Subject – Math AI(Higher Level)
Topic - Calculus
Year - May 2021 – Nov 2022
Paper -2
Answers

Question 1

(a) $\frac{dv}{dt} = 9.81 - 0.9v$ **M1**

$\int \frac{1}{9.81 - 0.9v} dv = \int 1 dt$ **M1**

$-\frac{1}{0.9}\ln(9.81 - 0.9v) = t + c$ **A1**

$9.81 - 0.9v = Ae^{-0.9t}$ **A1**

$v = \frac{9.81 - Ae^{-0.9t}}{0.9}$ **A1**

when $t = 0$, $v = 0$ hence $A = 9.81$ **A1**

$v = \frac{9.81(1 - e^{-0.9t})}{0.9}$

$v = 10.9(1 - e^{-0.9t})$ **A1**

[7 marks]

(b) **either** let t tend to infinity, or $\frac{dv}{dt} = 0$ ***(M1)***

$v = 10.9$ **A1**

[2 marks]

(c) $\frac{dx}{dt} = y$ **M1**

$\frac{dy}{dt} = 9.81 - 0.9y^2$ **A1**

[2 marks]

(d) $x_{n+1} = x_n + 0.2y_n$, $y_{n+1} = y_n + 0.2(9.81 - 0.9(y_n)^2)$ ***(M1)(A1)***

$x = 1.04$, $\frac{dx}{dt} = 3.31$ ***(M1)A1***

[4 marks]

(e) 3.3015 ***A1***

[1 mark]

(f) $0 = 9.81 - 0.9(v)^2$ ***M1***

$\Rightarrow v = \sqrt{\dfrac{9.81}{0.9}} = 3.301511\ldots\ (= 3.30)$ ***A1***

[2 marks]

(g) the model found the terminal velocity very accurately, so good approximation ***R1***

intermediate values had object exceeding terminal velocity so not good approximation ***R1***

[2 marks]

Total [20 marks]

Question 2

(a) $2(8\times4+3\times4+3\times8)$ **M1**

$=136\ (\text{cm}^2)$ **A1**

[2 marks]

(b) $\sqrt{8^2+4^2+3^2}$ **M1**

$(\text{AG}=)\ 9.43\ (\text{cm})\ (9.4339\ldots,\ \sqrt{89})$ **A1**

[2 marks]

(c) $-2x+220=0$ **M1**

$x=110$ **A1**

110 000 (boxes) **A1**

[3 marks]

(d) $P(x)=\int -2x+220\,\mathrm{d}x$ **M1**

Note:	Award ***M1*** for evidence of integration.

$P(x)=-x^2+220x+c$ **A1A1**

Note:	Award ***A1*** for either $-x^2$ or $220x$ award ***A1*** for both correct terms and constant of integration.

$1700=-(20)^2+220(20)+c$ **M1**

$c=-2300$

$P(x)=-x^2+220x-2300$ **A1**

[5 marks]

(e) $-x^2+220x-2300=0$ **M1**

$x=11.005$ **A1**

11 006 (boxes) **A1**

Note:	Award ***M1*** for their $P(x)=0$, award ***A1*** for their correct solution to x. Award the final ***A1*** for expressing their solution to the minimum number of boxes. Do not accept 11 005, the nearest integer, nor 11 000, the answer expressed to 3 significant figures, as these will not satisfy the demand of the question.

[3 marks]

Total [15 marks]

Question 3

(a) $\int \frac{1}{x} \mathrm{d}x = \int 2 \mathrm{d}t$ *(M1)*

$\ln x = 2t + c$

$x = A\mathrm{e}^{2t}$ *(A1)*

$x(0) = 100 \Rightarrow A = 100$ *(M1)*

$x = 100\mathrm{e}^{2t}$ *(A1)*

$x(1) = 739$ ***A1***

Note: Accept 738 for the final ***A1***.

[5 marks]

(b) $t_{n+1} = t_n + 0.25$ *(A1)*

Note: This may be inferred from a correct t column, where this is seen.

$x_{n+1} = x_n + 0.25x_n(2 - 0.01y_n)$ *(A1)*

$y_{n+1} = y_n + 0.25y_n(0.0002x_n - 0.8)$ *(A1)*

t	x	y
0	1000	100
0.25	1250	85
0.5	1609	73
0.75	2119	65
1	2836	58

(A1)

Note: Award ***A1*** for whole line correct when $t = 0.5$ or $t = 0.75$. The t column may be omitted and implied by the correct x and y values. The formulas are implied by the correct x and y columns.

(i) 2840 (2836 **OR** 2837) ***A1***

(ii) 58 **OR** 59 ***A1***

[6 marks]

(c) (i) both populations are increasing ***A1***

(ii) rabbits are decreasing and foxes are increasing ***A1A1***

[3 marks]

(d) setting at least one DE to zero *(M1)*

$x = 4000$, $y = 200$ ***A1A1***

[3 marks]

Total [17 marks]

Question 4

(a) $\int \frac{1}{x}\,dx = \int 2\,dt$ *(M1)*

$\ln x = 2t + c$

$x = Ae^{2t}$ *(A1)*

$x(0) = 100 \Rightarrow A = 100$ *(M1)*

$x = 100e^{2t}$ *(A1)*

$x(1) = 739$ ***A1***

Note: Accept 738 for the final ***A1***.

[5 marks]

(b) $t_{n+1} = t_n + 0.25$ *(A1)*

Note: This may be inferred from a correct t column, where this is seen.

