ***(A1)***

$\frac{\mathrm{d}y}{\mathrm{d}t}+5y+6x=3t+4$ ***A1***

(ii) recognition that $h=0.1$ in any recurrence formula ***(M1)***

$(t_{n+1}=t_n+0.1)$

$x_{n+1}=x_n+0.1y_n$ ***(A1)***

$y_{n+1}=y_n+0.1(3t_n+4-5y_n-6x_n)$ ***(A1)***

(when $t=1$,) $x=0.64402\ldots\approx0.644\ \mathrm{m}$ ***A2***

(iii) recognizing that y is the velocity

$0.5\ \mathrm{m\,s^{-1}}$ ***A1***

[8 marks]

[Total 13 marks]

Question 12

(a) (i) $\left(\frac{1}{2}\hat{AOB} =\right) \arccos\left(\frac{4}{4.5}\right) = 27.266...$ ***(M1)(A1)***

$\hat{AOB} = 54.532... \approx 54.5^{\circ}$ $(0.951764... \approx 0.952$ radians) ***A1***

Note: Other methods may be seen; award ***(M1)(A1)*** for use of a correct trigonometric method to find an appropriate angle and then ***A1*** for the correct answer.

(ii) a finding area of triangle

EITHER

area of triangle = $\frac{1}{2} \times 4.5^2 \times \sin(54.532...)$ ***(M1)***

Note: Award ***M1*** for correct substitution into formula.

$= 8.24621... \approx 8.25 \text{ m}^2$ ***(A1)***

OR

$AB = 2 \times \sqrt{4.5^2 - 4^2} = 4.1231\ldots$ ***(M1)***

area triangle $= \frac{4.1231\ldots \times 4}{2}$

$= 8.24621... \approx 8.25\ (\text{m}^2)$ ***(A1)***

finding area of sector

EITHER

area of sector = $\frac{54.532...}{360} \times \pi \times 4.5^2$ ***(M1)***

$= 9.63661... \approx 9.64 \text{ m}^2$ ***(A1)***

OR

area of sector = $\frac{1}{2} \times 0.9517641.. \times 4.5^2$ ***(M1)***

$= 9.63661... \approx 9.64 \text{ m}^2$ ***(A1)***

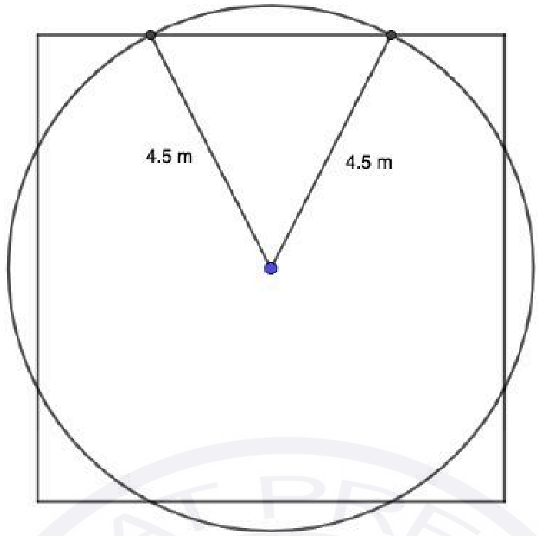
THEN

area of segment = $9.63661... - 8.24621...$

$= 1.39 \text{ m}^2$ $(1.39040...)$ ***A1***

[8 marks]

(b) **METHOD 1**



$\pi \times 4.5^2$ (63.6172…) *(A1)*

$4 \times 1.39040...$ (5.56160) *(A1)*

subtraction of four segments from area of circle *(M1)*

$= 58.1 \text{ m}^2$ (58.055...) ***A1***

METHOD 2

angle of sector = $90 - 54.532... \left(\frac{\pi}{2} - 0.951764...\right)$ *(A1)*

area of sector = $\frac{90 - 54.532...}{360} \times \pi \times 4.5^2$ $(= 6.26771...)$ *(A1)*

area is made up of four triangles and four sectors *(M1)*

total area = $(4 \times 8.2462...) + (4 \times 6.26771...)$

$= 58.1 \text{ m}^2$ (58.055...) ***A1***

[4 marks]

(c) sketch of $\frac{dV}{dt}$ **OR** $\frac{dV}{dt} = 0.110363...$ **OR** attempt to find where $\frac{d^2V}{dt^2} = 0$ *(M1)*

$t = 1$ hour ***A1***

[2 marks]

(d) recognizing $V = \int \frac{dV}{dt}\, dt$ *(M1)*

$\int_0^8 0.3te^{-t} dt$ *(A1)*

volume eaten is $0.299... \text{ m}^3$ (0.299094...) ***A1***

[3 marks]
[Total 17 marks]

Question13

(a) differentiating first equation. *M1*

$$\frac{d^2x}{dt^2} = \frac{dy}{dt}$$

substituting in for $\frac{dy}{dt}$ *M1*

$$= -2x - 3y = -2x - 3\frac{dx}{dt}$$

therefore $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$ *AG*

Note: The ***AG*** line must be seen to award the final ***M1*** mark.

[2 marks]

(b) the relevant matrix is $\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ ***(M1)***

Note: $\begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$ is also possible.

(this has characteristic equation) $-\lambda(-3-\lambda)+2=0$ ***(A1)***

$\lambda = -1, -2$ ***A1***

[3 marks]

(c) **EITHER**

the general solution is $x = Ae^{-t} + Be^{-2t}$ ***M1***

Note: Must have constants, but condone sign error for the ***M1***.

so $\frac{dx}{dt} = -Ae^{-t} - 2Be^{-2t}$ ***M1A1***

OR

attempt to find eigenvectors ***(M1)***

respective eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (or any multiple)

$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ ***(M1)A1***

THEN
the initial conditions become:

$0 = A + B$

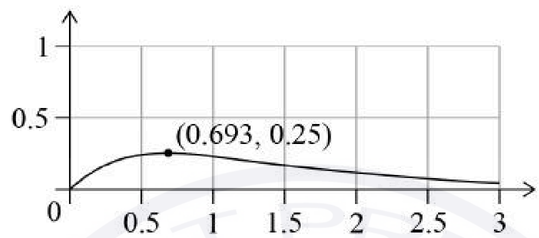
$1 = -A - 2B$ **M1**

this is solved by $A = 1,\ B = -1$

so the solution is $x = e^{-t} - e^{-2t}$ **A1**

[5 marks]

(d)

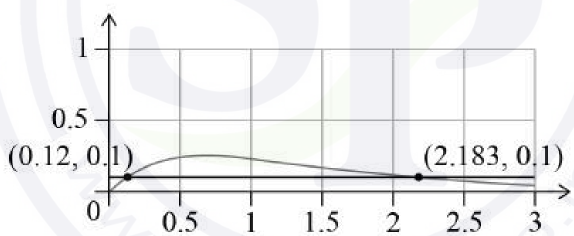


A1A1

> **Note:** Award ***A1*** for correct shape (needs to go through origin, have asymptote at $y = 0$ and a single maximum; condone $x < 0$). Award ***A1*** for correct coordinates of maximum.

