Subject – Math AI(Higher Level)
Topic - Calculus
Year - May 2021 – Nov 2022
Paper -3
Questions

Question 1

[Maximum mark: 28]

The number of brown squirrels, x, in an area of woodland can be modelled by the following differential equation.

$$\frac{dx}{dt} = \frac{x}{1000}(2000 - x)\text{, where } x > 0$$

(a) (i) Find the equilibrium population of brown squirrels suggested by this model.

(ii) Explain why the population of squirrels is increasing for values of x less than this value. [3]

One year conservationists notice that some black squirrels are moving into the woodland. The two species of squirrel are in competition for the same food supplies. Let y be the number of black squirrels in the woodland.

Conservationists wish to predict the likely future populations of the two species of squirrels. Research from other areas indicates that when the two populations come into contact the growth can be modelled by the following differential equations, in which t is measured in tens of years.

$$\frac{dx}{dt} = \frac{x}{1000}(2000 - x - 2y),\ x,\ y \geq 0$$
$$\frac{dy}{dt} = \frac{y}{1000}(3000 - 3x - y),\ x,\ y \geq 0$$

An equilibrium point for the populations occurs when both $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.

(b) (i) Verify that $x = 800$, $y = 600$ is an equilibrium point.

(ii) Find the other three equilibrium points. [6]

When the two populations are small the model can be reduced to the linear system

$$\frac{dx}{dt} = 2x$$
$$\frac{dy}{dt} = 3y\ .$$

(c) (i) By using separation of variables, show that the general solution of $\frac{dx}{dt} = 2x$ is $x = Ae^{2t}$.

(ii) Write down the general solution of $\frac{dy}{dt} = 3y$.

(iii) If both populations contain 10 squirrels at $t = 0$ use the solutions to parts (c) (i) and (ii) to estimate the number of black and brown squirrels when $t = 0.2$. Give your answers to the nearest whole numbers. [7]

For larger populations, the conservationists decide to use Euler's method to find the long-term outcomes for the populations. They will use Euler's method with a step length of 2 years ($t = 0.2$).

(d) (i) Write down the expressions for x_{n+1} and y_{n+1} that the conservationists will use.

(ii) Given that the initial populations are $x = 100$, $y = 100$, find the populations of each species of squirrel when $t = 1$.

(iii) Use further iterations of Euler's method to find the long-term population for each species of squirrel from these initial values.

(iv) Use the same method to find the long-term populations of squirrels when the initial populations are $x = 400$, $y = 100$. [7]

(e) Use Euler's method with step length 0.2 to sketch, on the same axes, the approximate trajectories for the populations with the following initial populations.

(i) $x = 1000$, $y = 1500$

(ii) $x = 1500$, $y = 1000$ [3]

(f) Given that the equilibrium point at $(800, 600)$ is a saddle point, sketch the phase portrait for $x \geq 0$, $y \geq 0$ on the same axes used in part (e). [2]

Question 2

[Maximum mark: 31]

Alessia is an ecologist working for Mediterranean fishing authorities. She is interested in whether the mackerel population density is likely to fall below 5000 mackerel per km^3, as this is the minimum value required for sustainable fishing. She believes that the primary factor affecting the mackerel population is the interaction of mackerel with sharks, their main predator.

The population densities of mackerel (M thousands per km^3) and sharks (S per km^3) in the Mediterranean Sea are modelled by the coupled differential equations:

$$\frac{dM}{dt} = \alpha M - \beta MS$$

$$\frac{dS}{dt} = \gamma MS - \delta S$$

where t is measured in years, and α, β, γ and δ are parameters.

This model assumes that no other factors affect the mackerel or shark population densities.

The term αM models the population growth rate of the mackerel in the absence of sharks.
The term βMS models the death rate of the mackerel due to being eaten by sharks.

(a) Suggest similar interpretations for the following terms.

(i) γMS [1]

(ii) δS [1]

(b) An equilibrium point is a set of values of M and S, such that $\frac{dM}{dt} = 0$ and $\frac{dS}{dt} = 0$.

Given that both species are present at the equilibrium point,

(i) show that, at the equilibrium point, the value of the mackerel population density is $\frac{\delta}{\gamma}$; [3]

(ii) find the value of the shark population density at the equilibrium point. [2]

(c) The equilibrium point found in part (b) gives the average values of M and S over time.