$x_{n+1} = x_n + 0.25x_n(2 - 0.01y_n)$ *(A1)*

$y_{n+1} = y_n + 0.25y_n(0.0002x_n - 0.8)$ *(A1)*

t	x	y
0	1000	100
0.25	1250	85
0.5	1609	73
0.75	2119	65
1	2836	58

(A1)

Note: Award ***A1*** for whole line correct when $t = 0.5$ or $t = 0.75$. The t column may be omitted and implied by the correct x and y values. The formulas are implied by the correct x and y columns.

(i) 2840 (2836 **OR** 2837) ***A1***

(ii) 58 **OR** 59 ***A1***

[6 marks]

(c) (i) both populations are increasing ***A1***

(ii) rabbits are decreasing and foxes are increasing ***A1A1***

[3 marks]

(d) setting at least one DE to zero *(M1)*

$x = 4000$, $y = 200$ ***A1A1***

[3 marks]

Total [17 marks]

Question 5

(a) (i) evidence of power rule (at least one correct term seen) ***(M1)***

$\frac{dy}{dx} = -0.3x^2 + 1.6x$ ***A1***

(ii) $-0.3x^2 + 1.6x = 0$ ***M1***

$x = 5.33 \left(5.33333..., \frac{16}{3}\right)$ ***A1***

$y = -0.1 \times 5.33333...^3 + 0.8 \times 5.33333...^2$ ***(M1)***

Note: Award ***M1*** for substituting their zero for $\frac{dy}{dx}$ $(5.333\ldots)$ into y.

7.59 m $(7.58519...)$ ***A1***

Note: Award ***M0A0M0A0*** for an unsupported 7.59.
Award at most ***M0A0M1A0*** if only the last two lines in the solution are seen.
Award at most ***M1A0M1A1*** if their $x = 5.33$ is not seen.

[6 marks]

(b) One correct substitution seen ***(M1)***

(i) 6.4 m ***A1***

(ii) 7.2 m ***A1***

[3 marks]

(c) $A=\frac{1}{2}\times 2\left((2.4+0)+2(6.4+7.2)\right)$ ***(A1)(M1)***

Note:	Award ***A1*** for $h=2$ seen. Award ***M1*** for correct substitution into the trapezoidal rule (the zero can be omitted in working).

$=29.6\text{ m}^2$ ***A1***

[3 marks]

(d) (i) $A=\int_2^8 -0.1x^3+0.8x^2\ \mathrm{d}x$ **OR** $A=\int_2^8 y\ \mathrm{d}x$ ***A1A1***

Note:	Award ***A1*** for a correct integral, ***A1*** for correct limits in the correct location. Award at most ***A0A1*** if $\mathrm{d}x$ is omitted.

(ii) $A=32.4\text{ m}^2$ ***A2***

Note:	As per the marking instructions, ***FT*** from their integral in part (d)(i). Award at most ***A1FTA0*** if their area is >48, this is outside the constraints of the question (a $6\text{x}8$ rectangle).

[4 marks]

Total [16 marks]

Question 6

(a) $\begin{vmatrix} -4-\lambda & 0 \\ 3 & -2-\lambda \end{vmatrix} = 0$ ***(M1)***

$(-4-\lambda)(-2-\lambda)=0$ ***(A1)***

$\lambda=-4$ **OR** $\lambda=-2$ ***A1***

$\lambda=-4$

$\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -4x \\ -4y \end{pmatrix}$ ***(M1)***

Note: This ***M1*** can be awarded for attempting to find either eigenvector.

$3x-2y=-4y$

$3x=-2y$

possible eigenvector is $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ (or any real multiple) ***A1***

$\lambda=-2$

$\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -2x \\ -2y \end{pmatrix}$

$x=0,\ y=1$

possible eigenvector is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (or any real multiple) ***A1***

[6 marks]

(b) $\begin{pmatrix} x \\ y \end{pmatrix}=Ae^{-4t}\begin{pmatrix} -2 \\ 3 \end{pmatrix}+Be^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ***(M1)A1***

Note: Award ***M1A1*** for $x=-2Ae^{-4t}$, $y=3Ae^{-4t}+Be^{-2t}$, ***M1A0*** if LHS is missing or incorrect.

[2 marks]

(c) two (distinct) real negative eigenvalues ***R1***

(or equivalent (*eg* both $e^{-4t}\to 0, e^{-2t}\to 0$ as $t\to\infty$))

$\Rightarrow$ stable equilibrium point ***A1***

Note: Do not award ***R0A1***.

[2 marks]

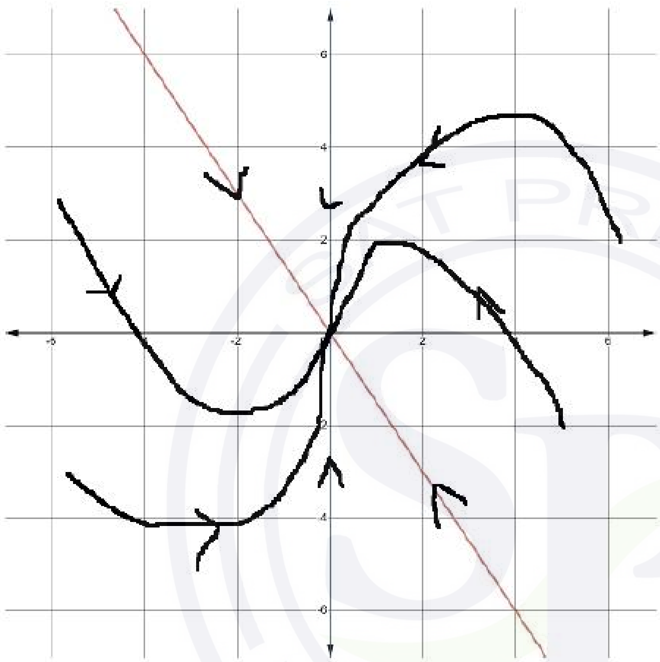
(d) $\frac{dy}{dx} = \frac{3x - 2y}{-4x}$ ***(M1)***