[2 marks]

(e) intersecting graph with $y = 0.1$ ***(M1)***



so the time fishing is stopped between $2.1830\ldots$ and $0.11957\ldots$ ***(A1)***

$= 2.06(343\ldots)$ days ***A1***

[3 marks]

(f) *Any reasonable answer. For example:*
There are greater downsides to allowing fishing when the levels may be dangerous than preventing fishing when the levels are safe.
The concentration of mercury may not be uniform across the river due to natural variation / randomness.
The situation at the power plant might get worse.
Mercury levels are low in water but still may be high in fish. ***R1***

> **Note:** Award ***R1*** for a reasonable answer that refers to this specific context (and not a generic response that could apply to *any* model).

[1 mark]
Total [16 marks]

Question 14

(a) attempt to use chain rule, including the differentiation of $\frac{1}{T}$ **(M1)**

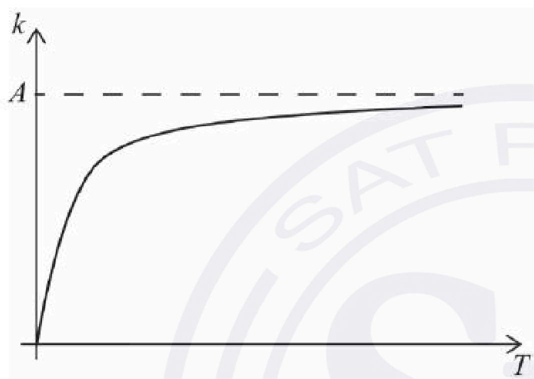
$\frac{\mathrm{d}k}{\mathrm{d}T} = A \times \frac{c}{T^2} \times \mathrm{e}^{-\frac{c}{T}}$ **A1**

this is the product of positive quantities so must be positive **R1**

Note: The ***R1*** may be awarded for correct argument from **their** derivative. ***R1*** is not possible if their derivative is not always positive.

[3 marks]

(b)



A1A1A1

Note: Award ***A1*** for an increasing graph, entirely in first quadrant, becoming concave down for larger values of T, ***A1*** for tending towards the origin and ***A1*** for asymptote labelled at $k = A$.

[3 marks]

(c) taking ln of both sides **OR** substituting $y = \ln x$ and $x = \frac{1}{T}$ ***(M1)***

$\ln k = \ln A - \frac{c}{T}$ **OR** $y = -cx + \ln A$ ***(A1)***

(i) so gradient is $-c$ ***A1***

(ii) y-intercept is $\ln A$ ***A1***

Note: The implied ***(M1)*** and ***(A1)*** can only be awarded if **both** correct answers are seen. Award zero if only one value is correct **and** no working is seen.

[4 marks]

(d) an attempt to convert data to $\frac{1}{T}$ and $\ln k$ *(M1)*

e.g. at least one correct row in the following table

$\frac{1}{T}$	$\ln k$
$1.69491\ldots\times10^{-3}$	$-7.60090\ldots$
$1.66666\ldots\times10^{-3}$	$-7.41858\ldots$
$1.63934\ldots\times10^{-3}$	$-6.90775\ldots$
$1.61290\ldots\times10^{-3}$	$-6.57128\ldots$
$1.58730\ldots\times10^{-3}$	$-6.21460\ldots$
1.5625×10^{-3}	$-5.84304\ldots$
$1.53846\ldots\times10^{-3}$	$-5.62682\ldots$

line is $\ln k = -13400\times\frac{1}{T}+15.0$ $\left(=-13383.1\ldots\times\frac{1}{T}+15.0107\ldots\right)$ ***A1***

[2 marks]

(e) (i) $c = 13400$ $(13383.1\ldots)$ ***A1***

(ii) attempt to rearrange or solve graphically $\ln A = 15.0107\ldots$ ***(M1)***

$A = 3300000$ $(3304258\ldots)$ ***A1***

Note: Accept an A value of $3269017\ldots$ from use of 3sf value.

[3 marks]

Total [15 marks]

Question 15

(a) attempt to expand given expression **OR** attempt at product rule ***(M1)***

$C = \frac{xk^2}{10} - \frac{3x^3}{1000}$

$\frac{dC}{dx} = \frac{k^2}{10} - \frac{9x^2}{1000}$ ***M1A1***

Note:	Award ***M1*** for power rule correctly applied to at least one term and ***A1*** for correct answer.

[3 marks]

(b) equating their $\frac{dC}{dx}$ to zero ***(M1)***

$\frac{k^2}{10} - \frac{9x^2}{1000} = 0$

$x^2 = \frac{100k^2}{9}$

$x = \frac{10k}{3}$ ***(A1)***

substituting their x back into given expression ***(M1)***

$C_{max} = \frac{10k}{30}\left(k^2 - \frac{300k^2}{900}\right)$

$C_{max} = \frac{2k^3}{9}\ (0.222\ldots k^3)$ ***A1***

[4 marks]

(c) (i) substituting 20 into given expression and equating to 426 ***M1***

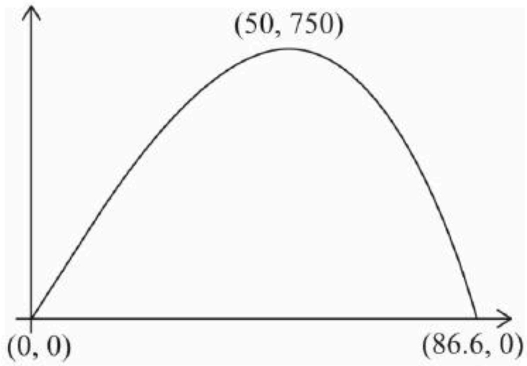
$426 = \frac{20}{10}\left(k^2 - \frac{3}{100}(20)^2\right)$

$k = 15$ ***A1***

(ii) 50 ***A1***

[3 marks]

(d)



A1A1A1

Note: Award ***A1*** for graph indicating an increasing and then decreasing function (drawn in first quadrant), ***A1*** for maximum labelled and ***A1*** for graph drawn for positive x, passing through the origin and 86.6 which is marked on the x-axis or its coordinates are given.

[3 marks]

(e) setting their expression for C to zero **OR** choosing correct x-intercept on their graph of C ***(M1)***

$x_{\max} = 86.6 \quad (86.6025\ldots)$ litres ***A1***

[2 marks]

Total [15 marks]

Question 16

(a) (i) use of chain rule *(M1)*

$v = -9\sin(3t)i + 12\cos(3t)j$ **A1**

Note:	Award ***(M1)*** for at least one correct term seen but condone omission of i or j.

