Subject – Math AI(Higher Level)
Topic - Geometry and Trigonometry
Year - May 2021 – Nov 2022
Paper -3
Questions

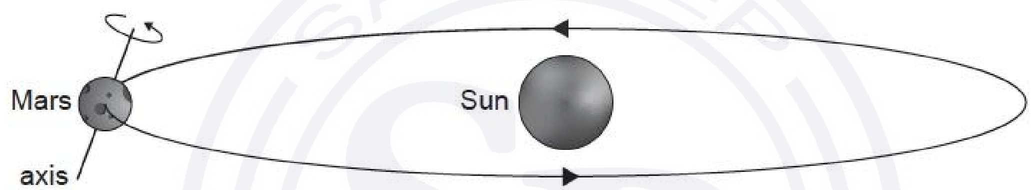
Question 1

[Maximum mark: 27]

A suitable site for the landing of a spacecraft on the planet Mars is identified at a point, A. The shortest time from sunrise to sunset at point A must be found.

Radians should be used throughout this question. All values given in the question should be treated as exact.

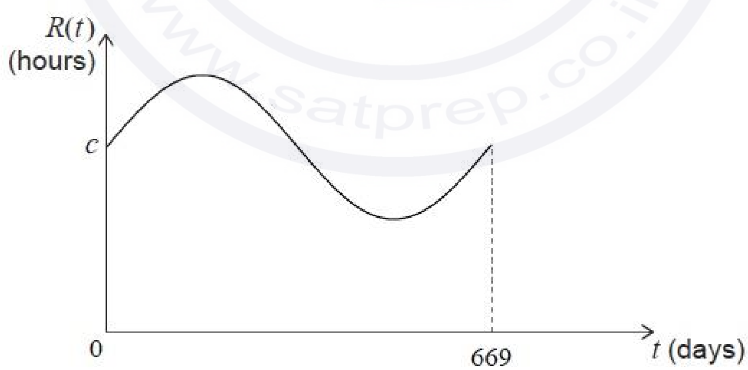
Mars completes a full orbit of the Sun in 669 Martian days, which is one Martian year.



On day t, where $t \in \mathbb{Z}$, the length of time, in hours, from the start of the Martian day until sunrise at point A can be modelled by a function, $R(t)$, where

$$R(t) = a\sin(bt) + c,\ t \in \mathbb{R}.$$

The graph of R is shown for one Martian year.



(a) Show that $b \approx 0.00939$. [2]

Mars completes a full rotation on its axis in 24 hours and 40 minutes.

(b) Find the angle through which Mars rotates on its axis each hour. [3]

The time of sunrise on Mars depends on the angle, δ, at which it tilts towards the Sun. During a Martian year, δ varies from -0.440 to 0.440 radians.

The angle, ω, through which Mars rotates on its axis from the start of a Martian day to the moment of sunrise, at point A, is given by $\cos\omega = 0.839\tan\delta,\ 0 \le \omega \le \pi$.

(c) (i) Show that the maximum value of $\omega = 1.98$, correct to three significant figures. [3]

(ii) Find the minimum value of ω. [1]

(d) Use your answers to parts (b) and (c) to find

(i) the maximum value of $R(t)$; [2]

(ii) the minimum value of $R(t)$. [1]

(e) Hence show that $a = 1.6$, correct to two significant figures. [2]

(f) Find the value of c. [2]

Let $S(t)$ be the length of time, in hours, from the start of the Martian day until **sunset** at point A on day t. $S(t)$ can be modelled by the function

$$S(t) = 1.5\sin(0.00939t + 2.83) + 18.65.$$

The length of time between sunrise and sunset at point A, $L(t)$, can be modelled by the function

$$L(t) = 1.5\sin(0.00939t + 2.83) - 1.6\sin(0.00939t) + d.$$

(g) Find the value of d. [2]

Let $f(t) = 1.5\sin(0.00939t + 2.83) - 1.6\sin(0.00939t)$ and hence $L(t) = f(t) + d$.

$f(t)$ can be written in the form $\mathrm{Im}(z_1 - z_2)$, where z_1 and z_2 are complex functions of t.

