Subject – Math AI(Higher Level)
Topic - Number and Algebra
Year - May 2021 – Nov 2024
Paper -2
Questions

Question 1

[Maximum mark: 14]

A city has two cable companies, X and Y. Each year 20% of the customers using company X move to company Y and 10% of the customers using company Y move to company X.
All additional losses and gains of customers by the companies may be ignored.

(a) Write down a transition matrix $\boldsymbol{T}$ representing the movements between the two companies in a particular year. [2]

(b) Find the eigenvalues and corresponding eigenvectors of $\boldsymbol{T}$. [4]

(c) Hence write down matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ such that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$. [2]

Initially company X and company Y both have 1200 customers.

(d) Find an expression for the number of customers company X has after n years, where $n \in \mathbb{N}$. [5]

(e) Hence write down the number of customers that company X can expect to have in the long term. [1]

Question 2

[Maximum mark: 18]

In this question, give all answers to two decimal places.

Bryan decides to purchase a new car with a price of €14 000, but cannot afford the full amount. The car dealership offers two options to finance a loan.

Finance option A:

A 6 year loan at a nominal annual interest rate of 14% **compounded quarterly**. No deposit required and repayments are made each quarter.

(a) (i) Find the repayment made each quarter.

(ii) Find the total amount paid for the car.

(iii) Find the interest paid on the loan. [7]

Finance option B:

A 6 year loan at a nominal annual interest rate of r% **compounded monthly**. Terms of the loan require a 10% deposit and monthly repayments of €250.

(b) (i) Find the amount to be borrowed for this option.

(ii) Find the annual interest rate, r. [5]

(c) State which option Bryan should choose. Justify your answer. [2]

Bryan chooses option B. The car dealership invests the money Bryan pays as soon as they receive it.

(d) If they invest it in an account paying 0.4% interest per month and inflation is 0.1% per month, calculate the real amount of money the car dealership has received by the end of the 6 year period. [4]

Question 3

[Maximum mark: 13]

Long term experience shows that if it is sunny on a particular day in Vokram, then the probability that it will be sunny the following day is 0.8. If it is not sunny, then the probability that it will be sunny the following day is 0.3.

The transition matrix $\boldsymbol{T}$ is used to model this information, where $\boldsymbol{T} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$.

(a) It is sunny today. Find the probability that it will be sunny in three days' time. [2]

(b) Find the eigenvalues and eigenvectors of $\boldsymbol{T}$. [5]

The matrix $\boldsymbol{T}$ can be written as a product of three matrices, $\boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix.

(c) (i) Write down the matrix $\boldsymbol{P}$.

(ii) Write down the matrix $\boldsymbol{D}$. [2]

(d) Hence find the long-term percentage of sunny days in Vokram. [4]

Question 4

[Maximum mark: 19]

Give your answers in parts (a), (d)(i), (e) and (f) to the nearest dollar.

Daisy invested $37\,000$ Australian dollars (AUD) in a fixed deposit account with an annual interest rate of 6.4% compounded **quarterly**.

(a) Calculate the value of Daisy's investment after 2 years. [3]

After m months, the amount of money in the fixed deposit account has appreciated to more than $50\,000$ AUD.

(b) Find the minimum value of m, where $m \in \mathbb{N}$. [4]

Daisy is saving to purchase a new apartment. The price of the apartment is $200\,000$ AUD.

Daisy makes an initial payment of 25% and takes out a loan to pay the rest.

(c) Write down the amount of the loan. [1]

The loan is for 10 years, compounded monthly, with equal monthly payments of 1700 AUD made by Daisy at the end of each month.

(d) For this loan, find

(i) the amount of interest paid by Daisy.

(ii) the annual interest rate of the loan. [5]

After 5 years of paying off this loan, Daisy decides to pay the **remainder** in one final payment.