(ii) $|v| = \sqrt{(-9\sin(9))^2 + (12\cos(9))^2}$ *(M1)*

$= 11.5\ \text{m s}^{-1}\ (11.5455\ldots)$ **A1**

[4 marks]

(b) (i) $a = -27\cos(3t)i - 36\sin(3t)j$ **A1**

(ii) $a = -9(3\cos(3t)i - 4\sin(3t)j)$ **M1**

$a = -9r$ (where r is a position vector from the origin) **A1**

a is in opposite direction to the position vector **R1**

hence a is always directed towards the origin **AG**

[4 marks]

(c) relative position $d = r_2 - r_1$ *(M1)*

distance between particles $= |d|\ (= |r_2 - r_1|)$ *(M1)*

$|d| = \sqrt{(-4\sin(4t) - 3\cos(3t))^2 + (3\cos(4t) - 4\sin(3t))^2}$ *(A1)*

minimum value of $|d|$ when $t = 4.71\text{(s)}\ \left(4.71238\ldots, \frac{3\pi}{2}\right)$ *(M1)***A1**

[5 marks]

(d) (i) for 2nd particle, $v = -16\cos(4t)i - 12\sin(4t)j$ *(A1)*

EITHER

consider the gradient of either v *(M1)*

$m_1 = -\frac{12\cos(3t)}{9\sin(3t)}$ and $m_2 = \frac{12\sin(4t)}{16\cos(4t)}$ *(A1)*

attempt to solve $m_1 = m_2$ *(M1)*

OR

vectors are parallel therefore one is a multiple of the other, $v_2 = l\,v_1$ *(M1)*

$(l =)\ \frac{16\cos(4t)}{9\sin(3t)} = -\frac{\sin(4t)}{\cos(3t)}$ *(A1)*

attempt to solve *(M1)*

THEN

$t = 1.30\ \text{s}\ (1.30135\ldots)$ **A1**

(ii) **EITHER**

at $t = 1.30$, $\boldsymbol{v}_1 = 6.22\boldsymbol{i} - 8.68\boldsymbol{j}$ and $\boldsymbol{v}_2 = -7.57\boldsymbol{i} + 10.6\boldsymbol{j}$ ***A1***

OR

$l = -1.22$ (following second method in part (d)(i)) ***A1***

THEN

$\boldsymbol{v}_2$ is a negative multiple of $\boldsymbol{v}_1$ $(\boldsymbol{v}_2 = -1.22\boldsymbol{v}_1)$ ***R1***

the two particles are moving in the opposite direction ***AG***

[7 marks]

Total [20 marks]

Question 17

(a) $\frac{1}{2}x^3 + 1 = (x-1)^4$ ***(M1)***

$(p =)$ 2.91 cm (2.91082...) ***A1***

[2 marks]

(b) attempt to make x (or x^2) the subject of $y = \frac{1}{2}x^3 + 1$ ***(M1)***

$x = \sqrt[3]{2(y-1)}$ (or $x^2 = (2(y-1))^{\frac{2}{3}}$) ***(A1)***

(upper limit $=$) $13.3(315\ldots)$ ***(A1)***

$V = \int_1^{13.3315\ldots} \pi(2(y-1))^{\frac{2}{3}}\, dy$ ***(M1)***

Note:	Award ***(M1)*** for setting up correct integral squaring their expression for x with both correct lower limit and their upper limit, and π. Condone omission of dy.

$= 197 \text{ cm}^3$ (196.946...) ***A1***

[5 marks]

(c) $x = y^{\frac{1}{4}} + 1$ (or $x^2 = \left(y^{\frac{1}{4}} + 1\right)^2$) ***(A1)***

$V_2 = \int_0^{13.3315\ldots} \pi(y^{\frac{1}{4}} + 1)^2\, dy$ ***(M1)(A1)***

Note:	Award ***(M1)*** for setting up correct integral squaring their expression for x with their upper limit, and π. Award ***(A1)*** for lower limit of 0, dependent on ***M1***. Condone omission of dy. If a candidate found an area in part (b), do not award ***FT*** for another area calculation seen in part (c).

$= 271.87668\ldots$ ***(A1)***

Note:	Accept $271.038\ldots$ from use of 3sf in the upper limit.

subtracting their volumes ***(M1)***

$271.87668\ldots - 196.946\ldots$

$= 74.9 \text{ cm}^3$ $(74.93033\ldots)$ ***A1***

Note:	Accept any answer that rounds to $75\ (\text{cm}^3)$. If a candidate found an area in part (b), do not award ***FT*** for another area calculation seen in part (c).

[6 marks]

[13 marks]

Question 18

(a) (i) $f'(x) = \frac{-2x}{50} + 2 \quad \left(= \frac{-x}{25} + 2,\ -0.04x + 2 \right)$ ***A1A1***

Note: Award ***A1*** for each correct term. Award at most ***A0A1*** if extra terms are seen.

(ii) $0 = \frac{-x}{25} + 2$ **OR** sketch of $f'(x)$ with x-intercept indicated ***M1***

$x = 50$ ***A1***

$y = 80$ ***A1***

$(50,\ 80)$

Note: Award ***M0A0A1*** for the coordinate $(50,\ 80)$ seen either with no working or found from a graph of $f(x)$.

[5 marks]

(b) (i) $\int_0^{70} \frac{-x^2}{50} + 2x + 30\ \mathrm{d}x$ ***A1A1***

Note: Award ***A1*** for a correct integral, ***A1*** for correct limits in the correct location. Award at most ***A0A1*** if $\mathrm{d}x$ is omitted.

(ii) (Area =) $4710\ \mathrm{m}^2 \quad \left(4713.33\ldots,\ \frac{14140}{3}\right)$ ***A2***

[4 marks]

(c) (i) $\frac{11.4}{4713.33\ldots} \times 100\%$ **OR** $\left|\frac{4701.93\ldots - 4713.33\ldots}{4713.33\ldots}\right| \times 100\%$ ***(M1)***

Note: Award ***(M1)*** for their correct substitution into the percentage error formula.

$0.242\% \quad (0.241867\ldots\%)$ ***A1***

Note: Percentage sign is required. Accept $0.242038\ldots\%$ if 4710 is used.

(ii) **EITHER**

reduce the width of the intervals (trapezoids) ***A1***

OR

increase the number of intervals (trapezoids) ***A1***

Note: Accept equivalent statements. Award ***A0*** for the ambiguous answer "increase the intervals".

[3 marks]

(d) (i) width of the square is $70-x$ **OR** the length of the square is $\frac{-x^2}{50}+2x+30$

(M1)

Note: Award ***(M1)*** for $70-x$ seen anywhere. Accept $\frac{-x^2}{50}+2x+30$ but only if this expression is explicitly identified as a dimension of the square.

in term of x, equating the length to the width ED ***(M1)***

$$\frac{-x^2}{50}+2x+30=70-x$$

$(x=14.7920\ldots$ or $135.21)$

$(x=)\ 14.8\text{ m}\ (14.7920\ldots)$ ***A1***

Note: Award ***M0M0A0*** for an unsupported answer of 15. Award at most ***M1M0A0*** for an approach which leads to $A'(x)=0$. This will lead to a square base which extends beyond the east boundary of the property. Similar for any solution where F is not on the northern boundary, or GH is not on the east boundary.