Use the model to predict how the following events would affect the average value of M. Justify your answers.

(i) Toxic sewage is added to the Mediterranean Sea. Alessia claims this reduces the shark population growth rate and hence the value of γ is halved. No other parameter changes. [2]

(ii) Global warming increases the temperature of the Mediterranean Sea. Alessia claims that this promotes the mackerel population growth rate and hence the value of α is doubled. No other parameter changes. [2]

(d) To estimate the value of α, Alessia considers a situation where there are no sharks and the initial mackerel population density is M_0.

(i) Write down the differential equation for M that models this situation. [1]

(ii) Show that the expression for the mackerel population density after t years is $M = M_0 e^{\alpha t}$. [4]

(iii) Alessia estimates that the mackerel population density increases by a factor of three every two years. Show that $\alpha = 0.549$ to three significant figures. [3]

Based on additional observations, it is believed that

$$\alpha = 0.549,$$

$$\beta = 0.236,$$

$$\gamma = 0.244,$$

$$\delta = 1.39.$$

Alessia decides to use Euler's method to estimate future mackerel and shark population densities. The initial population densities are estimated to be $M_0 = 5.7$ and $S_0 = 2$.
She uses a step length of 0.1 years.

(e) (i) Write down expressions for M_{n+1} and S_{n+1} in terms of M_n and S_n. [3]

(ii) Use Euler's method to find an estimate for the mackerel population density after one year. [2]

(f) Alessia will use her model to estimate whether the mackerel population density is likely to fall below the minimum value required for sustainable fishing, 5000 per km^3, during the first nine years.

(i) Use Euler's method to sketch the trajectory of the phase portrait, for $4 \le M \le 7$ and $1.5 \le S \le 3$, over the first nine years. [3]

(ii) Using your phase portrait, or otherwise, determine whether the mackerel population density would be sufficient to support sustainable fishing during the first nine years. [2]

(iii) State **two** reasons why Alessia's conclusion, found in part (f)(ii), might not be valid. [2]

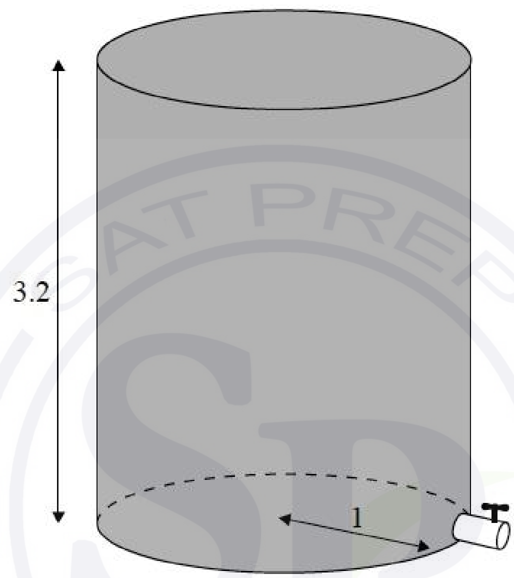
Question 3

[Maximum mark: 30]

This question explores models for the height of water in a cylindrical container as water drains out.

The diagram shows a cylindrical water container of height 3.2 metres and base radius 1 metre. At the base of the container is a small circular valve, which enables water to drain out.

diagram not to scale



Eva closes the valve and fills the container with water.

At time $t = 0$, Eva opens the valve. She records the height, h metres, of water remaining in the container every 5 minutes.

Time, t (minutes)	Height, h (metres)
0	3.2
5	2.4
10	1.6
15	1.1
20	0.5

Eva first tries to model the height using a linear function, $h(t) = at + b$, where a, $b \in \mathbb{R}$.

(a) (i) Find the equation of the regression line of h on t. [2]

(ii) Interpret the meaning of parameter a in the context of the model. [1]

Eva uses the equation of the regression line of h on t, to predict the time it will take for all the water to drain out of the container.

(iii) Suggest why Eva's use of the linear regression equation in this way could be unreliable. [1]

Eva thinks she can improve her model by using a quadratic function, $h(t) = pt^2 + qt + r$, where $p,\ q,\ r \in \mathbb{R}$.