(e) Find the amount of Daisy's final payment. [3]

(f) Find how much money Daisy saved by making one final payment after 5 years. [3]

Question 5

[Maximum mark: 18]

In a small village there are two doctors' clinics, one owned by Doctor Black and the other owned by Doctor Green. It was noted after each year that 3.5% of Doctor Black's patients moved to Doctor Green's clinic and 5% of Doctor Green's patients moved to Doctor Black's clinic. All additional losses and gains of patients by the clinics may be ignored.

At the start of a particular year, it was noted that Doctor Black had 2100 patients on their register, compared to Doctor Green's 3500 patients.

(a) Write down a transition matrix $\boldsymbol{T}$ indicating the annual population movement between clinics. [2]

(b) Find a prediction for the ratio of the number of patients Doctor Black will have, compared to Doctor Green, after two years. [2]

(c) Find a matrix $\boldsymbol{P}$, with integer elements, such that $\boldsymbol{T}=\boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix. [6]

(d) Hence, show that the long-term transition matrix $\boldsymbol{T}^{\infty}$ is given by $\boldsymbol{T}^{\infty}=\begin{pmatrix}\frac{10}{17} & \frac{10}{17}\\ \frac{7}{17} & \frac{7}{17}\end{pmatrix}$. [6]

(e) Hence, or otherwise, determine the expected ratio of the number of patients Doctor Black would have compared to Doctor Green in the long term. [2]

Question 6

[Maximum mark: 18]

A flying drone is programmed to complete a series of movements in a horizontal plane relative to an origin O and a set of x-y-axes.

In each case, the drone moves to a new position represented by the following transformations:

- a rotation anticlockwise of $\frac{\pi}{6}$ radians about O
- a reflection in the line $y = \frac{x}{\sqrt{3}}$.
- a rotation clockwise of $\frac{\pi}{3}$ radians about O.

All the movements are performed in the listed order.

(a) (i) Write down each of the transformations in matrix form, clearly stating which matrix represents each transformation.

(ii) Find a single matrix $\boldsymbol{P}$ that defines a transformation that represents the overall change in position.

(iii) Find $\boldsymbol{P}^2$.

(iv) Hence state what the value of $\boldsymbol{P}^2$ indicates for the possible movement of the drone. [12]

(b) Three drones are initially positioned at the points A, B and C. After performing the movements listed above, the drones are positioned at points A′, B′ and C′ respectively.

Show that the area of triangle ABC is equal to the area of triangle A′B′C′. [2]

(c) Find a single transformation that is equivalent to the three transformations represented by matrix $\boldsymbol{P}$. [4]

Question 7

[Maximum mark: 13]

(a) Let $z = 1 - \mathrm{i}$.

(i) Plot the position of z on an Argand Diagram.

(ii) Express z in the form $z = a\mathrm{e}^{\mathrm{i}b}$, where a, $b \in \mathbb{R}$, giving the exact value of a and the exact value of b. [3]

(b) Let $w_1 = \mathrm{e}^{\mathrm{i}x}$ and $w_2 = \mathrm{e}^{\mathrm{i}\left(x - \frac{\pi}{2}\right)}$, where $x \in \mathbb{R}$.

(i) Find $w_1 + w_2$ in the form $\mathrm{e}^{\mathrm{i}x}(c + \mathrm{i}d)$.

(ii) Hence find $\mathrm{Re}(w_1 + w_2)$ in the form $A\cos(x - \alpha)$, where $A > 0$ and $0 < \alpha \le \frac{\pi}{2}$. [6]

The current, I, in an AC circuit can be modelled by the equation $I = a\cos(bt - c)$ where b is the frequency and c is the phase shift.

Two AC voltage sources of the same frequency are independently connected to the same circuit. If connected to the circuit alone they generate currents I_A and I_B. The maximum value and the phase shift of each current is shown in the following table.

Current	Maximum value	Phase shift
I_A	12 amps	0
I_B	12 amps	$\frac{\pi}{2}$

When the two voltage sources are connected to the circuit at the same time, the total current I_T can be expressed as $I_A + I_B$.

(c) (i) Find the maximum value of I_T.