(ii) **EITHER**

$(70-14.7920\ldots)^2$ ***(M1)***

OR

$(55.2079\ldots)^2$ ***(M1)***

OR

$$\left(\frac{-(14.7920\ldots)^2}{50}+2(14.7920\ldots)+30\right)^2$$ ***(M1)***

THEN

$(\text{Area}=)\ 3050\text{ m}^2\ (3047.92\ldots)$ ***A1***

Note: Follow through from part (d)(i), provided x is between 0 and 70. Award at most ***M1A0*** if their answer is outside the range of their $[0, 4713.33\ldots]$ from part (b).

[5 marks]
Total [17 marks]

Question 19

(a) $\begin{vmatrix} -4-\lambda & 6 \\ 9 & -1-\lambda \end{vmatrix} = 0$ ***(M1)***

Note: Do not accept $\det(\boldsymbol{A}-\lambda\boldsymbol{I})=0$ or similar as evidence of a correct method unless A is explicitly defined to be the given matrix.

$(-4-\lambda)(-1-\lambda)-54=0$

$\lambda=-10,\ \lambda=5$ ***A1A1***

For $\lambda=-10$

$\begin{pmatrix} -4 & 6 \\ 9 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -10x \\ -10y \end{pmatrix}$ ***(M1)***

$-4x+6y=-10x$

$x+y=0$

possible eigenvector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (or equivalent) ***A1***

for $\lambda=5$

$\begin{pmatrix} -4 & 6 \\ 9 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 5x \\ 5y \end{pmatrix}$

$-4x+6y=5x$

$3x=2y$

possible eigenvector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (or equivalent) ***A1***

Note: If both eigenvalues are incorrect then award at most ***M1A0A0M1A0A0***.

[6 marks]

(b) attempt to substitute their eigenvalues and eigenvectors equation ***(M1)***

$\begin{pmatrix} x \\ y \end{pmatrix}=Ae^{-10t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}+Be^{5t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ ***A1***

Note: Award at most ***(M1)A0*** if $\begin{pmatrix} x \\ y \end{pmatrix}$ not seen.

[2 marks]

(c) At $t = 0$, $x = 500$ and $y = 125$

$x = -A + 2B$ and $y = A + 3B$

Solving simultaneously: ***(M1)***

$A = -250$ and $B = 125$ ***A1***

$$\left(\begin{pmatrix} x \\ y \end{pmatrix} = -250e^{-10t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 125e^{5t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)$$

Note: Follow through from their eigenvectors.
Accept equivalent values for A and B based on the direction of their eigenvectors and the order of their eigenvalues in the equation.

[2 marks]

(d) $2:3$ ***A1***

[1 mark]

(e) attempt to eliminate $\mathrm{d}t$ from the two differential equations ***M1***

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{9x - y}{-4x + 6y}$$

substituting initial conditions ***(M1)***

$$= \frac{9(500) - 125}{-4(500) + 6(125)}$$

$= -3.5$ ***A1***

Note: Award ***M1*** for $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4x + 6y}{9x - y}$.

[3 marks]

(f) trajectory or trajectories that are consistent with their eigenvalues **A1**

a trajectory that passes through the point $(500, 125)$ with gradient that is consistent with the response to part (e) **A1**

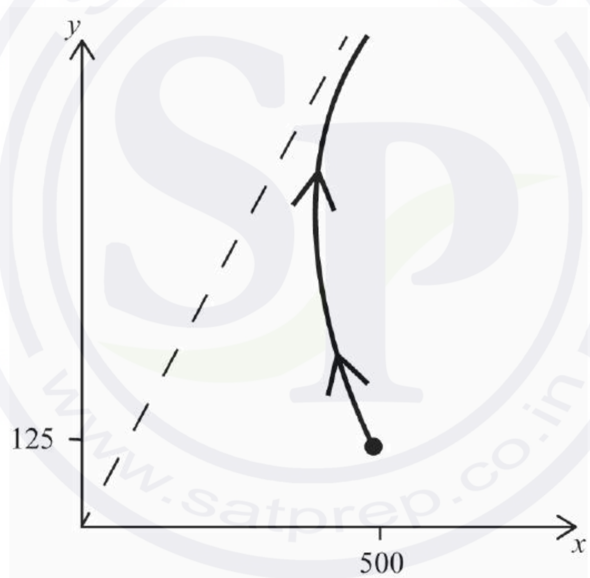
the diagram contains at least one of their eigenvectors **A1**

(e.g. labelled $y = 1.5x$; $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\lambda = 5$ etc.)

the trajectory that passes through $(125, 500)$ tends towards an oblique asymptote that corresponds to their eigenvector and the direction is indicated by at least one arrow on the trajectory **A1**

Note: For the second ***A1***, the point $(500, 125)$ may not be labelled but there should be a point marked on the trajectory that is consistent with these coordinates.

The final ***A1*** will depend on their eigenvalues. Follow through can be awarded as long as the direction of the trajectory is consistent with the nature of their eigenvalues and eigenvectors.



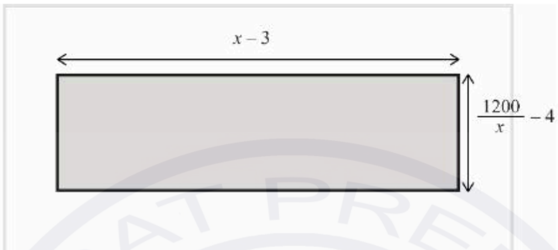
[4 marks]
Total [18 marks]

Question 20

(a)

Note:	In methods 1 and 2, full marks are available for candidates who work with a dummy variable, e.g. y, that represents the width of the park and hence is equal to $\frac{1200}{x}$. The substitution to express an answer in only x may come as late as the final line.

METHOD 1 (finding dimensions of garden)



(width of park =) $\frac{1200}{x}$ ***(A1)***

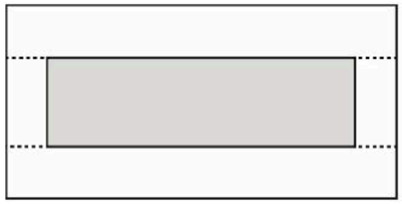
(length of garden =) $x-3$, (width of garden =) $\frac{1200}{x}-4$ ***(A1)(A1)***

$A=(x-3)\times\left(\frac{1200}{x}-4\right)$ ***A1***

$=1200-4x-\frac{3600}{x}+12$ ***A1***

$=1212-4x-\frac{3600}{x}$ ***AG***

METHOD 2 (subtracting the area of the path)



width of park = $\frac{1200}{x}$ ***(A1)***

attempt to cut path into 4 (or 8) pieces ***(M1)***

four (or eight) areas of the path expressed in terms of x ***(A1)***

$A = 1200 - 2x - 2x - 1.5\left(\frac{1200}{x} - 4\right) - 1.5\left(\frac{1200}{x} - 4\right)$ ***A1***

correct manipulation leading to given result ***A1***

$= 1212 - 4x - \frac{1800}{x} - \frac{1800}{x}$

$= 1212 - 4x - \frac{3600}{x}$ ***AG***

Note: To award ***(M1)(A1)*** without a diagram the division of the park must be clear.