(i) $(4, 0) \Rightarrow \frac{dy}{dx} = -\frac{3}{4}$ ***A1***

(ii) $(-4, 0) \Rightarrow \frac{dy}{dx} = -\frac{3}{4}$ ***A1***

[3 marks]

(e)



A1A1A1A1

Note: Award ***A1*** for a phase plane, with correct axes (condone omission of labels) and at least three non-overlapping trajectories. Award ***A1*** for all trajectories leading to a stable node at $(0, 0)$. Award ***A1*** for showing gradient is negative at $x = 4$ and -4. Award ***A1*** for both eigenvectors on diagram.

[4 marks]

Total [17 marks]

Question 7

(a) solving $v=0$ **M1**

$t=2,\ t=6$ **A1**

[2 marks]

(b) use of power rule ***(M1)***

$\frac{dv}{dt}=-4t+16$ ***(A1)***

$(t=6)$

$\Rightarrow a=-8$ ***(A1)***

magnitude $=8\ \text{ms}^{-2}$ ***A1***

[4 marks]

(c) using a sketch graph of v ***(M1)***

$24\ \text{ms}^{-1}$ ***A1***

[2 marks]

(d) **METHOD ONE**

$x=\int v\ dt$

attempt at integration of v ***(M1)***

$-\frac{2t^3}{3}+8t^2-24t\ (+c)$ ***A1***

attempt to find c (use of $t=0,\ x=0$) ***(M1)***

$c=0$ ***A1***

$\left(x=-\frac{2t^3}{3}+8t^2-24t\right)$

METHOD TWO

$x=\int_0^t v\ dt$

attempt at integration of v ***(M1)***

$\left[-\frac{2t^3}{3}+8t^2-24t\right]_0^t$ ***A1***

attempt to substituted limits into their integral ***(M1)***

$x=-\frac{2t^3}{3}+8t^2-24t$ ***A1***

[4 marks]

(e) $\int_0^4 |v|\ dt$ ***(M1)(A1)***

Note: Award ***M1*** for using the absolute value of v, or separating into two integrals, ***A1*** for the correct expression.

$=32\ \text{m}$ ***A1***

[3 marks]

Total [15 marks]

Question 8

(a) evidence of splitting diagram into equilateral triangles **M1**

area $= 6\left(\frac{1}{2}x^2 \sin 60°\right)$ **A1**

$= \frac{3\sqrt{3}x^2}{2}$ **AG**

Note: The ***AG*** line must be seen for the final ***A1*** to be awarded.

[2 marks]

(b) total surface area of prism $1200 = 2\left(3x^2 \frac{\sqrt{3}}{2}\right) + 6xh$ **M1A1**

Note: Award ***M1*** for expressing total surface areas as a sum of areas of rectangles and hexagon(s), and ***A1*** for a correctly substituted formula, equated to 1200.

[5 marks]

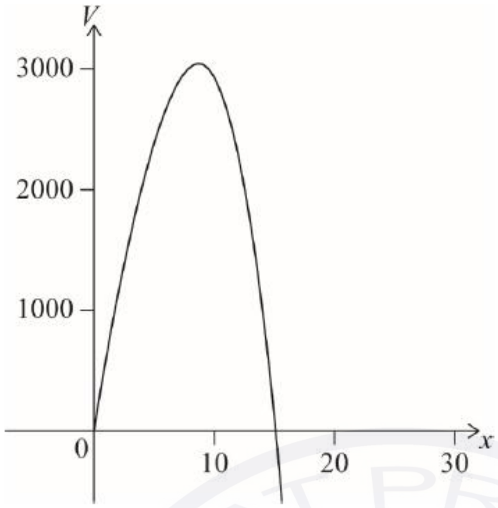
$h = \frac{400 - \sqrt{3}x^2}{2x}$ **A1**

volume of prism $= \frac{3\sqrt{3}}{2}x^2 h$ ***(A1)***

$= \frac{3\sqrt{3}}{2}x^2\left(\frac{400 - \sqrt{3}x^2}{2x}\right)$ **A1**

$= 300\sqrt{3}x - \frac{9}{4}x^3$ ***(AG)***

(c)



A1A1

Note: Award ***A1*** for correct shape, ***A1*** for roots in correct place with some indication of scale (indicated by a labelled point).

[2 marks]

(d) $\frac{dV}{dx} = 300\sqrt{3} - \frac{27}{4}x^2$ ***A1A1***

Note: Award ***A1*** for a correct term.

[2 marks]

(e) from the graph of V or $\frac{dV}{dx}$ **OR** solving $\frac{dV}{dx} = 0$ ***(M1)***

$x = 8.77 \ \ (8.77382...)$ ***A1***

[2 marks]

(f) from the graph of V **OR** substituting their value for x into V ***(M1)***

$V_{max} = 3040 \text{ cm}^3 \ \ (3039.34...)$ ***A1***

[2 marks]

Total [15 marks]

Question 9

(a) $y = \dot{x} \Rightarrow \dot{y} = \ddot{x}$ **A1**

$\dot{y} + 3(y) + 1.25x = 0$ **R1**

Note:	If no explicit reference is made to $\dot{y} = \ddot{x}$, or equivalent, award ***A0R1*** if second line is seen. If $\frac{\mathrm{d}y}{\mathrm{d}x}$ used instead of $\frac{\mathrm{d}y}{\mathrm{d}t}$, award ***A0R0***.