(ii) Find the phase shift of I_T. [4]

Question 8

[Maximum mark: 18]

A transformation, T, of a plane is represented by $\boldsymbol{r}' = \boldsymbol{Pr} + \boldsymbol{q}$, where $\boldsymbol{P}$ is a 2×2 matrix, $\boldsymbol{q}$ is a 2×1 vector, $\boldsymbol{r}$ is the position vector of a point in the plane and $\boldsymbol{r}'$ the position vector of its image under T.

The triangle OAB has coordinates $(0, 0)$, $(0, 1)$ and $(1, 0)$. Under T, these points are transformed to $(0, 1)$, $\left(\frac{1}{4}, 1 + \frac{\sqrt{3}}{4}\right)$ and $\left(\frac{\sqrt{3}}{4}, \frac{3}{4}\right)$ respectively.

(a) (i) By considering the image of $(0, 0)$, find $\boldsymbol{q}$.

(ii) By considering the image of $(1, 0)$ and $(0, 1)$, show that [6]

$$\boldsymbol{P} = \begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix}.$$

$\boldsymbol{P}$ can be written as $\boldsymbol{P} = \boldsymbol{RS}$, where $\boldsymbol{S}$ and $\boldsymbol{R}$ are matrices.

$\boldsymbol{S}$ represents an enlargement with scale factor 0.5, centre $(0, 0)$.

$\boldsymbol{R}$ represents a rotation about $(0, 0)$.

(b) Write down the matrix $\boldsymbol{S}$. [1]

(c) (i) Use $\boldsymbol{P} = \boldsymbol{RS}$ to find the matrix $\boldsymbol{R}$.

(ii) Hence find the angle and direction of the rotation represented by $\boldsymbol{R}$. [7]

The transformation T can also be described by an enlargement scale factor $\frac{1}{2}$, centre (a, b), followed by a rotation about the same centre (a, b).

(d) (i) Write down an equation satisfied by $\begin{pmatrix} a \\ b \end{pmatrix}$.

(ii) Find the value of a and the value of b. [4]

Question 9

[Maximum mark: 12]

A geneticist uses a Markov chain model to investigate changes in a specific gene in a cell as it divides. Every time the cell divides, the gene may mutate between its normal state and other states.

The model is of the form

$$\begin{pmatrix} X_{n+1} \\ Z_{n+1} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} X_n \\ Z_n \end{pmatrix}$$

where X_n is the probability of the gene being in its normal state after dividing for the nth time, and Z_n is the probability of it being in another state after dividing for the nth time, where $n \in \mathbb{N}$.

Matrix $\boldsymbol{M}$ is found to be $\begin{pmatrix} 0.94 & b \\ 0.06 & 0.98 \end{pmatrix}$.

(a) (i) Write down the value of b.

(ii) What does b represent in this context? [2]

(b) Find the eigenvalues of $\boldsymbol{M}$. [3]

(c) Find the eigenvectors of $\boldsymbol{M}$. [3]

(d) The gene is in its normal state when $n = 0$. Calculate the probability of it being in its normal state

(i) when $n = 5$;

(ii) in the long term. [4]

Question 10

[Maximum mark: 13]

Scott purchases food for his dog in large bags and feeds the dog the same amount of dog food each day. The amount of dog food left in the bag at the end of each day can be modelled by an arithmetic sequence.

On a particular day, Scott opened a new bag of dog food and fed his dog. By the end of the third day there were 115.5 cups of dog food remaining in the bag and at the end of the eighth day there were 108 cups of dog food remaining in the bag.

(a) Find the number of cups of dog food

(i) fed to the dog per day;

(ii) remaining in the bag at the end of the first day. [4]

(b) Calculate the number of days that Scott can feed his dog with one bag of food. [2]

In 2021, Scott spent $\$625$ on dog food. Scott expects that the amount he spends on dog food will increase at an annual rate of 6.4%.

(c) Determine the amount that Scott expects to spend on dog food in 2025. Round your answer to the nearest dollar. [3]

(d) (i) Calculate the value of $\sum_{n=1}^{10}(625 \times 1.064^{(n-1)})$.