[5 marks]

(b) setting $1212 - 4x - \frac{3600}{x} = 800$ (accept a sketch) ***(M1)***

$x = 9.64\ (9.64011\ldots)\ (\text{m})$ **OR** $x = 93.4\ (93.3598\ldots)\ (\text{m})$ ***A1***

(width $=$) $124\ (124.479\ldots)\ (\text{m})$ ***A1***

(width $=$) $12.9\ (12.8534\ldots)\ (\text{m})$ ***A1***

Note: To award the final ***A1*** both values of x **and** both values of the width must be seen. Accept 12.8 for second value of width from candidate dividing 1200 by 3 sf value of 93.4.

[4 marks]

(c) $\left(\frac{\mathrm{d}A}{\mathrm{d}x} =\right) -4 + \frac{3600}{x^2}$ **OR** $-4 + 3600x^{-2}$ ***A1A1A1***

Note: Award ***A1*** for -4, ***A1*** for $+3600$, and ***A1*** for x^{-2} or x^2 in denominator.

[3 marks]

(d) setting *their* $\frac{dA}{dx}$ equal to 0 **OR** sketch of *their* $\frac{dA}{dx}$ with x-intercept highlighted ***M1***

$(x=)\ 30\ (\text{m})$ ***A1***

Note:	To award ***A1FT*** the candidate's value of x must be within the domain given in the problem $(3 < x < 300)$.

[2 marks]

(e) **EITHER**

evidence of using GDC to find maximum of graph of $A = 1212 - 4x - \frac{3600}{x}$ ***(M1)***

OR

substitution of *their* x into A ***(M1)***

OR

dividing 1200 by *their* x to find width of park **and** subtracting 3 from *their* x and 4 from the width to find park dimensions ***(M1)***

Note:	For the last two methods, only follow through if $3 <$ their $x < 300$.

THEN

$(A=)\ 972\ (\text{m}^2)$ ***A1***

[2 marks]

Total [16 marks]

Question 21

(a) attempt to solve $\det(A-\lambda I)=0$ *(M1)*

$(-0.05-\lambda)^2+25=0$ *(A1)*

$-0.05-\lambda=\pm 5\mathrm{i}$ *(A1)*

$\lambda=-0.05\pm 5\mathrm{i}$ ***A1***

[4 marks]

(b) (i) spiral ***A1***

(ii) inwards / towards O ***A1***

[2 marks]

(c) (i) attempt to substitute $(20,\ 0)$ into expression for $\frac{\mathrm{d}y}{\mathrm{d}t}$ *(M1)*

$-5(20)-0.05(0)$

$\frac{\mathrm{d}y}{\mathrm{d}t}=-100\ (\mathrm{ms}^{-1})$ ***A1***

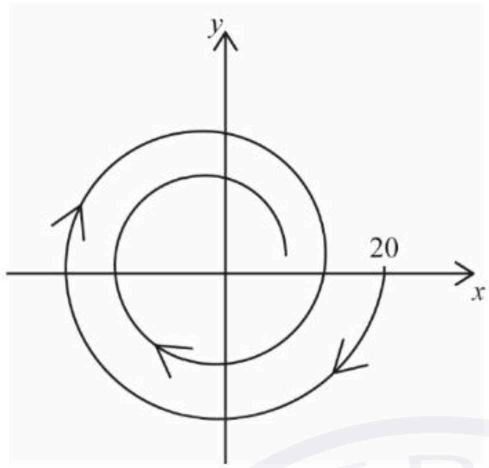
(ii) $\frac{\mathrm{d}x}{\mathrm{d}t}=-1$ *(A1)*

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t}$ **OR** $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}$ *(M1)*

$(=-100\div-1=)\ 100$ ***A1***

[5 marks]

(d)



A4

Note: Award ***A1*** for starting at $(20,\ 0)$, ***A1*** for spiral inwards, ***A1*** for clockwise, ***A1*** for non-negative gradient at $(20,\ 0)$.

[4 marks]

[Total: 15 marks]

Question 22

(a) (i) $h(0) = 0.00623$ (km) $(= 0.00622517)$ ***A1***

(ii) this is the height of the nose of the plane (above the runway), when the plane is on the runway ***A1***

[2 marks]

(b) (i) $y = 9.94$ ***A1***

Note: Accept $h = 9.94$.

(ii) **EITHER**

this is the height that the (nose of the) plane approaches (but does not reach) ***A1***

OR

this is the maximum possible height of the (nose of the) plane ***A1***

OR

the (nose of the) plane does not exceed this height ***A1***

[2 marks]

(c) **METHOD 1 (chain rule)**

$h(x) = 10\left(1+150\mathrm{e}^{-0.07x}\right)^{-1} - 0.06$ ***(M1)***

find $h'(x) = -10\left(1+150\mathrm{e}^{-0.07x}\right)^{-2} \times 150\mathrm{e}^{-0.07x} \times -0.07$ ***A1M1A1***

$\left(= \dfrac{105\mathrm{e}^{-0.07x}}{(1+150\mathrm{e}^{-0.07x})^2}\right)$

> **Note:** Award ***A1*** for correct first term $\left(-10\left(1+150\mathrm{e}^{-0.07x}\right)^{-2}\right)$, ***M1*** for attempt to use the chain rule, ***A1*** for correct use of chain rule $\left(\times 150\mathrm{e}^{-0.07x} \times -0.07\right)$. Award at most ***A1M1A0*** if additional terms are seen. The answer is not required to be simplified beyond what is shown in the markscheme.

METHOD 2 (quotient rule)

$$\frac{\left(1+150\,\mathrm{e}^{-0.007x}\right)(0) - 10\left(150\mathrm{e}^{-0.007x} \times -0.007\right)}{\left(1+150\,\mathrm{e}^{-0.007x}\right)^2}$$ ***M1A1***

> **Note:** Award ***M1*** for attempt to use quotient rule, ***A1*** for correct use.

$$= \frac{-10\left(150\mathrm{e}^{-0.007x} \times -0.007\right)}{\left(1+150\,\mathrm{e}^{-0.007x}\right)^2} \quad \left(= \frac{105\mathrm{e}^{-0.07x}}{(1+150\mathrm{e}^{-0.07x})^2}\right)$$ ***A1A1***

> **Note:** Award ***A1*** for correct numerator and ***A1*** for correct denominator.