(h) (i) Write down z_1 and z_2 in exponential form, with a constant modulus. [3]

(ii) Hence or otherwise find an equation for L in the form $L(t) = p\sin(qt + r) + d$, where p, q, r, $d \in \mathbb{R}$. [4]

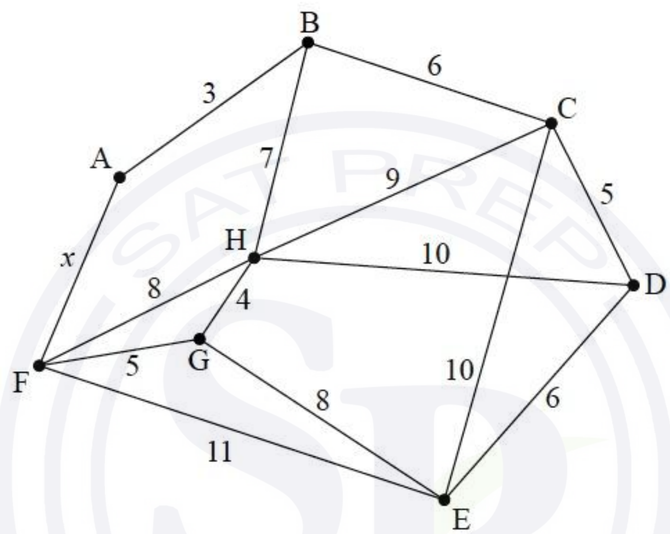
(iii) Find, in hours, the shortest time from sunrise to sunset at point A that is predicted by this model. [2]

Question 2

[Maximum mark: 25]

This question explores how graph algorithms can be applied to a graph with an unknown edge weight.

Graph W is shown in the following diagram. The vertices of W represent tourist attractions in a city. The weight of each edge represents the travel time, to the nearest minute, between two attractions. The route between A and F is currently being resurfaced and this has led to a variable travel time. For this reason, AF has an unknown travel time x minutes, where $x \in \mathbb{Z}^+$.



(a) Write down a Hamiltonian cycle in W. [1]

Daniel plans to visit all the attractions, starting and finishing at A. He wants to minimize his travel time.

To find a lower bound for Daniel's travel time, vertex A and its adjacent edges are first deleted.

(b) (i) Use Prim's algorithm, starting at vertex B, to find the weight of the minimum spanning tree of the remaining graph. You should indicate clearly the order in which the algorithm selects each edge. [5]

(ii) Hence, for the case where $x < 9$, find a lower bound for Daniel's travel time, in terms of x. [2]

Daniel makes a table to show the minimum travel time between each pair of attractions.

	A	B	C	D	E	F	G	H
A		3	9	14	19 **or** $(11+x)$	18 **or** x	14 **or** $(5+x)$	10 **or** $(8+x)$
B			6	11	16 **or** $(14+x)$	15 **or** $(3+x)$	11 **or** $(8+x)$	7
C				5	10	17 **or** $(9+x)$	p	9
D					6	q **or** $(r+x)$	14	10
E						11	8	12
F							5	8
G								4
H								

(c) Write down the value of

(i) p; [1]

(ii) q; [1]

(iii) r. [1]

To find an upper bound for Daniel's travel time, the nearest neighbour algorithm is used, starting at vertex A.

(d) Consider the case where $x = 3$.

(i) Use the nearest neighbour algorithm to find two possible cycles. [3]

(ii) Find the best upper bound for Daniel's travel time. [2]

(e) Consider the case where $x > 3$.

(i) Find the least value of x for which the edge AF will definitely not be used by Daniel. [2]

(ii) Hence state the value of the upper bound for Daniel's travel time for the value of x found in part (e)(i). [2]

The tourist office in the city has received complaints about the lack of cleanliness of some routes between the attractions. Corinne, the office manager, decides to inspect all the routes between all the attractions, starting and finishing at H. The sum of the weights of all the edges in graph W is $(92+x)$.

Corinne inspects all the routes as quickly as possible and takes 2 hours.

(f) Find the value of x during Corinne's inspection. [5]

Question 3

[Maximum mark: 27]

This question is about a metropolitan area council planning a new town and the location of a new toxic waste dump.

A metropolitan area in a country is modelled as a square. The area has four towns, located at the corners of the square. All units are in kilometres with the x-coordinate representing the distance east and the y-coordinate representing the distance north from the origin at $(0, 0)$.

- Edison is modelled as being positioned at $\mathrm{E}(0, 40)$.
- Fermitown is modelled as being positioned at $\mathrm{F}(40, 40)$.
- Gaussville is modelled as being positioned at $\mathrm{G}(40, 0)$.
- Hamilton is modelled as being positioned at $\mathrm{H}(0, 0)$.