(ii) Describe what the value in part (d)(i) represents in this context. [3]

(e) Comment on the appropriateness of modelling this scenario with a geometric sequence. [1]

Question 11

[Maximum mark: 14]

The three countries of Belgium, Germany and The Netherlands meet at a single point called Vaalserberg.

To support future transport planning, a 10 km circle was drawn around Vaalserberg on a map. A study was carried out over five years to determine what percentage of people living in each of these countries (within the 10 km circular region) either stayed in their own country or moved to another country within the circle.

From this study, the following movements were observed during the five years.

- From Belgium, 5% moved to Germany, and 0.5% moved to The Netherlands.
- From Germany, 2% moved to The Netherlands, and 1.5% moved to Belgium.
- From The Netherlands, 3% moved to Germany, and 2% moved to Belgium.

All additional population changes within the circular region may be ignored.

(a) Represent the above information in a transition matrix $\boldsymbol{T}$. [3]

At the end of the study, the population of the Belgian side was $26\,000$, the population of the German side was $240\,000$, and the population of The Netherlands side was $50\,000$.

(b) By using $\boldsymbol{T}$, find the expected population of the German side of Vaalserberg 30 years after the end of the study. [4]

For matrix $\boldsymbol{T}$ there exists a steady state vector

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

where u_1, u_2 and u_3 are the proportions of the total population on the Belgian side, the German side and The Netherlands side respectively.

The steady state vector $\boldsymbol{u}$ may be found by solving a system of equations.

(c) (i) Determine these equations that are to be solved.

(ii) By solving your system of equations, find $\boldsymbol{u}$. [3]

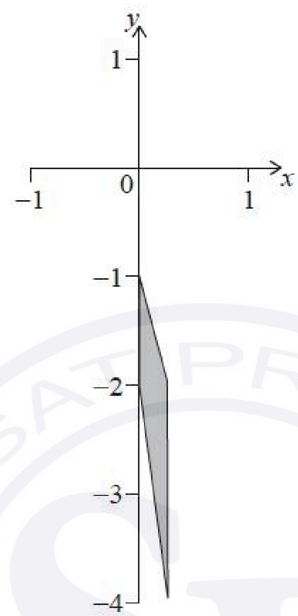
(d) Use your answer to part (c)(ii) to determine the long-term expected population of the German side. [2]

(e) Suggest two reasons why your answer to part (d) is not likely to be accurate. You may comment on both the model and the situation in context. [2]

Question 12

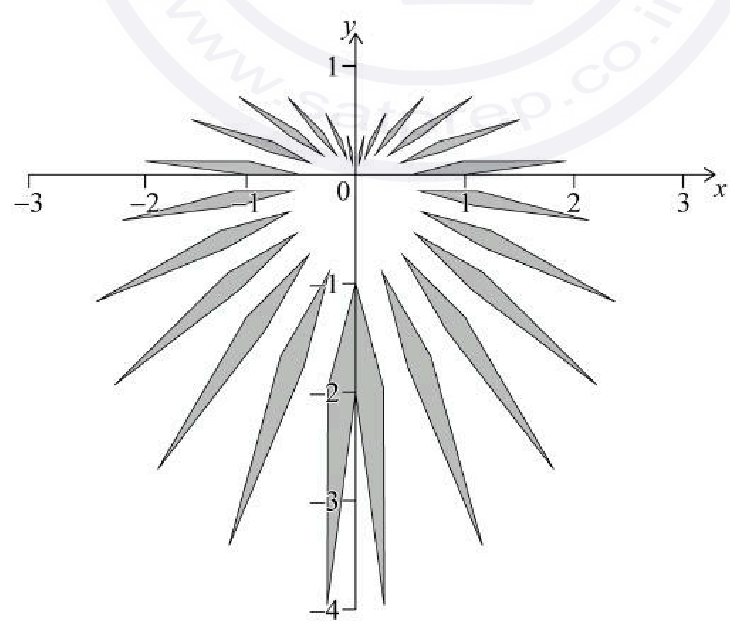
[Maximum mark: 18]

A trapezoid, Q, has vertices $(0, -1)$, $(0, -2)$, $(\sin 15°, -3-\cos 15°)$, $(\sin 15°, -1-\cos 15°)$ as shown.