(b) (i) Find the equation of the least squares quadratic regression curve. [1]

Eva uses this equation to predict the time it will take for all the water to drain out of the container and obtains an answer of k minutes.

(ii) Find the value of k. [2]

(iii) Hence, write down a suitable domain for Eva's function $h(t) = pt^2 + qt + r$. [1]

Let V be the volume, in cubic metres, of water in the container at time t minutes.
Let R be the radius, in metres, of the circular valve.

Eva does some research and discovers a formula for the rate of change of V.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\pi R^2 \sqrt{70\,560\,h}$$

(c) Show that $\frac{\mathrm{d}h}{\mathrm{d}t} = -R^2 \sqrt{70\,560\,h}$. [3]

(d) By solving the differential equation $\frac{\mathrm{d}h}{\mathrm{d}t} = -R^2 \sqrt{70\,560\,h}$, show that the general solution is given by $h = 17\,640\,(c - R^2 t)^2$, where $c \in \mathbb{R}$. [5]

Eva measures the radius of the valve to be 0.023 metres. Let T be the time, in minutes, it takes for all the water to drain out of the container.

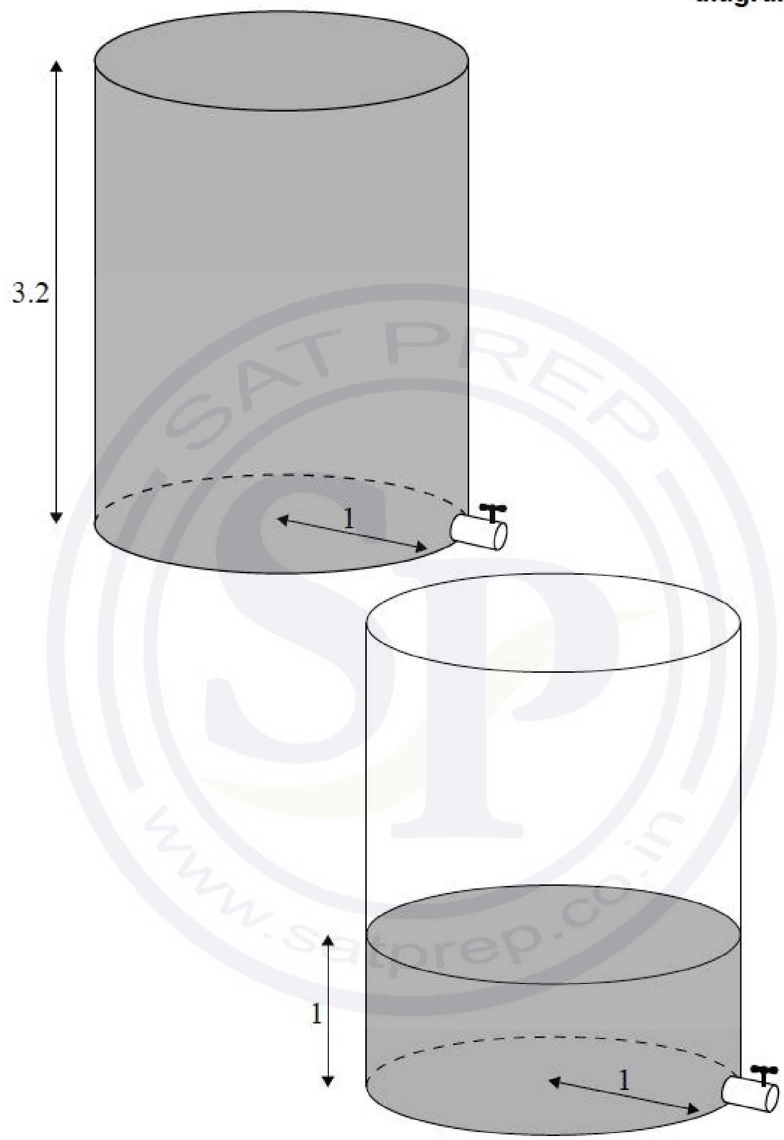
(e) Use the general solution from part (d) and the initial condition $h(0) = 3.2$ to predict the value of T. [4]

Eva wants to use the container as a timer. She adjusts the initial height of water in the container so that all the water will drain out of the container in 15 minutes.