[4 marks]

(d) evidence of a graph of $h'(x)$ ***(M1)***

maximum at $x = 71.6 \quad (= 71.58051...)$ ***(A1)***

$h'(71.58051...) = 0.175$ ***A1***

maximum gradient is less than 0.2
and hence the regulation is being followed ***A1***

[4 marks]
[Total 12 marks]

Question 23

(a) (i) correct approach to find missing length *(A1)*

$\sqrt{4^2 - 1^2} \quad (= \sqrt{15})$

attempt to find cross-section *(M1)*

e.g. use of area of trapezoid formula or rectangle+triangle or rectangle – triangle

use of volume of prism formula *(M1)*

(their cross-section multiplied by 3)

$3\left[\frac{1}{2}(10+11)\left(\sqrt{4^2-1^2}\right)\right]$

$= 122\,(\text{m}^3)\ \ (121.998\ldots)$ ***A1***

(ii) correct approach to find missing height *(A1)*

$\sqrt{4^2 - 3.2^2} \quad (= 2.4)$

attempt to find volume *(M1)*

(multiplication by 3.2 and 3 seen)

$3\left[\frac{1}{2}\left(10+10+\sqrt{4^2-3.2^2}\right)(3.2)\right]$

$= 108\,(\text{m}^3)\ \ (107.52\ldots)$ ***A1***

(iii) correct approach to find missing lengths *(A1)*

$\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{3}\right)$ **OR** $\sin\left(\frac{\pi}{3}\right)$ and Pythagoras etc seen in work

$3\left[\frac{1}{2}\left(10+10+4\cos\left(\frac{\pi}{3}\right)\right)4\sin\left(\frac{\pi}{3}\right)\right]$

$= 114\,(\text{m}^3)\ \ (114.315\ldots)$ ***A1***

[9 marks]

(b) $V = 3\left[\frac{1}{2}(10+10+4\cos(\theta))4\sin(\theta)\right]$ ***A1***

all correct intermediate working leading to given answer ***A1***

e.g. $V = 6\sin(\theta)(20+4\cos(\theta))$

$V = 24\sin(\theta)(5+\cos(\theta))$ ***AG***

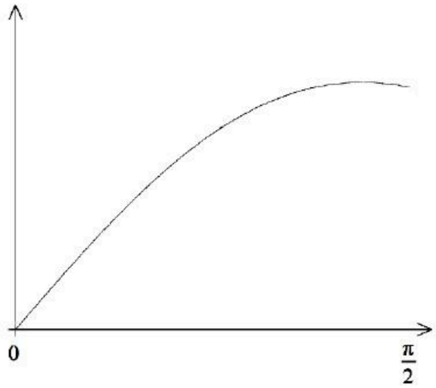
Note: The ***AG*** line must be seen for the final ***A1*** to be awarded.

[2 marks]

(c) *accept any reasoning along the lines:* "skip would have zero volume" or "if the angle is zero, then the contents would fall out" ***R1***

[1 mark]

(d) (i)



A1A1

Note: Award ***A1*** for the correct shape and ***A1*** for the graph on the correct, labelled, domain. Condone omission of θ / V labels (or x/y).

(ii) $\theta = 1.38$ $(1.38356...)$ $(79.3^\circ$ $(79.2723...^\circ))$ ***A1***

$V_{\max} = 122$ $(122.292...)$ ***A1***

Note: Award ***A0A1*** if values are reversed and ***A0A0*** for a coordinate pair.

[4 marks]

(e) recognizing that derivative is equal to zero (seen at any stage) ***M1***

$\frac{dV}{d\theta} = 0$ (accept $\frac{dy}{dx} = 0$)

(from graph, turning point is a global maximum)

use of product rule ***M1***

$\left(\frac{dV}{d\theta} =\right) 24\cos(\theta)(5+\cos(\theta)) + 24\sin(\theta)(-\sin(\theta))$ ***A1***

$= 120\cos(\theta) + 24\cos^2(\theta) - 24\sin^2(\theta)$ $(=0)$ (or equivalent) ***A1***

substituting $1-\cos^2(\theta)$ for $\sin^2(\theta)$ ***M1***

e.g $120\cos(\theta) + 24\cos^2(\theta) - 24(1-\cos^2(\theta))$ $(=0)$

correct intermediate steps leading to given answer ***A1***

$2\cos^2(\theta) + 5\cos(\theta) - 1 = 0$ ***AG***

[6 marks]
[Total: 22 marks]

Question 23

(a) heights, $0, 4, 1.75, 3$ and 3.75 seen *(A2)*

Note: Award ***A1A0*** if **two** of 1.75, 3 or 3.75 are seen.

attempt to use trapezoidal rule formula for their heights *(M1)*

$\frac{1}{2}\times 1\times\{0+4+2(1.75+3+3.75)\}$ *(A1)*

Note: Award ***(M1)(A1)*** for correctly expressing this as 3 trapezoids and a triangle. The "$\times 1$" need not be seen.

$=10.5\ (\text{m}^2)$ ***A1***

[5 marks]

(b) $-\frac{1}{12}x^3+x^2+c$ ***A1A1A1***

[3 marks]

(c) $\int_0^4\left(-\frac{1}{4}x^2+2x\right)dx+1\times 4+\frac{1}{2}\times 7\times 4$ ***(A1)(M1)(A1)***

Note: Award ***A1*** for correct area of rectangle **OR** triangle, ***M1*** for substituting correct limits into given integral (may be seen in part (b)), and ***A1*** for entire expression correct.

$=10.6666\ldots + 4 + 14$

$= 28\frac{2}{3}\ (\text{m}^2)\ \left(\frac{86}{3}\right)$ ***A1***

Note: The answer must be **exact** for the ***A1*** to be awarded. For an answer of 28.7 or 28.66 award ***(A1)(M1)(A1)A0***.

[4 marks]

(d) (Total area using part (a) =) 28.5 ***(A1)***

Percentage error $=\left|\frac{28.5-28.6666\ldots}{28.6666\ldots}\right|\times 100$ ***(M1)***

Note: if their trapezoid value is incorrect but is used correctly in the percentage error formula, award at most ***A0M1A0***. If it is clear from the answer that ×100 has been used, then condone the omission and award the ***M*** mark.

$=0.581\ (\%)\ (0.581395\ldots)$ ***A1***

(accept 0.697 from use of 28.7)

[3 marks]

[Total: 15 marks]

Question 24

(a) attempt to use $V = \pi\int x^2\,dy$ **(M1)**

$x^2 = 2y + 2$ or any reasonable attempt to find x in terms of y **(M1)**

$V = \pi\int_0^h 2y + 2\,dy$ **(A1)**

:Correct limits must be seen for the ***A1*** to be awarded however the dy may be omitted (as not a final answer). If this is given as the final answer to this part the remaining marks can be awarded if seen in part (b).