(a) Show that the area of the trapezoid is $\frac{3}{2}\sin 15°$. [2]

A design is created with 24 elements. Each element is obtained by transforming the trapezoid Q. These elements are shaded in the following diagram such that the y-axis is a line of symmetry.



The transformation that produces each of the elements on the **right side** of the design can be represented by a matrix of the form

$$M_k = \begin{pmatrix} \left(1-\frac{k}{12}\right)\cos\left(k\times 15^\circ\right) & -\left(1-\frac{k}{12}\right)\sin\left(k\times 15^\circ\right) \\ \left(1-\frac{k}{12}\right)\sin\left(k\times 15^\circ\right) & \left(1-\frac{k}{12}\right)\cos\left(k\times 15^\circ\right) \end{pmatrix}$$

where $k = 0, 1, 2, 3, \ldots, 11$.

(b) (i) Find the matrix M_6. Give your answer in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{Q}$.

(ii) Hence find the coordinates of the image of the vertex $(0, -1)$ after it is transformed by the matrix M_6. [4]

The matrix M_k can be expressed as the product of a rotation matrix and an enlargement matrix.

(c) Write down, in terms of k,

(i) the rotation matrix;

(ii) the enlargement matrix;

(iii) the angle of the rotation;

(iv) the scale factor of the enlargement. [4]

(d) Using your answer to part (c)(iv), or otherwise, find the determinant of the matrix M_k in terms of k. [2]

(e) Hence, or otherwise, find the total area of the elements in the **whole** design. [4]

Each element on the **left side** of the design can be obtained through a transformation of the trapezoid Q by applying the matrix N_k, where $k = 0, 1, 2, 3, \ldots, 11$.

(f) Write down the matrix N_k as a product of two matrices. [2]

Question 13

[Maximum mark: 14]

François is a video game designer. He designs his games to take place in two dimensions, relative to an origin O. In one of his games, an object travels on a straight line L_1 with vector equation

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(a) Write down L_1 in the form $x = x_0 + \lambda l$ and $y = y_0 + \lambda m$, where $l,\ m \in \mathbb{Z}$. [1]

François uses the matrix $\boldsymbol{T} = \begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix}$ to transform L_1 into a new straight line L_2. The object will then travel along L_2.

(b) Find the vector equation of L_2. [4]

François knows that the transformation given by matrix $\boldsymbol{T}$ is made up of the following three separate transformations (in the order listed):

- A rotation of $\frac{\pi}{4}$, anticlockwise (counter-clockwise) about the origin O
- An enlargement of scale factor $5\sqrt{2}$, centred at O
- A reflection in the straight line $y = mx$, where $m = \tan\alpha$, $0 \le \alpha < \pi$

(c) Write down the matrix that represents

(i) the rotation.

(ii) the enlargement. [2]

(d) The matrix $\boldsymbol{R}$ represents the reflection. Write down $\boldsymbol{R}$ in terms of α. [1]

(e) Given that $\boldsymbol{T} = \boldsymbol{RX}$,

(i) use your answers to part (c) to find matrix $\boldsymbol{X}$.

(ii) hence, find the value of α. [6]

Question 14

[Maximum mark: 16]

Tiffany wants to buy a house for a price of $285\,000$ US Dollars (USD). She goes to a bank to get a loan to buy the house. To be eligible for the loan, Tiffany must make an initial down payment equal to 15% of the price of the house.

The bank offers her a 30-year loan for the remaining balance, with a 4% nominal interest rate per annum, compounded monthly. Tiffany will pay the loan in fixed payments at the end of each month.

(a) (i) Find the original amount of the loan after the down payment is paid. Give the exact answer.