$\dot{y} = -3y - 1.25x$ **AG**

[2 marks]

(b) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -1.25 & -3 \end{pmatrix}$ **A1**

[1 mark]

(c) (i) $\begin{vmatrix} -\lambda & 1 \\ -1.25 & -3-\lambda \end{vmatrix} = 0$ ***(M1)***

$\lambda(\lambda + 3) + 1.25 = 0$ ***(A1)***

$\lambda = -2.5 \;;\; \lambda = -0.5$ **A1**

(ii) $\begin{pmatrix} 2.5 & 1 \\ -1.25 & -0.5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ***(M1)***

$2.5a + b = 0$

$\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ **A1**

$\begin{pmatrix} 0.5 & 1 \\ -1.25 & -2.5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$0.5a + b = 0$

$\boldsymbol{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ **A1**

Note:	Award ***M1*** for a valid attempt to find either eigenvector. Accept equivalent forms of the eigenvectors. Do not award ***FT*** for eigenvectors that do not satisy both rows of the matrix.

[6 marks]

(d) $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-2.5t}\begin{pmatrix} -2 \\ 5 \end{pmatrix} + Be^{-0.5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ ***M1A1***

$t = 0 \;\Rightarrow\; x = 8,\; \dot{x} = y = 0$ ***(M1)***

$-2A - 2B = 8$
$5A + B = 0$ ***(M1)***

$A = 1;\; B = -5$ ***A1***

$x = -2e^{-2.5t} + 10e^{-0.5t}$ ***A1***

Note: Do not award the final ***A1*** if the answer is given in the form $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-2.5t}\begin{pmatrix} -2 \\ 5 \end{pmatrix} + Be^{-0.5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

[6 marks]
Total: [15 marks]

Question 10

(a) (i) $y = x^{\frac{1}{2}}$ ***(M1)***

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ ***A1***

(ii) gradient at $x = 0.16$ is $\frac{1}{2} \times \frac{1}{\sqrt{0.16}}$ ***M1***

$= 1.25$

EITHER

$y - 0.4 = 1.25(x - 0.16)$ ***M1***

OR

$0.4 = 1.25(0.16) + b$ ***M1***

Note: Do not allow working backwards from the given answer.

THEN

hence $y = 1.25x + 0.2$ ***AG***

[4 marks]

(b) $p = 0.45$, $q = 0.4125$ (or 0.413) (accept "$(0.45, 0.4125)$") ***A1A1***

[2 marks]

(c) (i) $(h(x) =)\ \frac{1}{2}\sqrt{2(x - 0.2)}$ ***A2***

Note: Award ***A1*** if only two correct transformations are seen.

(ii) $(a =)\ 0.28$ ***A1***

(iii) **EITHER**

Correct substitution of their part (b) (or $(0.28, 0.2)$) into the given expression ***(M1)***

OR

$\frac{1}{2}(1.25 \times 2(x - 0.2) + 0.2)$ ***(M1)***

Note: Award ***M1*** for transforming the equivalent expression for *f* correctly.

THEN

$(b =)\ -0.15$ ***A1***

[5 marks]

(d) (i) recognizing need to add two integrals *(M1)*

$\int_0^{0.16} \sqrt{x}\ \mathrm{d}x + \int_{0.16}^{0.5} (1.25x + 0.2)\ \mathrm{d}x$ *(A1)*

> **Note:** The second integral could be replaced by the formula for the area of a trapezoid $\frac{1}{2} \times 0.34\,(0.4 + 0.825)$.

$0.251\ \mathrm{m}^2\ (0.250916\ldots)$ ***A1***

(ii) **EITHER**

area of trapezoid $\frac{1}{2} \times 0.05\,(0.4125 + 0.825) = 0.0309375$ *(M1)(A1)*

OR

$\int_{0.45}^{0.5} (8.25x - 3.3)\,\mathrm{d}x = 0.0309375$ *(M1)(A1)*

> **Note:** If the rounded answer of 0.413 from part (b) is used, the integral is $\int_{0.45}^{0.5} (8.24x - 3.295)\,\mathrm{d}x = 0.03095$ which would be awarded ***(M1)(A1)***.

THEN

shaded area $= 0.250916\ldots - 0.0627292 - 0.0309375$ *(M1)*

> **Note:** Award ***(M1)*** for the subtraction of both $0.0627292\ldots$ and their area for the trapezoid from their answer to (a)(i).

$= 0.157\ \mathrm{m}^2\ (0.15725)$ ***A1***

[7 marks]

[Total 18 marks]

Question 11

(a) (i) $y = \frac{dx}{dt} \Rightarrow \frac{dy}{dt} + 5\frac{dx}{dt} + 6x = 0$ **OR** $\frac{dy}{dt} + 5y + 6x = 0$ ***M1***

Note: Award ***M1*** for substituting $\frac{dy}{dt}$ for $\frac{d^2x}{dt^2}$.

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$ ***AG***

(ii) $\det\begin{pmatrix} -\lambda & 1 \\ -6 & -5-\lambda \end{pmatrix} = 0$ ***(M1)***

Note: Award ***M1*** for an attempt to find eigenvalues. Any indication that $\det(\boldsymbol{M} - \lambda \boldsymbol{I}) = 0$ has been used is sufficient for the ***(M1)***.

$-\lambda(-5-\lambda) + 6 = 0$ **OR** $\lambda^2 + 5\lambda + 6 = 0$ ***(A1)***

$\lambda = -2, -3$ ***A1***

(iii) (on a phase portrait the particle approaches $(0, 0)$ as t increases so long term velocity (v) is)

0 ***A1***

Note: Only award ***A1*** for 0 if both eigenvalues in part (a)(ii) are negative. If at least one is positive accept an answer of '*no limit*' or '*infinity*', or in the case of one positive and one negative also accept '*no limit or 0 (depending on initial conditions)*'.