$\int 2y + 2\,dy = y^2 + 2y$ **(A1)**

: Accept equivalent with alternate variable

$V = \pi\left[y^2 + 2y\right]_0^h$

$= \pi(h^2 + 2h)$ ***A1***

: The final two ***A1*** marks can be awarded independently of the first ***A1***. If $h^2 + 2h$ or $y^2 + 2y$ is the final (unsupported) answer award at most ***(M1)(M1)(A0)(A1)A0***.

[5 marks]

(b) volume of vase $= \pi(15^2 + 2\times 15)$ $(= 801.106...)$ **(A1)**

(time to fill vase $= \dfrac{801.106...}{20}$ =) 40.1 $(40.0553...)$ (seconds) ***A1***

e: Accept exact answers in terms of π, e.g. 12.75π or $\dfrac{51\pi}{4}$

[2 marks]

(c) **EITHER**

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$ ***(M1)***

$$\frac{dV}{dh} = \pi(2h + 2)$$ ***(A1)***

OR

differentiating $V = \pi(h^2 + 2h)$ implicitly ***(M1)***

$$\frac{dV}{dt} = \pi(2h + 2)\frac{dh}{dt}$$ ***(A1)***

THEN

$$\frac{dh}{dt} = 20 \times \frac{1}{\pi(2h + 2)}$$ ***(M1)(A1)***

Note: Award ***M1*** for attempting to solve for $\frac{dh}{dt}$, ***A1*** for a correct expression.

substituting $h = 10$ seen anywhere ***(M1)***

$0.289 \quad (0.289372\ldots)\ \text{cm s}^{-1}$ ***A1A1***

Note: Award ***A1*** for the correct value. Award ***A1*** for the correct units, independent of other marks.

[7 marks]
[Total 14 marks]

Question 25

(a) C. **A1**

Any valid reason for accepting C. or rejecting A. and B. **R1**

for example:

- when $x=0$ slopes have (or appear to have) zero gradient
- (slope field is) always positive for $x>0$

: Allow ***A1R0***.

[2 marks]

(b) $\int e^{2y}dy = \int x dx$ ***(M1)***

$\frac{1}{2}e^{2y} = \frac{1}{2}x^2 \ (+c)$ ***(A1)(A1)***

: ***A1*** for left hand side, ***A1*** for right hand side.

substituting in $x=0$, $y=0$ ***(M1)***

$\frac{1}{2}=c$ ***(A1)***

: The substitution may be seen and credited later, however at that point the constant term may be 1.

$e^{2y} = x^2+1$

$y=\frac{1}{2}\ln(x^2+1)$ ***M1A1***

:e: Award ***M1*** for use of log law.

[7 marks]

(c) $\frac{dy}{dx} = \frac{1}{2}\times 2x \times \frac{1}{x^2+1} \left(=\frac{x}{x^2+1}\right)$ ***M1A1***

:e: Award ***M1*** for use of chain rule, or use of implicit differentiation of the penultimate line of the answer to (b).

[2 marks]

(d) substitution of $e^{2y}=x^2+1$ from part (b) into part(c)(i) or original differential equation ***M1***

$\frac{dy}{dx} = \frac{x}{x^2+1} = \frac{x}{e^{2y}}$ ***A1***

and hence $y=\frac{1}{2}\ln(x^2+1)$ is a solution for the differential equation ***AG***

:e: Only award the ***A1*** as follow-through if their $\frac{dy}{dx}$ is of the form $\frac{x}{x^2+c}$.

[2 marks]

[Total 13 marks]

Question 26

(a) $\frac{40000}{x^2} = 400$ ***(M1)***

$x = 10$ (pesos) (since x is positive) ***A1***

[2 marks]

(b) (i) $\left(\frac{40000}{50^2} =\right) 16$ ***A1***

(ii) $(50 \times 16 =)\ 800$ (pesos) ***A1***

[2 marks]

(c) (i) **EITHER**

profit for each smoothie = $x - 20$ ***(M1)***

$P = \frac{40000}{x^2} \times (x - 20)$ ***A1***

OR

profit = revenue – costs = $nx - 20n$ ***(M1)***

$P = x \times \frac{40000}{x^2} - 20 \times \frac{40000}{x^2}$ ***A1***

e: Do not award ***A1*** if $\frac{40000}{x}$ seen as first term unless explained (in part (a) or (b)), as it is given in question.

THEN

$P = \frac{40000}{x} - \frac{800000}{x^2}$ ***AG***

(ii) attempt to express P ready for power rule **(M1)**

$P = 40000x^{-1} - 800000x^{-2}$

$\frac{dP}{dx} = -\frac{40000}{x^2} + \frac{1600000}{x^3}$ **OR** $\frac{dP}{dx} = -40000x^{-2} + 1600000x^{-3}$ ***A1A1***

:The ***(M1)*** can be awarded for either of the correct terms seen.
A1 for each correct term.
At most ***M1A1A0*** if additional terms seen.

(iii) attempt to find x-value ***(M1)***

e.g. sketch of $\frac{dP}{dx}$ with x-intercept indicated **OR** recognition that it occurs at the maximum of P **OR** algebraic approach (requires multiplication by x^3)

$x = 40$ ***A1***

: $\frac{-40000}{x^2} + \frac{1600000}{x^3} = 0$ is insufficient to award ***M1***, this is given in the question. There must be an "attempt to find x-value".
Award ***M1A0*** for a coordinate pair $(40, 500)$.

(iv) attempt to substitue their x-value into equation for n ***(M1)***

$n = \frac{40000}{40^2}$
$= 25$ ***A1***

: Given the nature of the function P, the local maximum is also the global maximum. This is often the case in examinations, but should not always be assumed.

[9 marks]
[Total 13 marks]

Question 27

(a) $\dfrac{dy}{dt}=\dfrac{d^2x}{dt^2}$

A1

$\dfrac{dy}{dt}+ay+bx=0$ **A1**

$\dfrac{dy}{dt}=-bx-ay$ **AG**

[2 marks]

(b) $\begin{pmatrix}0 & 1\\ -77 & -18\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=-7\begin{pmatrix}x\\ y\end{pmatrix}$ **OR** $\begin{pmatrix}0 & 1\\ -77 & -18\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=-11\begin{pmatrix}x\\ y\end{pmatrix}$ ***(M1)***

$y=-7x$

eigenvector is $\begin{pmatrix}1\\ -7\end{pmatrix}$ (or any multiple) ***A1***

$y=-11x$

eigenvector is $\begin{pmatrix}1\\ -11\end{pmatrix}$ (or any multiple) ***A1***

[3 marks]