(a) The model assumes that each town is positioned at a single point. Describe possible circumstances in which this modelling assumption is reasonable. [1]

(b) Sketch a Voronoi diagram showing the regions within the metropolitan area that are closest to each town. [1]

The metropolitan area council decides to build a new town called Isaacopolis located at $\mathrm{I}(30, 20)$.

A new Voronoi diagram is to be created to include Isaacopolis. The equation of the perpendicular bisector of [IE] is $y = \frac{3}{2}x + \frac{15}{2}$.

(c) (i) Find the equation of the perpendicular bisector of [IF]. [4]

(ii) Given that the coordinates of one vertex of the new Voronoi diagram are $(20, 37.5)$, find the coordinates of the other two vertices within the metropolitan area. [4]

(iii) Sketch this new Voronoi diagram showing the regions within the metropolitan area which are closest to each town. [2]

The metropolitan area is divided into districts based on the Voronoi regions found in part (c).

(d) A car departs from a point due north of Hamilton. It travels due east at constant speed to a destination point due North of Gaussville. It passes through the Edison, Isaacopolis and Fermitown districts. The car spends 30% of the travel time in the Isaacopolis district.

Find the distance between Gaussville and the car's destination point. [4]

A toxic waste dump needs to be located within the metropolitan area. The council wants to locate it as far as possible from the nearest town.

(e) (i) Find the location of the toxic waste dump, given that this location is not on the edge of the metropolitan area. [4]

(ii) Make one possible criticism of the council's choice of location. [1]

(f) The toxic waste dump, T, is connected to the towns via a system of sewers.

The connections are represented in the following matrix, $\boldsymbol{M}$, where the order of rows and columns is (E, F, G, H, I, T).

$$\boldsymbol{M} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

A leak occurs from the toxic waste dump and travels through the sewers. The pollution takes one day to travel between locations that are directly connected.

The digit 1 in $\boldsymbol{M}$ represents a direct connection. The values of 1 in the leading diagonal of $\boldsymbol{M}$ mean that once a location is polluted it will stay polluted.

(i) Find which town is last to be polluted. Justify your answer. [3]

(ii) Write down the number of days it takes for the pollution to reach the last town. [1]

(iii) A sewer inspector needs to plan the shortest possible route through each of the connections between different locations. Determine an appropriate start point and an appropriate end point of the inspection route.

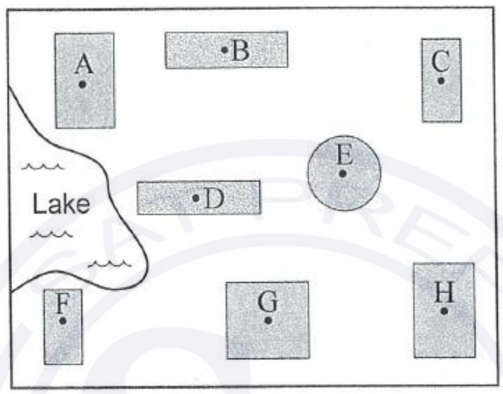
Note that the fact that each location is connected to itself does not correspond to a sewer that needs to be inspected. [2]

Question 4

[Maximum mark: 28]

This question compares possible designs for a new computer network between multiple school buildings, and whether they meet specific requirements.

A school's administration team decides to install new fibre-optic internet cables underground. The school has eight buildings that need to be connected by these cables. A map of the school is shown below, with the internet access point of each building labelled A–H.



Jonas is planning where to install the underground cables. He begins by determining the distances, in metres, between the underground access points in each of the buildings.

He finds $AD = 89.2\,m$, $DF = 104.9\,m$ and $A\hat{D}F = 83^\circ$.

(a) Find AF. [3]