[5 marks]

(b) (i) $y = \frac{dx}{dt}$

$\frac{d^2x}{dt^2} = \frac{dy}{dt}$ ***(A1)***

$\frac{dy}{dt} + 5y + 6x = 3t + 4$ ***A1***

(ii) recognition that $h = 0.1$ in any recurrence formula ***(M1)***

$(t_{n+1} = t_n + 0.1)$

$x_{n+1} = x_n + 0.1y_n$ ***(A1)***

$y_{n+1} = y_n + 0.1(3t_n + 4 - 5y_n - 6x_n)$ ***(A1)***

(when $t = 1$,) $x = 0.64402... \approx 0.644$ m ***A2***

(iii) recognizing that y is the velocity

$0.5\ \mathrm{m\,s^{-1}}$ ***A1***

[8 marks]

[Total 13 marks]

Question 12

(a) (i) $\left(\frac{1}{2}\hat{AOB} =\right) \arccos\left(\frac{4}{4.5}\right) = 27.266...$ ***(M1)(A1)***

$\hat{AOB} = 54.532... \approx 54.5^\circ$ ($0.951764... \approx 0.952$ radians) ***A1***

Note: Other methods may be seen; award ***(M1)(A1)*** for use of a correct trigonometric method to find an appropriate angle and then ***A1*** for the correct answer.

(ii) a finding area of triangle

EITHER

area of triangle = $\frac{1}{2} \times 4.5^2 \times \sin(54.532...)$ ***(M1)***

Note: Award ***M1*** for correct substitution into formula.

$= 8.24621... \approx 8.25 \text{ m}^2$ ***(A1)***

OR

$AB = 2 \times \sqrt{4.5^2 - 4^2} = 4.1231...$ ***(M1)***

area triangle $= \frac{4.1231... \times 4}{2}$

$= 8.24621... \approx 8.25 \ (\text{m}^2)$ ***(A1)***

finding area of sector

EITHER

area of sector = $\frac{54.532...}{360} \times \pi \times 4.5^2$ ***(M1)***

$= 9.63661... \approx 9.64 \text{ m}^2$ ***(A1)***

OR

area of sector = $\frac{1}{2} \times 0.9517641.. \times 4.5^2$ ***(M1)***

$= 9.63661... \approx 9.64 \text{ m}^2$ ***(A1)***

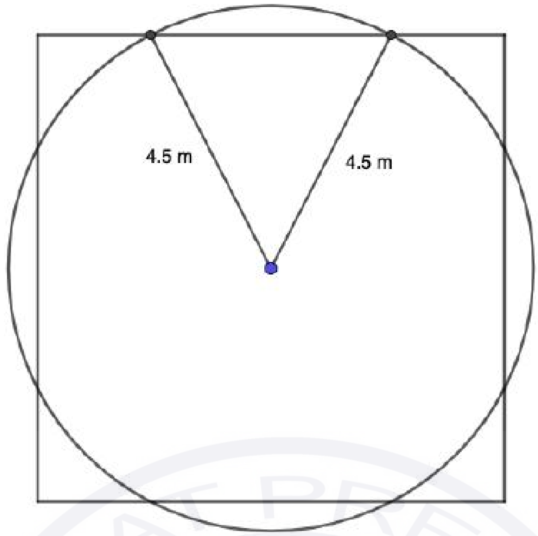
THEN

area of segment = $9.63661... - 8.24621...$

$= 1.39 \text{ m}^2$ $(1.39040...)$ ***A1***

[8 marks]

(b) **METHOD 1**



$\pi \times 4.5^2$ (63.6172…) *(A1)*

$4 \times 1.39040...$ (5.56160) *(A1)*

subtraction of four segments from area of circle *(M1)*

$= 58.1 \text{ m}^2$ (58.055...) ***A1***

METHOD 2

angle of sector = $90 - 54.532...$ $\left(\frac{\pi}{2} - 0.951764...\right)$ *(A1)*

area of sector = $\frac{90 - 54.532...}{360} \times \pi \times 4.5^2$ $(= 6.26771...)$ *(A1)*

area is made up of four triangles and four sectors *(M1)*

total area = $(4 \times 8.2462...) + (4 \times 6.26771...)$

$= 58.1 \text{ m}^2$ $(58.055...)$ ***A1***

[4 marks]

(c) sketch of $\frac{dV}{dt}$ **OR** $\frac{dV}{dt} = 0.110363...$ **OR** attempt to find where $\frac{d^2V}{dt^2} = 0$ *(M1)*

$t = 1$ hour ***A1***

[2 marks]

(d) recognizing $V = \int \frac{dV}{dt}\, dt$ *(M1)*

$\int_0^8 0.3te^{-t} dt$ *(A1)*

volume eaten is $0.299... \text{ m}^3$ $(0.299094...)$ ***A1***

[3 marks]

[Total 17 marks]

Question13

(a) differentiating first equation. ***M1***

$\frac{d^2x}{dt^2} = \frac{dy}{dt}$

substituting in for $\frac{dy}{dt}$ ***M1***

$= -2x - 3y = -2x - 3\frac{dx}{dt}$

therefore $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$ ***AG***

Note: The ***AG*** line must be seen to award the final ***M1*** mark.

[2 marks]

(b) the relevant matrix is $\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ ***(M1)***

Note: $\begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$ is also possible.