(f) Find this new height. [3]

Eva has another water container that is identical to the first one. She places one water container above the other one, so that all the water from the highest container will drain into the lowest container. Eva completely fills the highest container, but only fills the lowest container to a height of 1 metre, as shown in the diagram.

diagram not to scale



At time $t=0$ Eva opens both valves. Let H be the height of water, in metres, in the lowest container at time t.

(g) (i) Show that $\frac{dH}{dt} \approx 0.2514 - 0.009873t - 0.1405\sqrt{H}$, where $0 \le t \le T$. [4]

(ii) Use Euler's method with a step length of 0.5 minutes to estimate the maximum value of H. [3]

Question 4

[Maximum mark: 28]

This question is about modelling the spread of a computer virus to predict the number of computers in a city which will be infected by the virus.

A systems analyst defines the following variables in a model:

- t is the number of days since the first computer was infected by the virus.
- $Q(t)$ is the total number of computers that have been infected up to and including day t.

The following data were collected:

t	10	15	20	25	30	35	40
$Q(t)$	20	90	403	1806	8070	32 667	120 146

(a) (i) Find the equation of the regression line of $Q(t)$ on t. [2]

(ii) Write down the value of r, Pearson's product-moment correlation coefficient. [1]

(iii) Explain why it would not be appropriate to conduct a hypothesis test on the value of r found in (a)(ii). [1]

A model for the early stage of the spread of the computer virus suggests that

$$Q'(t) = \beta N Q(t)$$

where N is the total number of computers in a city and β is a measure of how easily the virus is spreading between computers. Both N and β are assumed to be constant.

(b) (i) Find the general solution of the differential equation $Q'(t) = \beta N Q(t)$. [4]

(ii) Using the data in the table write down the equation for an appropriate non-linear regression model. [2]

(iii) Write down the value of R^2 for this model. [1]

(iv) Hence comment on the suitability of the model from (b)(ii) in comparison with the linear model found in part (a). [2]

(v) By considering large values of t write down one criticism of the model found in (b)(ii). [1]

(c) Use your answer from part (b)(ii) to estimate the time taken for the number of infected computers to double. [2]

The data above are taken from city X which is estimated to have 2.6 million computers. The analyst looks at data for another city, Y. These data indicate a value of $\beta = 9.64 \times 10^{-8}$.

(d) Find in which city, X or Y, the computer virus is spreading more easily. Justify your answer using your results from part (b). [3]

An estimate for $Q'(t)$, $t \geq 5$, can be found by using the formula:

$$Q'(t) \approx \frac{Q(t+5) - Q(t-5)}{10}.$$

The following table shows estimates of $Q'(t)$ for city X at different values of t.

t	10	15	20	25	30	35	40
$Q(t)$	20	90	403	1806	8070	32 667	120 146
$Q'(t)$		a	171.6	766.7	b	11 207.6	

(e) Determine the value of a and of b. Give your answers correct to one decimal place. [2]

An improved model for $Q(t)$, which is valid for large values of t, is the logistic differential equation

$$Q'(t) = kQ(t)\left(1 - \frac{Q(t)}{L}\right)$$

where k and L are constants.

Based on this differential equation, the graph of $\frac{Q'(t)}{Q(t)}$ against $Q(t)$ is predicted to be a straight line.

(f) (i) Use linear regression to estimate the value of k and of L. [5]

(ii) The solution to the differential equation is given by

$$Q(t) = \frac{L}{1 + Ce^{-kt}}$$

where C is a constant.

Using your answer to part (f)(i), estimate the percentage of computers in city X that are expected to have been infected by the virus over a long period of time. [2]

Question 5

[Maximum mark: 26]

Some medical conditions require patients to take medication regularly for long periods of time. In this question, you will explore the concentration of a medicinal drug in the body, when the drug is given repeatedly.

Once a drug enters the body, it is absorbed into the blood. As the body breaks down the drug over time, the concentration of the drug decreases. Let $C(t)$, measured in milligrams per millilitre (mg ml^{-1}), be the concentration of the drug, t hours after the drug is given to the patient. The rate at which the drug is broken down is modelled as directly proportional to its concentration, leading to the differential equation

$$\frac{\mathrm{d}C}{\mathrm{d}t} = -kC \text{, where } k \in \mathbb{R}^+ .$$

The initial concentration is $d\ \text{mg ml}^{-1}$, $d > 0$.

