Subject – Math AI(Higher Level)
Topic - Geometry and Trigonometry
Year - May 2021 – Nov 2022
Paper -3
Answers

Question 1

(a) recognition that period $= 669$ *(M1)*

$b = \frac{2\pi}{669}$ **OR** $b = 0.00939190\ldots$ **A1**

> **Note:** Award ***A1*** for a correct expression leading to the given value or for a correct value of b to 4 sf or greater accuracy.

$b \approx 0.00939$ **AG**

[2 marks]

(b) length of day = $24\frac{2}{3}$ hours *(A1)*

> **Note:** Award ***A1*** for $\frac{2}{3}$, $0.666\ldots$, $0.\overline{6}$ or 0.667.

$\frac{2\pi}{24\frac{2}{3}}$ *(M1)*

> **Note:** Accept $\left(\frac{360}{24\frac{2}{3}}\right)$.

$= 0.255$ radians $\left(= 0.254723\ldots, \frac{3\pi}{37}, 14.5945\ldots^\circ\right)$ **A1**

[3 marks]

(c) (i) substitution of either value of δ into equation *(M1)*
correct use of arccos to find a value for ω *(M1)*

> **Note:** Both ***(M1)*** lines may be seen in either part (c)(i) or part (c)(ii).

$\cos\omega = 0.839\tan(-0.440)$ **A1**
$\omega = 1.97684\ldots$

≈ 1.98 **AG**

> **Note:** For substitution of 1.98 award ***M0A0***.

[3 marks]

(ii) $\delta = 0.440$
$\omega = 1.16$ $(1.16474\ldots)$ **A1**

[1 mark]

(d) (i) $R_{\max} = \dfrac{1.97684...}{0.25472...}$ **(M1)**

$= 7.76$ hours $(7.76075...)$ **A1**

Note: Accept 7.70 from use of 1.98.

[2 marks]

(ii) $R_{\min} = \dfrac{1.16474...}{0.25472...}$

$= 4.57$ hours $(4.57258...)$ **A1**

[1 mark]

Note: Accept 4.55 and 4.56 from use of rounded values.

(e) $a = \dfrac{7.76075... - 4.57258...}{2}$ **M1**

$\approx 1.59408...$ **A1**

Note: Award ***M1*** for substituting their values into a correct expression.
Award ***A1*** for a correct value of a from their expression which has at least 3 significant figures and rounds correctly to 1.6.

≈ 1.6 (correct to 2 sf) **AG**

[2 marks]

(f) **EITHER**

$c = \dfrac{7.76075... + 4.57258...}{2} \left(= \dfrac{12.333...}{2} \right)$ **(M1)**

OR

$c = 4.57258... + 1.59408...$ or $c = 7.76075... - 1.59408...$

THEN

$= 6.17$ $(6.16666...)$ **A1**

Note: Accept 6.16 from use of rounded values.
Follow through on their answers to part (d) and 1.6.

[2 marks]

(g) $d = 18.65 - 6.16666...$ **(M1)**

$= 12.5$ $(12.4833...)$ **A1**

Note: Follow through for 18.65 minus their answer to part (f).

[2 marks]

(h) (i) at least one expression in the form $r\mathrm{e}^{g(t)\mathrm{i}}$ ***(M1)***

$z_1 = 1.5\mathrm{e}^{(0.00939t+2.83)\mathrm{i}}$, $z_2 = 1.6\mathrm{e}^{(0.00939t)\mathrm{i}}$ ***A1A1***

[3 marks]

(ii) **EITHER**

$z_1 - z_2 = 1.5\mathrm{e}^{(0.00939t+2.83)\mathrm{i}} - 1.6\mathrm{e}^{(0.00939t)\mathrm{i}}$

$= \mathrm{e}^{0.00939t\mathrm{i}}\left(1.5\mathrm{e}^{2.83\mathrm{i}} - 1.6\right)$ ***(M1)***

$= \mathrm{e}^{0.00939t\mathrm{i}}\left(3.06249\ldots\mathrm{e}^{2.99086\ldots\mathrm{i}}\right)$ ***(A1)(A1)***

OR

graph of L or f

$p = 3.06249\ldots$ ***(A1)***

$r = -0.150729\ldots$ **OR** $r = 2.99086\ldots$ ***(M1)(A1)***

Note: The p and r variables (or equivalent) must be seen.

THEN

$L(t) = 3.06\sin(0.00939t + 2.99) + 12.5$ ***A1***

$\left(L(t) = 3.06248\ldots\sin(0.00939t + 2.99086\ldots) + 12.4833\ldots\right)$

Note: Accept equivalent forms, e.g. $L(t) = 3.06\sin(0.00939t - 0.151) + 12.5$.
Follow through on their answer to part (g) replacing 12.5.