(ii) Calculate Tiffany's monthly payment for this loan, to two decimal places. [5]

(b) Using your answer from part (a)(ii), calculate the total amount Tiffany will pay over the life of the loan, to the nearest dollar. Do **not** include the initial down payment. [2]

Tiffany would like to repay the loan faster and increases her payments such that she pays 1300 USD each month.

(c) Find the total number of monthly payments she will need to make to pay off the loan. [2]

This strategy will result in Tiffany's final payment being less than 1300 USD.

(d) Determine the amount of Tiffany's final payment, to two decimal places. [4]

(e) Hence, determine the total amount Tiffany will save, to the nearest dollar, by making the higher monthly payments. [3]

Question 15

[Maximum mark: 16]

The drivers of a delivery company can park their vans overnight either at its headquarters or at home.

Urvashi is a driver for the company. If Urvashi has parked her van overnight at headquarters on a given day, the probability that she parks her van at headquarters on the following day is 0.88. If Urvashi has parked her van overnight at her home on a given day, the probability that she parks her van at home on the following day is 0.92.

(a) Write down a transition matrix, $\boldsymbol{T}$, that shows the movement of Urvashi's van between headquarters and home. [2]

On Monday **morning** she collected her van from headquarters where it was parked overnight.

(b) Find the probability that Urvashi's van will be parked at home at the end of the week on Friday **evening**. [3]

(c) Write down the characteristic polynomial for the matrix $\boldsymbol{T}$. Give your answer in the form $\lambda^2 + b\lambda + c$. [2]

(d) Calculate eigenvectors for the matrix $\boldsymbol{T}$. [4]

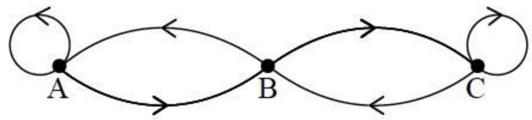
(e) Write down matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ such that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix. [2]

(f) Hence find the long-term probability that Urvashi's van is parked at home. [3]

Question 16

[Maximum mark: 18]

(a) Write down the adjacency matrix for the directed graph shown below. [2]



(b) Find the total number of walks of length 5 from A to B. [3]

A bird sits on one of three posts, labelled A, B and C, with B between A and C. When the bird moves, it will either fly to an adjacent post or return to the same post according to the following pattern.

- If it is on B, it will fly to A or C, each with a probability of 0.5.
- If it is on A or C, it will return to the same post with a probability of 0.5 or fly to B with a probability of 0.5.

The possible flights of the bird can be represented by the graph in part (a).

(c) (i) Every possible sequence of 5 flights by the bird has the same probability of occurring. State this probability.

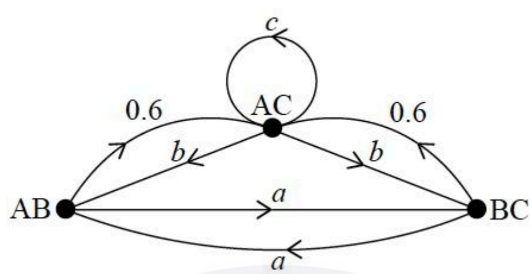
(ii) Use your answer to part (b) to find the probability that if the bird was initially on post A, it will be on post B after 5 flights. [3]

A second bird often joins the first on the posts. Their flights now follow the pattern given below.

- The birds will never sit on the same post.
- They will always fly from the posts at the same time.
- If they are on adjacent posts, the bird on post B will **always** fly to the vacant end post. The other bird will fly to post B with a probability of 0.4 or return to the same post with a probability of 0.6.
- If they are each on one of the end posts, they will fly to post B with a probability of 0.5 or return to the same post with a probability of 0.5. However, if they both try to fly to post B at the same time, they will see the other one doing so and both will immediately return to the post they were previously on.

The possible flights of the two birds can be represented by the following transition diagram, where the three vertices represent which posts are **occupied**.

diagram not to scale



(d) Write down the value of

(i) a.

(ii) b.

(iii) c. [3]

(e) Given that the birds are initially on posts A and B, find the probability they will be on posts B and C after 5 flights. [4]

The birds continue this pattern of flights for a long period.