(this has characteristic equation) $-\lambda(-3-\lambda)+2=0$ ***(A1)***

$\lambda = -1, -2$ ***A1***

[3 marks]

(c) **EITHER**

the general solution is $x = Ae^{-t} + Be^{-2t}$ ***M1***

Note: Must have constants, but condone sign error for the ***M1***.

so $\frac{dx}{dt} = -Ae^{-t} - 2Be^{-2t}$ ***M1A1***

OR

attempt to find eigenvectors ***(M1)***

respective eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (or any multiple)

$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ ***(M1)A1***

THEN
the initial conditions become:

$0 = A + B$

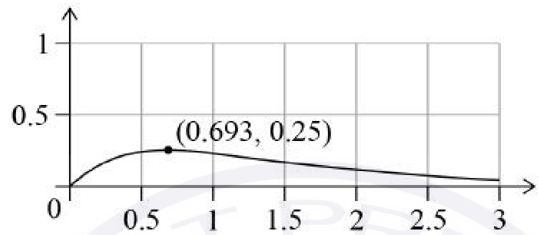
$1 = -A - 2B$ **M1**

this is solved by $A = 1,\ B = -1$

so the solution is $x = e^{-t} - e^{-2t}$ **A1**

[5 marks]

(d)

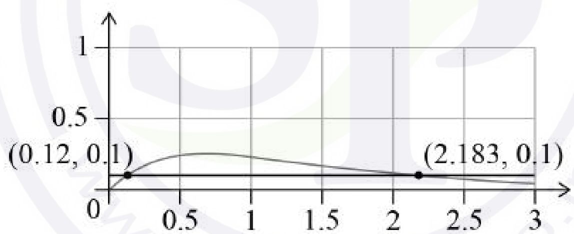


A1A1

Note: Award ***A1*** for correct shape (needs to go through origin, have asymptote at $y = 0$ and a single maximum; condone $x < 0$). Award ***A1*** for correct coordinates of maximum.

[2 marks]

(e) intersecting graph with $y = 0.1$ ***(M1)***



so the time fishing is stopped between $2.1830\ldots$ and $0.11957\ldots$ ***(A1)***

$= 2.06(343\ldots)$ days ***A1***

[3 marks]

(f) *Any reasonable answer. For example:*
There are greater downsides to allowing fishing when the levels may be dangerous than preventing fishing when the levels are safe.
The concentration of mercury may not be uniform across the river due to natural variation / randomness.
The situation at the power plant might get worse.
Mercury levels are low in water but still may be high in fish. ***R1***

Note: Award ***R1*** for a reasonable answer that refers to this specific context (and not a generic response that could apply to *any* model).

[1 mark]
Total [16 marks]

Question 14

(a) attempt to use chain rule, including the differentiation of $\frac{1}{T}$ ***(M1)***

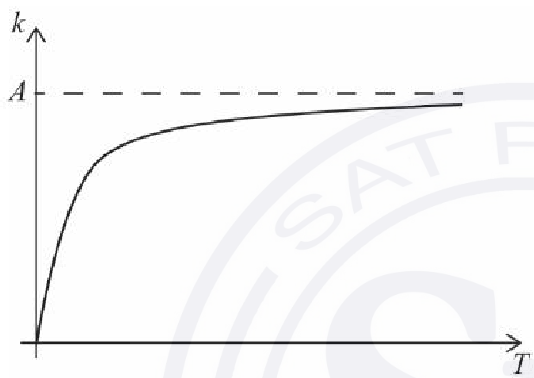
$\frac{\mathrm{d}k}{\mathrm{d}T} = A \times \frac{c}{T^2} \times \mathrm{e}^{-\frac{c}{T}}$ ***A1***

this is the product of positive quantities so must be positive ***R1***

Note: The ***R1*** may be awarded for correct argument from **their** derivative. ***R1*** is not possible if their derivative is not always positive.

[3 marks]

(b)



A1A1A1

Note: Award ***A1*** for an increasing graph, entirely in first quadrant, becoming concave down for larger values of T, ***A1*** for tending towards the origin and ***A1*** for asymptote labelled at $k = A$.

[3 marks]

(c) taking ln of both sides **OR** substituting $y = \ln x$ and $x = \frac{1}{T}$ ***(M1)***

$\ln k = \ln A - \frac{c}{T}$ **OR** $y = -cx + \ln A$ ***(A1)***

(i) so gradient is $-c$ ***A1***

(ii) y-intercept is $\ln A$ ***A1***

Note: The implied ***(M1)*** and ***(A1)*** can only be awarded if **both** correct answers are seen. Award zero if only one value is correct **and** no working is seen.

[4 marks]

(d) an attempt to convert data to $\frac{1}{T}$ and $\ln k$ ***(M1)***

e.g. at least one correct row in the following table

$\frac{1}{T}$	$\ln k$
$1.69491\ldots\times10^{-3}$	$-7.60090\ldots$
$1.66666\ldots\times10^{-3}$	$-7.41858\ldots$
$1.63934\ldots\times10^{-3}$	$-6.90775\ldots$
$1.61290\ldots\times10^{-3}$	$-6.57128\ldots$
$1.58730\ldots\times10^{-3}$	$-6.21460\ldots$
1.5625×10^{-3}	$-5.84304\ldots$
$1.53846\ldots\times10^{-3}$	$-5.62682\ldots$

line is $\ln k = -13400\times\frac{1}{T}+15.0 \quad \left(=-13383.1\ldots\times\frac{1}{T}+15.0107\ldots\right)$ ***A1***

[2 marks]

(e) (i) $c=13400 \quad (13383.1\ldots)$ ***A1***

(ii) attempt to rearrange or solve graphically $\ln A = 15.0107\ldots$ ***(M1)***

$A = 3300000 \quad (3304258\ldots)$ ***A1***

Note: Accept an A value of $3269017\ldots$ from use of 3sf value.