[4 marks]

(iii) shortest time between sunrise and sunset

$12.4833\ldots - 3.06249\ldots$ ***(M1)***

$= 9.42$ hours $(9.420843\ldots)$ ***A1***

Note: Accept 9.44 from use of 3 sf values.

[2 marks]

[Total 27 marks]

Question 2

(a) e.g. ABCDEGHFA **A1**

Note: Accept any other correct answers starting at any vertex.

(b) (i) 7 vertices, so 6 edges required for MST ***(M1)***

Note: To award ***(M1)***, their 6 edges should not form a cycle.

Order selected	Edge	
1	BC	
2	CD	
3	DE	***M1***
4	BH	***A1***
5	GH	
6	FG	***A1***

Note: Award ***M1*** for the first three edges in correct order, ***A1*** for BH in correct order and ***A1*** for all of the edges correct.

weight of MST = 33 ***A1***

Note: The final ***A1*** can be awarded independently of previous marks.

(ii) lower bound = $33+3+x$ ***(M1)***
$=36+x$ ***A1***

(c) (i) $p=13$ ***A1***

(ii) $q=17$ ***A1***

(iii) $r=14$ ***A1***

(d) (i) attempt to use nearest neighbour algorithm ***(M1)***

any two correct cycles from
ABCDEGHFA, AFGHBCDE(F)A, AB(A)FGHCDE(F)A ***A1A1***

Note: Bracketed vertices may be omitted in candidate's answer.
Award ***M1A0A1*** for candidates who list two correct sequences of vertices, but omit the final vertex A.

(ii) use ABCDEGHFA **OR** their shortest cycle from (d)(i) ***(M1)***
upper bound = 43 ***A1***

(e) (i) cycle starts: ABCDEGHF
return to A has two options, $\text{FA} = 18$ or x *(M1)*
hence least value of $x = 19$ ***A1***

(ii) upper bound $= 58$ ***A2***

(f) recognition that edges will be repeated / there are odd vertices *(M1)*
$\text{BH} + \text{DG} = 21$, $\text{BD} + \text{GH} = 15$, $\text{BG} + \text{DH} = 21$ **OR** $18 + x$ ***A1***
recognizing BD and GH is lowest weight and is repeated *(M1)*
solution to CPP $= 107 + x$ ***A1***
$x = 13$ ***A1***

Note: Award ***M1A0M0A1A1*** if only pairing BD and GH is considered, leading to a correct answer.

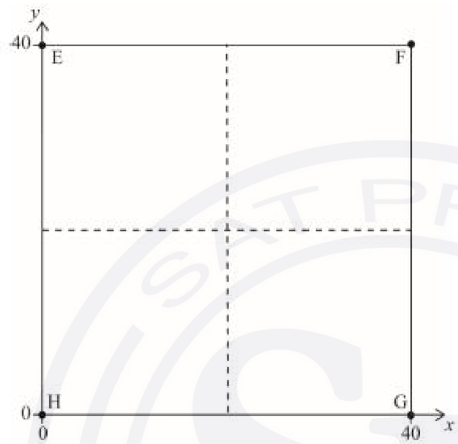
Question 3

(a) the size of each town is small (in comparison with the distance between the towns)
OR
if towns have an identifiable centre
OR
the centre of the town is at that point *R1*

Note: Accept a geographical landmark in place of "centre", e.g. "town hall" or "capitol".

[1 mark]

(b)



A1

Note: There is no need for a scale / coordinates here. Condone boundaries extending beyond the metropolitan area.

[1 mark]

(c) (i) the gradient of IF is $\frac{40-20}{40-30}=2$ *(A1)*

negative reciprocal of any gradient *(M1)*

gradient of perpendicular bisector $=-\frac{1}{2}$

Note: Seeing $-\frac{2}{3}$ (for example) used clearly as a gradient anywhere is evidence of the "negative reciprocal" method despite being applied to an inappropriate gradient.

midpoint is $\left(\frac{40+30}{2}, \frac{40+20}{2}\right)=(35,\ 30)$ *(A1)*

equation of perpendicular bisector is $y-30=-\frac{1}{2}(x-35)$ *A1*

Note: Accept equivalent forms e.g. $y=-\frac{1}{2}x+\frac{95}{2}$ or $2y+x-95=0$.
Allow ***FT*** for the final ***A1*** from their midpoint and gradient of perpendicular bisector, as long as the ***M1*** has been awarded.