(f) Given that the time between flights is always the same, find the post which is sat on least and the proportion of the time it is free. [3]

Question 17

[Maximum mark: 13]

Juan is creating animations for a website. He uses matrices to transform objects relative to the origin, O.

One matrix that he uses is $\boldsymbol{A}=\begin{pmatrix}\cos(15°) & -\sin(15°)\\ \sin(15°) & \cos(15°)\end{pmatrix}$.

(a) Describe fully the transformation represented by matrix $\boldsymbol{A}$. [1]

(b) Find the smallest value of n such that $\boldsymbol{A}^n=\boldsymbol{I}$, $n\in\mathbb{Z}^+$. [2]

Juan also uses matrix $\boldsymbol{B}$, which represents an enlargement with a scale factor of 1.05, centre $(0, 0)$.

(c) (i) Write down matrix $\boldsymbol{B}$.

(ii) Describe fully the transformation represented by $\boldsymbol{B}^n$, where n is the value found in part (b). [3]

Juan creates a new matrix, $\boldsymbol{C}=\boldsymbol{AB}$.

(d) Find matrix $\boldsymbol{C}$. [2]

Juan creates an animation by repeatedly transforming an object by $\boldsymbol{C}$.

A point, P, on the object is initially at $(1, 0)$. Juan sets the speed of the animation to 6 transformations per second.

(e) Sketch the path of P for the first 4 seconds of motion **and** label the coordinates of the start and end points. [2]

Juan uses a different transformation, $\boldsymbol{T}$, defined by

$$\boldsymbol{T}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0.9 & 0\\ 0 & 0.8\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}2\\ 1\end{pmatrix}.$$

To create his animation, he repeatedly transforms an object by $\boldsymbol{T}$. After many transformations, he notices that all points, (x, y), on the object tend towards a single point, (p, q), such that

$$\lim_{a\to\infty} T^a\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}p\\ q\end{pmatrix}, \text{ where } a\in\mathbb{Z}^+.$$

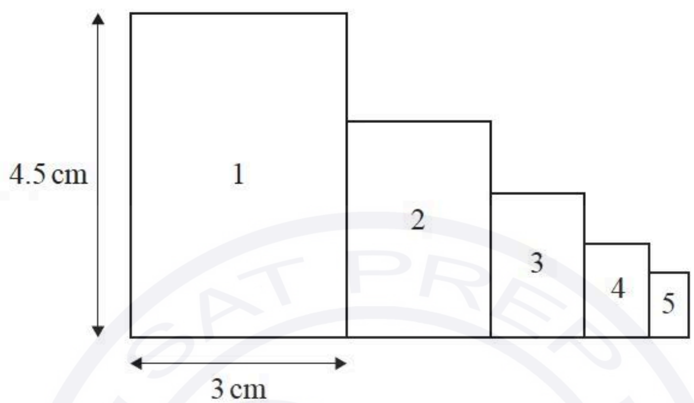
(f) Find $\begin{pmatrix}p\\ q\end{pmatrix}$, where $p, q\in\mathbb{R}$. [3]

Question 18

[Maximum mark: 15]

Ayaka is creating a design made from a sequence of rectangles. The diagram shows part of her design, using 5 rectangles.

diagram not to scale



The first rectangle has the following dimensions: height $4.5\,\text{cm}$ and width $3\,\text{cm}$.

The dimensions of each subsequent rectangle are $\frac{2}{3}$ of the dimensions of the previous rectangle.

(a) Calculate the width of the 5th rectangle. [2]

(b) Calculate the total width of the design that uses 5 rectangles. [2]

Ayaka continues to add rectangles to her design.

(c) Find the smallest total width that her design cannot exceed. [3]

The width of Ayaka's final design must be at least $8.5\,\text{cm}$ and use the least number of rectangles.

(d) Find the total number of rectangles in her final design. [3]

The decreasing **areas** of the rectangles form a geometric sequence.

(e) Find the common ratio for this sequence of areas. [2]

(f) Find the total area of Ayaka's final design. [3]