[3 marks]
Total [15 marks]

Question 15

(a) attempt to expand given expression **OR** attempt at product rule ***(M1)***

$$C = \frac{xk^2}{10} - \frac{3x^3}{1000}$$

$$\frac{dC}{dx} = \frac{k^2}{10} - \frac{9x^2}{1000}$$ ***M1A1***

Note:	Award ***M1*** for power rule correctly applied to at least one term and ***A1*** for correct answer.

[3 marks]

(b) equating their $\frac{dC}{dx}$ to zero ***(M1)***

$$\frac{k^2}{10} - \frac{9x^2}{1000} = 0$$

$$x^2 = \frac{100k^2}{9}$$

$$x = \frac{10k}{3}$$ ***(A1)***

substituting their x back into given expression ***(M1)***

$$C_{\max} = \frac{10k}{30}\left(k^2 - \frac{300k^2}{900}\right)$$

$$C_{\max} = \frac{2k^3}{9}\ \left(0.222\ldots k^3\right)$$ ***A1***

[4 marks]

(c) (i) substituting 20 into given expression and equating to 426 ***M1***

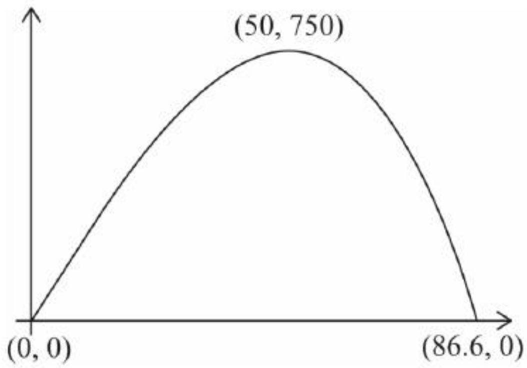
$$426 = \frac{20}{10}\left(k^2 - \frac{3}{100}(20)^2\right)$$

$$k = 15$$ ***A1***

(ii) 50 ***A1***

[3 marks]

(d)



A1A1A1

> **Note:** Award ***A1*** for graph indicating an increasing and then decreasing function (drawn in first quadrant), ***A1*** for maximum labelled and ***A1*** for graph drawn for positive x, passing through the origin and 86.6 which is marked on the x-axis or its coordinates are given.

[3 marks]

(e) setting their expression for C to zero **OR** choosing correct x-intercept on their graph of C ***(M1)***

$x_{\max} = 86.6 \quad (86.6025\ldots)$ litres ***A1***

[2 marks]

Total [15 marks]

Question 16

(a) (i) use of chain rule *(M1)*
$v = -9\sin(3t)\boldsymbol{i} + 12\cos(3t)\boldsymbol{j}$ **A1**

Note:	Award ***(M1)*** for at least one correct term seen but condone omission of $\boldsymbol{i}$ or $\boldsymbol{j}$.

(ii) $|v| = \sqrt{(-9\sin(9))^2 + (12\cos(9))^2}$ *(M1)*
$= 11.5 \text{ m s}^{-1}$ (11.5455...) **A1**

[4 marks]

(b) (i) $\boldsymbol{a} = -27\cos(3t)\boldsymbol{i} - 36\sin(3t)\boldsymbol{j}$ **A1**

(ii) $\boldsymbol{a} = -9(3\cos(3t)\boldsymbol{i} - 4\sin(3t)\boldsymbol{j})$ **M1**
$\boldsymbol{a} = -9\boldsymbol{r}$ (where $\boldsymbol{r}$ is a position vector from the origin) **A1**
$\boldsymbol{a}$ is in opposite direction to the position vector **R1**
hence $\boldsymbol{a}$ is always directed towards the origin **AG**

[4 marks]

(c) relative position $\boldsymbol{d} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ *(M1)*
distance between particles $= |\boldsymbol{d}|$ $(= |\boldsymbol{r}_2 - \boldsymbol{r}_1|)$ *(M1)*
$|\boldsymbol{d}| = \sqrt{(-4\sin(4t) - 3\cos(3t))^2 + (3\cos(4t) - 4\sin(3t))^2}$ *(A1)*
minimum value of $|\boldsymbol{d}|$ when $t = 4.71\text{(s)}$ $\left(4.71238..., \frac{3\pi}{2}\right)$ *(M1)***A1**

[5 marks]

(d) (i) for 2nd particle, $v = -16\cos(4t)\boldsymbol{i} - 12\sin(4t)\boldsymbol{j}$ *(A1)*

EITHER
consider the gradient of either v *(M1)*
$m_1 = -\frac{12\cos(3t)}{9\sin(3t)}$ and $m_2 = \frac{12\sin(4t)}{16\cos(4t)}$ *(A1)*
attempt to solve $m_1 = m_2$ *(M1)*

OR
vectors are parallel therefore one is a multiple of the other, $v_2 = l\,v_1$ *(M1)*
$(l =)\ \frac{16\cos(4t)}{9\sin(3t)} = -\frac{\sin(4t)}{\cos(3t)}$ *(A1)*
attempt to solve *(M1)*

THEN
$t = 1.30 \text{ s}$ (1.30135...) **A1**

(ii) **EITHER**

at $t = 1.30$, $\boldsymbol{v}_1 = 6.22\boldsymbol{i} - 8.68\boldsymbol{j}$ and $\boldsymbol{v}_2 = -7.57\boldsymbol{i} + 10.6\boldsymbol{j}$ ***A1***