[4 marks]

(ii) the perpendicular bisector of EH is $y = 20$ *(A1)*

> **Note:** Award this ***A1*** if seen in the y-coordinate of any final answer or if 20 is used as the y-value in the equation of any other perpendicular bisector.

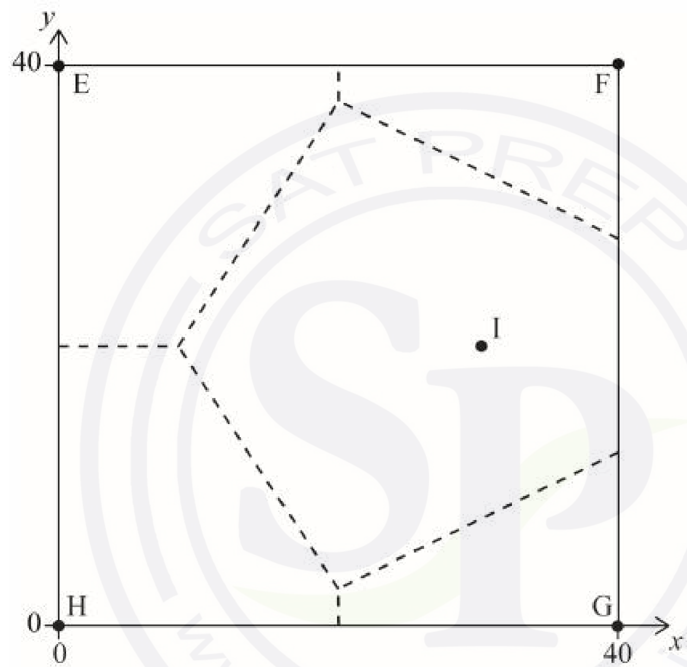
attempt to use symmetry **OR** intersecting two perpendicular bisectors *(M1)*

$\left(\frac{25}{3}, 20\right)$ ***A1***

$(20, 2.5)$ ***A1***

[4 marks]

(iii)



M1A1

> **Note:** Award ***M1*** for exactly four perpendicular bisectors around I (IE, IF, IG and IH) seen, even if not in exactly the right place.
> Award ***A1*** for a completely correct diagram. Scale / coordinates are NOT necessary. Vertices should be in approximately the correct positions but only penalized if clearly wrong (condone northern and southern vertices appearing to be very close to the boundary).
>
> Condone the Voronoi diagram extending outside of the square.
> Do not award follow-though marks in this part.

[2 marks]

(d) 30% of 40 is 12 **(A1)**

recognizing line intersects bisectors at $y = c$ (or equivalent) but different x-values **(M1)**

$c = \frac{3}{2}x_1 + \frac{15}{2}$ and $c = -\frac{1}{2}x_2 + \frac{95}{2}$

finding an expression for the distance in Isaacopolis in terms of one variable **(M1)**

$x_2 - x_1 = (95 - 2c) - \frac{2c - 15}{3} = 100 - \frac{8c}{3}$

equating their expression to 12

$100 - \frac{8c}{3} = 0.3 \times 40 = 12$

$c = 33$

distance $= 33$ (km) ***A1***

[4 marks]

(e) (i) must be a vertex (award if vertex given as a final answer) **(R1)**
attempt to calculate the distance of at least one town from a vertex **(M1)**

Note:	This must be seen as a calculation or a value.

correct calculation of distances ***A1***

$\frac{65}{3}$ **OR** 21.7 **AND** $\sqrt{406.25}$ **OR** 20.2

$\left(\frac{25}{3}, 20\right)$ ***A1***

Note:	Award ***R1M0A0A0*** for a vertex written with no other supporting calculations. Award ***R1M0A0A1*** for correct vertex with no other supporting calculations. The final ***A1*** is not dependent on the previous ***A1***. There is no follow-through for the final ***A1***. Do not accept an answer based on “uniqueness” in the question.

[4 marks]

(ii) *For example, any one of the following:*
decision does not take into account the different population densities
closer to a city will reduce travel time/help employees
it is closer to some cites than others ***R1***

Note:	Accept any correct reason that engages with the scenario. Do not accept any answer to do with ethical issues about whether toxic waste should ever be dumped, or dumped in a metropolitan area.