(a) By solving the differential equation, show that $C = d\mathrm{e}^{-kt}$. [3]

For the remainder of this question, you will consider a particular drug where it is known that $k = 0.2$. The first dose is given at time $t = 0$ and it is assumed that before this there is no drug present in the blood.

(b) Find the time, in hours, for this drug to reach 5% of its initial concentration. [2]

The drug is to be given every T hours and in constant doses, such that the concentration of the drug is increased by an amount $d\ \text{mg ml}^{-1}$. To simplify the model, it is assumed that each time the drug is given the concentration of the drug in the blood increases instantaneously.

(c) Show that the concentration of the drug is $d\left(1 + \mathrm{e}^{-0.2T} + \mathrm{e}^{-0.4T}\right)$ immediately after the third dose is given. [4]

Immediately after the nth dose is given, the concentration of the drug is

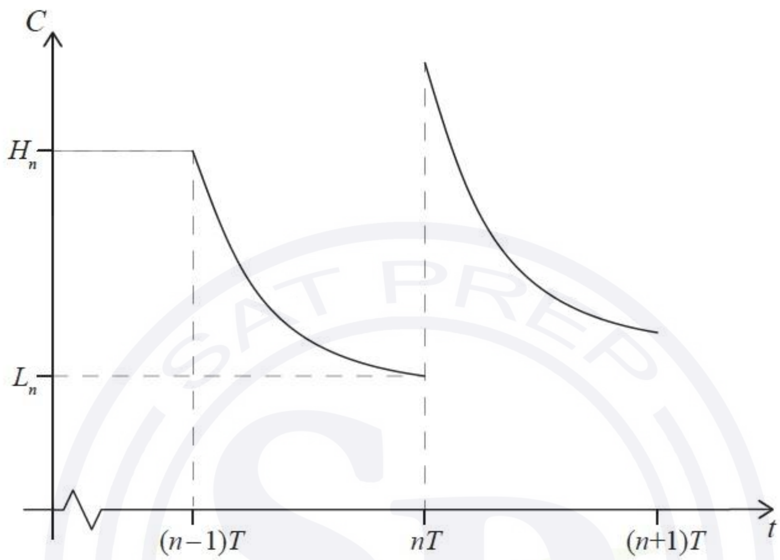
$$d\left(1 + \mathrm{e}^{-0.2T} + \mathrm{e}^{-0.4T} + \ldots + \mathrm{e}^{-0.2(n-1)T}\right).$$

(d) Show that this concentration can be expressed as $d\left(\frac{1 - \mathrm{e}^{-0.2nT}}{1 - \mathrm{e}^{-0.2T}}\right)$. [2]

After a patient has been taking this drug for a long time, it is required to keep the concentration within a particular range so that it is both safe and effective.

Let H_n be the highest concentration of the drug in the body for the interval $(n-1)T \le t < nT$.

Let L_n be the lowest concentration of the drug in the body for the interval $(n-1)T \le t < nT$. This is shown in the following graph.



H_∞ is defined as $\lim_{n\to\infty} H_n$ and L_∞ is defined as $\lim_{n\to\infty} L_n$.

(e) Find, in terms of d and T, an expression for

(i) H_∞. [2]

(ii) L_∞. [3]

(f) Show that

(i) $H_\infty - L_\infty = d$. [2]

(ii) $5\ln\left(\frac{H_\infty}{L_\infty}\right) = T$. [3]

It is known that this drug is ineffective if the long-term concentration is less than $0.06\,\mathrm{mg\,ml^{-1}}$ and safe if it never exceeds $0.28\,\mathrm{mg\,ml^{-1}}$.

(g) Hence, for this drug, find a suitable value for

(i) d. [1]

(ii) T. [1]

(h) For the values of d and T found in part (g), find the proportion of time for which the concentration of the drug is at least $0.06\,\mathrm{mg\,ml^{-1}}$ between the first and second doses. [2]

(i) Suggest a reason why the instructions on the label of the drug might use a different value for T to that found in part (g)(ii). [1]