OR

$l = -1.22$ (following second method in part (d)(i)) ***A1***

THEN

$\boldsymbol{v}_2$ is a negative multiple of $\boldsymbol{v}_1$ $(\boldsymbol{v}_2 = -1.22\boldsymbol{v}_1)$ ***R1***

the two particles are moving in the opposite direction ***AG***

[7 marks]

Total [20 marks]

Question 17

(a) $\frac{1}{2}x^3+1=(x-1)^4$ **(M1)**

$(p=)$ 2.91 cm (2.91082...) **A1**

[2 marks]

(b) attempt to make x (or x^2) the subject of $y=\frac{1}{2}x^3+1$ **(M1)**

$x=\sqrt[3]{2(y-1)}$ (or $x^2=\left(2(y-1)\right)^{\frac{2}{3}}$) **(A1)**

(upper limit $=$) $13.3(315\ldots)$ **(A1)**

$V=\int_1^{13.3315\ldots}\pi\left(2(y-1)\right)^{\frac{2}{3}}\,dy$ **(M1)**

> **Note:** Award ***(M1)*** for setting up correct integral squaring their expression for x with both correct lower limit and their upper limit, and π. Condone omission of dy.

$=197\text{ cm}^3$ (196.946...) **A1**

[5 marks]

(c) $x=y^{\frac{1}{4}}+1$ (or $x^2=\left(y^{\frac{1}{4}}+1\right)^2$) **(A1)**

$V_2=\int_0^{13.3315\ldots}\pi(y^{\frac{1}{4}}+1)^2\,dy$ **(M1)(A1)**

> **Note:** Award ***(M1)*** for setting up correct integral squaring their expression for x with their upper limit, and π. Award ***(A1)*** for lower limit of 0, dependent on ***M1***. Condone omission of dy. If a candidate found an area in part (b), do not award ***FT*** for another area calculation seen in part (c).

$=271.87668\ldots$ **(A1)**

> **Note:** Accept $271.038\ldots$ from use of 3sf in the upper limit.

subtracting their volumes **(M1)**

$271.87668\ldots-196.946\ldots$

$=74.9\text{ cm}^3$ $(74.93033\ldots)$ **A1**

> **Note:** Accept any answer that rounds to $75\ (\text{cm}^3)$. If a candidate found an area in part (b), do not award ***FT*** for another area calculation seen in part (c).

[6 marks]

[13 marks]

Question 18

(a) (i) $f'(x) = \frac{-2x}{50} + 2 \quad \left(= \frac{-x}{25} + 2,\ -0.04x + 2 \right)$ ***A1A1***

Note: Award ***A1*** for each correct term. Award at most ***A0A1*** if extra terms are seen.

(ii) $0 = \frac{-x}{25} + 2$ **OR** sketch of $f'(x)$ with x-intercept indicated ***M1***

$x = 50$ ***A1***

$y = 80$ ***A1***

$(50, 80)$

Note: Award ***M0A0A1*** for the coordinate $(50, 80)$ seen either with no working or found from a graph of $f(x)$.

[5 marks]

(b) (i) $\int_0^{70} \frac{-x^2}{50} + 2x + 30 \, dx$ ***A1A1***

Note: Award ***A1*** for a correct integral, ***A1*** for correct limits in the correct location. Award at most ***A0A1*** if dx is omitted.

(ii) (Area =) $4710 \text{ m}^2 \quad \left(4713.33\ldots, \frac{14140}{3} \right)$ ***A2***

[4 marks]

(c) (i) $\frac{11.4}{4713.33\ldots} \times 100\%$ **OR** $\left| \frac{4701.93\ldots - 4713.33\ldots}{4713.33\ldots} \right| \times 100\%$ ***(M1)***

Note: Award ***(M1)*** for their correct substitution into the percentage error formula.

$0.242\% \quad (0.241867\ldots\%)$ ***A1***

Note: Percentage sign is required. Accept $0.242038\ldots\%$ if 4710 is used.

(ii) **EITHER**

reduce the width of the intervals (trapezoids) ***A1***

OR

increase the number of intervals (trapezoids) ***A1***

Note: Accept equivalent statements. Award ***A0*** for the ambiguous answer "increase the intervals".

[3 marks]

(d) (i) width of the square is $70-x$ **OR** the length of the square is $\frac{-x^2}{50}+2x+30$

(M1)

Note: Award ***(M1)*** for $70-x$ seen anywhere. Accept $\frac{-x^2}{50}+2x+30$ but only if this expression is explicitly identified as a dimension of the square.

in term of x, equating the length to the width ED ***(M1)***

$\frac{-x^2}{50}+2x+30=70-x$

$(x=14.7920\ldots$ or $135.21)$

$(x=)\ 14.8\ \mathrm{m}\ (14.7920\ldots)$ ***A1***

Note: Award ***M0M0A0*** for an unsupported answer of 15. Award at most ***M1M0A0*** for an approach which leads to $A'(x)=0$. This will lead to a square base which extends beyond the east boundary of the property. Similar for any solution where F is not on the northern boundary, or GH is not on the east boundary.

(ii) **EITHER**

$(70-14.7920\ldots)^2$ ***(M1)***

OR

$(55.2079\ldots)^2$ ***(M1)***

OR

$\left(\frac{-(14.7920\ldots)^2}{50}+2(14.7920\ldots)+30\right)^2$ ***(M1)***

THEN

$(\text{Area}=)\ 3050\ \mathrm{m}^2\ (3047.92\ldots)$ ***A1***

Note: Follow through from part (d)(i), provided x is between 0 and 70. Award at most ***M1A0*** if their answer is outside the range of their $[0,\ 4713.33\ldots]$ from part (b).

[5 marks]
Total [17 marks]