(c) (i) $\begin{pmatrix}x\\ y\end{pmatrix}=Ae^{-7t}\begin{pmatrix}1\\ -7\end{pmatrix}+Be^{-11t}\begin{pmatrix}1\\ -11\end{pmatrix}$ ***(A1)***

substitution of initial values ***(M1)***

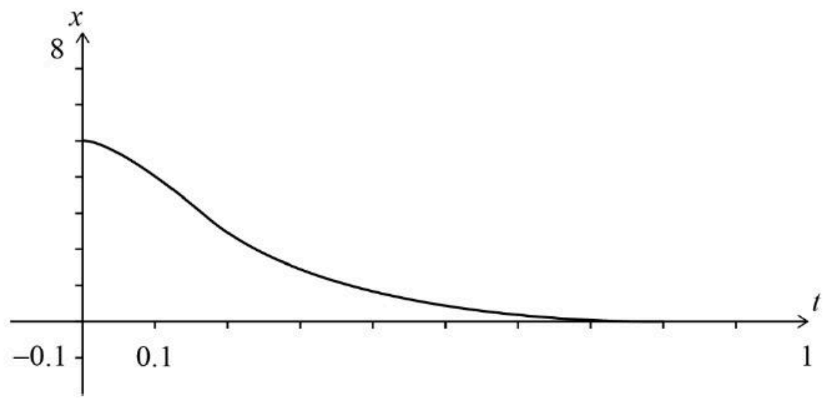
two correct equations (not in vector form) ***(A1)***

$5=A+B,\ 2=-7A-11B$

$A=\dfrac{57}{4},\ B=-\dfrac{37}{4}$ ***(A1)***

$x=\dfrac{57}{4}e^{-7t}-\dfrac{37}{4}e^{-11t}$ ($x=14.25e^{-7t}-9.25e^{-11t}$) ***A1***

(ii)



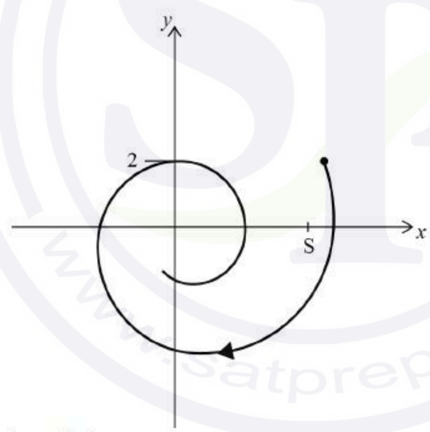
decreasing curve **A1**

Note: There is a maximum at $t = 0.00496$, but this does not need to be shown.

starting at $x=5$ and asymptote at $x = 0$ **A1**

[7 marks]

(d) (i)



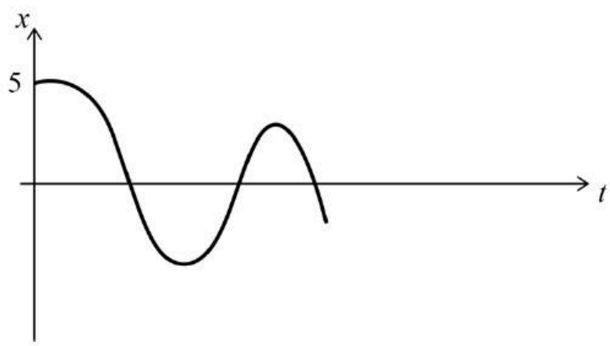
spiral towards origin **A1**

starting at $(5, 2)$ **A1**

$$\frac{dy}{dt} = -85(5) - 18(2) = -461 < 0$$

Clockwise spiral **A1**

(ii)



oscillations **A1**

starting at $x = 5$, decreasing amplitude **A1**

[5 marks]

(e) spring 1 is better as it gives a smoother ride (less oscillations, less bouncy) **R1**

[1 mark]

[Total:18 marks]

Question 28

(a) $\frac{1}{2}\times 1\times((4+0.7)+2(7.3+6.7+4+1.3))$ **(A1)(A1)**

$=21.7\ (21.65)\ (\text{cm}^2)$ **A1**

[3 marks]

(b) $22.1\ (22.08333\ldots,\frac{265}{12})\ (\text{cm}^2)$ **A2**

[2 marks]

(c) $600=\pi r^2 h$ **A1**

[1 mark]

(d) **METHOD 1** (Substitution of $h=\frac{600}{\pi r^2}$ in $A=2\pi r^2+2\pi rh$)

$A=2\pi r^2+2\pi rh$ **(A1)**

attempt to isolate h or πrh **(M1)**

$h=\frac{600}{\pi r^2}$ **OR** $\pi rh=\frac{600}{r}$

correct substitution of their h into correct expression **(M1)**

$A=2\pi r^2+2\pi r\left(\frac{600}{\pi r^2}\right)$ $\left(A=2\pi r^2+\frac{1200}{r}\right)$

$k=1200$ **A1**

METHOD 2 (Equating $2\pi rh$ and $\frac{k}{r}$)

$A=2\pi r^2+2\pi rh$ **OR** $2\pi rh=\frac{k}{r}$ **(A1)**

attempt to isolate h or $(\pi)rh$ or r or r^2 **(M1)**

$h=\frac{600}{\pi r^2}$ **OR** $\pi rh=\frac{600}{r}$ **OR** $r=\sqrt{\frac{600}{\pi h}}$ **OR** $r^2=\frac{600}{\pi h}$

correct substitution of their h or πrh or r or r^2 into $2\pi r^2+2\pi rh=2\pi r^2+\frac{k}{r}$ **(M1)**

$2\pi r\left(\frac{600}{\pi r^2}\right)=\frac{k}{r}$ **OR** $2\left(\frac{600}{r}\right)=\frac{k}{r}$ **OR** $2(600)=k$ **OR** $2\pi\left(\frac{600}{\pi h}\right)h=k$

$k=1200$ **A1**

[4 marks]

(e) (i) $\frac{dA}{dr} = 4\pi r - 1200r^{-2}$ ***A1(M1)A1***

Note: Award ***A1*** for $4\pi r$ seen, and ***(M1)*** for expressing $\frac{1200}{r}$ as $1200r^{-1}$ (can be implied through $\mp\frac{1200}{r^2}$ seen), ***A1*** for $-1200r^{-2}$. Award at most ***A1(M1)A0*** if any additional terms are seen.

(ii) $0 = 4\pi r - 1200r^{-2}$ **OR** $\frac{dA}{dr} = 0$ ***(M1)***

$r = 4.57 \left(4.570781\ldots, \sqrt[3]{\frac{300}{\pi}}\right)$ (cm) ***A1***

Note: Award at most ***M1A0*** if the final answer is in terms of k.

[5 marks]

(f) $(h =)$ 9.14 $(9.14156\ldots..)$ (cm) ***A1***

$(C =)$ 2π $(4.570781\ldots) \approx 28.7$ $(28.7190\ldots)$(cm) ***A1***

EITHER

the longest dimension of the label (9 cm) is less than both values and hence the label will fit (in any rotation) ***R1***

OR

$9 < 9.14$ and $5 < 28.7$ ***R1***

Note: Do not accept an argument based on the comparison of areas.

[3 marks]

[Total 18 marks]