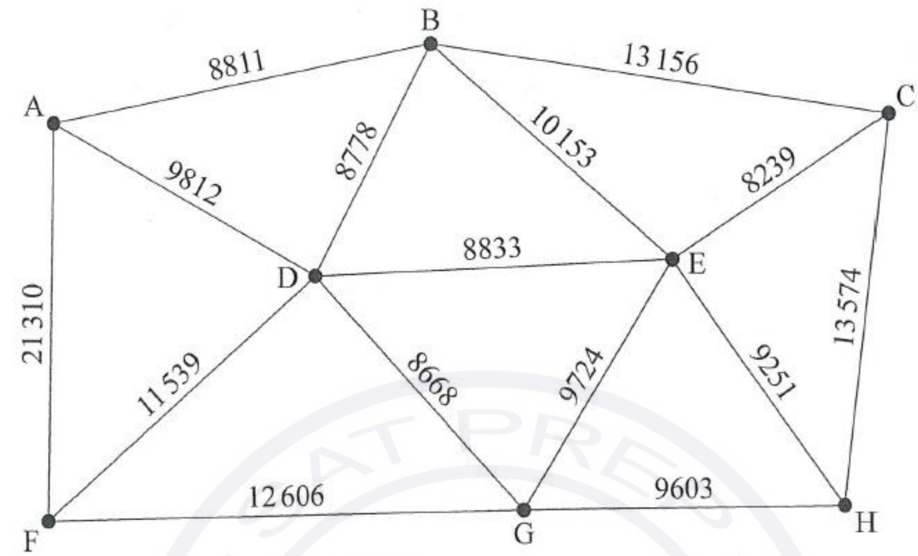
The cost for installing the cable directly between A and F is $21 310.

(b) Find the cost per metre of installing this cable. [2]

Jonas estimates that it will cost $110 per metre to install the cables between all the other buildings.

(c) State why the cost for installing the cable between A and F would be higher than between the other buildings. [1]

Jonas creates the following graph, S, using the cost of installing the cables between two buildings as the weight of each edge.



The computer network could be designed such that each building is directly connected to at least one other building and hence all buildings are indirectly connected.

(d) (i) By using Kruskal's algorithm, find the minimum spanning tree for S, showing clearly the order in which edges are added. [3]

(ii) Hence find the minimum installation cost for the cables that would allow all the buildings to be part of the computer network. [2]

The computer network fails if any part of it becomes unreachable from any other part. To help protect the network from failing, every building could be connected to at least two other buildings. In this way if one connection breaks, the building is still part of the computer network. Jonas can achieve this by finding a Hamiltonian cycle within the graph.

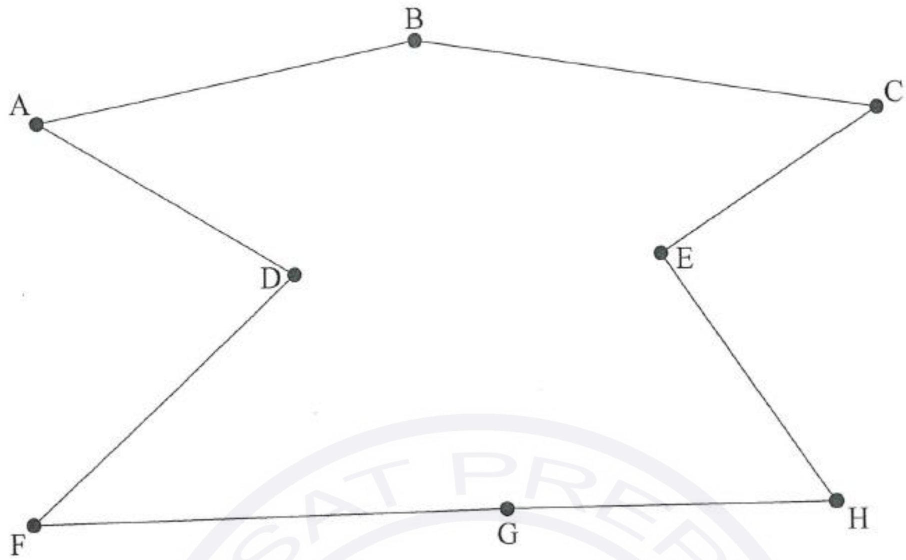
(e) State why a path that forms a Hamiltonian cycle does not always form an Eulerian circuit. [1]

(f) Starting at D, use the nearest neighbour algorithm to find the upper bound for the installation cost of a computer network in the form of a Hamiltonian cycle.

Note: Although the graph is not complete, in this instance it is not necessary to form a table of least distances. [5]

(g) By deleting D, use the deleted vertex algorithm to find the lower bound for the installation cost of the cycle. [6]

After more research, Jonas decides to install the cables as shown in the diagram below.



Each individual cable is installed such that each end of the cable is connected to a building's access point. The connection between each end of a cable and an access point has a 1.4% probability of failing after a power surge.

For the network to be successful, each building in the network must be able to communicate with every other building in the network. In other words, there must be a path that connects any two buildings in the network. Jonas would like the network to have less than a 2% probability of failing to operate after a power surge.

(h) Show that Jonas's network satisfies the requirement of there being less than a 2% probability of the network failing after a power surge. [5]