[1 mark]

(f) (i) **METHOD 1**

attempting $\boldsymbol{M}^3$ **_M1_**

attempting $\boldsymbol{M}^4$ **_M1_**

e.g.

last row/column of $\boldsymbol{M}^3 = \begin{pmatrix} 3 & 5 & 1 & 6 & 0 & 7 \end{pmatrix}$

last row/column of $\boldsymbol{M}^4 = \begin{pmatrix} 10 & 12 & 4 & 16 & 1 & 18 \end{pmatrix}$

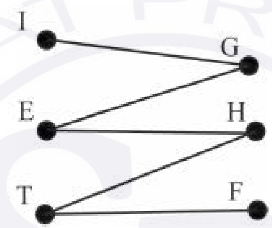
hence Isaacopolis is the last city to be polluted **_A1_**

Note: Do not award the ***A1*** unless both $\boldsymbol{M}^3$ and $\boldsymbol{M}^4$ are considered. Award ***M1M0A0*** for a claim that the shortest distance is from T to I and that it is 4, without any support.

METHOD 2

attempting to translate $\boldsymbol{M}$ to a graph or a list of cities polluted on each day ***(M1)***

correct graph or list **_A1_**



hence Isaacopolis is the last city to be polluted **_A1_**

Note: Award ***M1A1A1*** for a clear description of the graph in words leading to the correct answer.

[3 marks]

(ii) it takes 4 days **_A1_**

[1 mark]

(iii) **EITHER**

the orders of the different vertices are:

E	2
F	1
G	2
H	2
I	1
T	2

(A1)

Note: Accept a list where each order is 2 greater than listed above.

OR

a correct diagram/graph showing the connections between the locations ***(A1)***

Note: Accept a diagram with loops at each vertex.
This mark should be awarded if candidate is clearly using their correct diagram from the previous part.

THEN

"Start at F and end at I" **OR** *"Start at I and end at F"* ***A1***

Note: Award ***A1A0*** for *"it could start at either F or I"*.
Award ***A1A1*** for *"IGEHTF"* **OR** *"FTHEGI"*.
Award ***A1A1*** for *"F and I"* **OR** *"I and F"*.

[2 marks]

[Total 27 marks]

Question 4

(a) $AF^2 = 89.2^2 + 104.9^2 - 2(89.2)(104.9)\cos 83$ **(M1)(A1)**

Note: Award ***(M1)*** for substitution into the cosine rule and ***(A1)*** for correct substitution.

$AF = 129 \text{ m} \quad (129.150...)$ ***A1***

[3 marks]

(b) $21310 \div 129.150...$ ***(M1)***

$\$\,165$ ***A1***

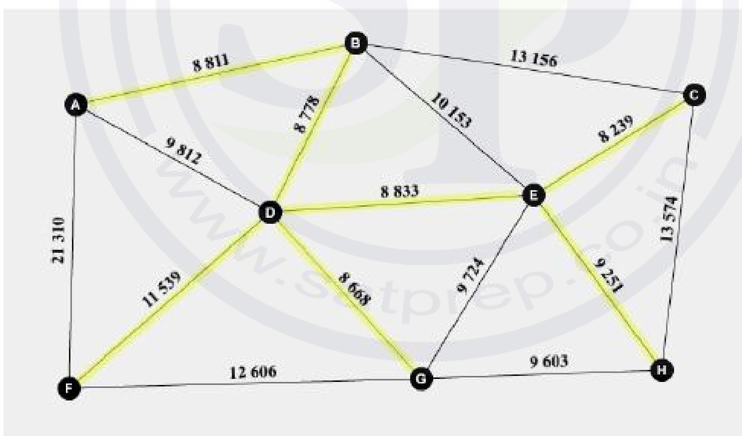
[2 marks]

(c) any reasonable statement referring to the lake ***R1***
(eg. there is a lake between A and F, the cables would need to be installed under/over/around the lake, special waterproof cables are needed for lake, etc.)

[1 mark]

(d) (i) edges (or weights) are chosen in the order

CE (8239)
DG (8668)
BD (8778)
AB (8811)
DE (8833)
EH (9251)
DF (11 539) ***A1A1A1***



Note: Award ***A1*** for the first two edges chosen in the correct order. Award ***A1A1*** for the first six edges chosen in the correct order. Award ***A1A1A1*** for all seven edges chosen in the correct order. Accept a diagram as an answer, provided the order of edges is communicated.

[3 marks]

(ii) Finding the sum of the weights of their edges ***(M1)***
$8239 + 8668 + 8778 + 8811 + 8833 + 9251 + 11539$

total cost = $\$64119$ ***A1***

[2 marks]

(g) attempt to find MST after deleting vertex D *(M1)*

these edges (or weights) (in any order)

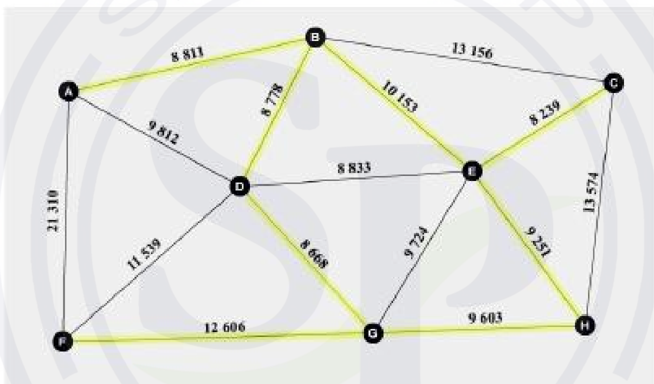
CE (8239)
AB (8811)
EH (9251)
GH (9603)
BE (10 153)
FG (12 606) *A1*

Note: Prim's or Kruskal's algorithm could be used at this stage.

reconnect D to MST with two different edges *(M1)*

DG (8668)
BD (8778) *A1*

Note: This ***A1*** is independent of the first ***A*** mark and can be awarded if both DG and BD are chosen to reconnect D to the MST, even if the MST is incorrect.



finding the sum of the weights of their edges *(M1)*

$8239+8811+9251+9603+10153+12606+8668+8778$

Note: For candidates with an incorrect MST or no MST, the weights of at least seven of the edges being summed (two of which must connect to D) must be shown to award this ***(M1)***.

lower bound = \$76109 *A1*

[6 marks]

(h) **METHOD 1**

recognition of a binomial distribution ***(M1)***

$X \sim \mathrm{B}(2,\ 0.014)$

finding the probability that a cable fails (at least one of its connections fails)

$\mathrm{P}(X > 0) = 0.027804$ **OR** $1 - \mathrm{P}(X = 0) = 0.027804$ ***A1***

recognition that **two** cables must fail for the network to go offline ***M1***

recognition of binomial distribution for network, $Y \sim \mathrm{B}(8,\ 0.027804)$ ***(M1)***

$\mathrm{P}(Y \geq 2) = 0.0194\ \ (0.0193602...)$ **OR** $1 - \mathrm{P}(Y < 2) = 0.0194\ \ (0.0193602\ldots)$ ***A1***

therefore, the diagram satisfies the requirement since $1.94\% < 2\%$ ***AG***

Note: Evidence of binomial distribution may be seen as combinations.

METHOD 2

recognition of a binomial distribution ***(M1)***

$X \sim \mathrm{B}(16,\ 0.014)$

finding the probability that at least **two** connections fail

$\mathrm{P}(X \geq 2) = 0.0206473\ldots$ **OR** $1 - \mathrm{P}(X < 2) = 0.0206473\ldots$ ***A1***

recognition that the previous answer is an overestimate ***M1***

finding probability of two ends of the same cable failing, $F \sim \mathrm{B}(2,\ 0.014)$,

and the ends of the other 14 cables not failing, $S \sim \mathrm{B}(14,\ 0.014)$

$\mathrm{P}(F = 2) \times \mathrm{P}(S = 0) = 0.0000160891\ldots$ ***(A1)***

$0.0000160891\ldots \times 8 = 0.00128713\ldots$

$0.0206473\ldots - 0.00128713\ldots = 0.0194\ \ (0.0193602\ldots)$ ***A1***

therefore, the diagram satisfies the requirement since $1.94\% < 2\%$ ***AG***

METHOD 3

recognition of a binomial distribution ***M1***

$X \sim \mathrm{B}(16,\ 0.014)$

finding the probability that the network remains secure if 0 or 1 connections fail or if 2 connections fail provided that the second failed connection occurs at the other end of the cable with the first failure ***(M1)***

$\mathrm{P}(\text{remains secure}) = \mathrm{P}(X \leq 1) + \frac{1}{15} \times \mathrm{P}(X = 2)$ ***A1***

$= 0.9806397625$ ***A1***

$\mathrm{P}(\text{network fails}) = 1 - 0.9806397625 = 0.0194\ \ (0.0193602...)$ ***A1***

therefore, the diagram satisfies the requirement since $1.94\% < 2\%$ ***AG***

METHOD 4

P(network failing)

$= 1 - \mathrm{P}$(0 connections failing) $-$ P(1 connection failing)

$-$ P(2 connections on the same cable failing) ***M1***

$= 1 - 0.986^{16} - {}^{16}C_1 \times 0.014 \times 0.986^{15} - {}^{8}C_1 \times 0.014^2 \times 0.986^{14}$ ***A1A1A1***

Note: Award ***A1*** for each of 2nd, 3rd and last terms.

$= 0.0194 \ (0.0193602\ldots)$ ***A1***

therefore, the diagram satisfies the requirement since $1.94\% < 2\%$ ***AG***

[5 marks]

[Total 28 marks]