Subject – Math AI(Higher Level)
Topic - Statistics and Probability
Year - May 2021 – Nov 2022
Paper -3
Answers

Question 1

(a) χ^2 (goodness of fit) ***A1***

[1 mark]

(b) ***EITHER***

because aim is to measure improvement

OR

because the students may be of different ability in the two schools ***R1***

[1 mark]

(c) (i) 0.1875 (accept 0.188, 0.19) ***A1***

(ii) 2.46 ***(M1)A1***

Note: Award ***(M1)A0*** for 2.63.

[3 marks]

(d) H_0: there has been no improvement

H_1: there has been an improvement ***A1***

attempt at a one-tailed paired t-test ***(M1)***

p-value $= 0.423$ ***A1***

there is no significant evidence that the students have improved ***R1***

Note: If the hypotheses are not stated award a maximum of ***A0M1A1R0***.

[4 marks]

(e) (i) H_0: there is no difference between the schools

H_1: school B did better than school A ***A1***

one-tailed 2 sample t-test ***(M1)***

p-value $= 0.0984$ ***A1***

$0.0984 > 0.05$ (not significant at the 5% level) so do not reject the null hypothesis ***R1A1***

Note: The final ***A1*** cannot be awarded following an incorrect reason. The final ***R1A1*** can follow through from their incorrect p-value. Award a maximum of ***A1(M1)A0R1A1*** for p-value $= 0.0993$.

(ii) sample too small for the central limit theorem to apply (and t-tests assume normal distribution) ***R1***

[6 marks]

(f) (i) $H_0: \rho = 0$

$H_1: \rho > 0$ *A1*

Note: Allow hypotheses to be expressed in words.

p-value $= 0.00157$ *A1*

$(0.00157 < 0.01)$ there is a significant evidence of a (linear) correlation between effort and improvement (so it is reasonable to assume a linear relationship) *R1*

(ii) (gradient of line of regression =) 6.6 *A1*

[4 marks]

(g) H_0: improvement and gender are independent

H_1: improvement and gender are not independent *A1*

choice of χ^2 test for independence *(M1)*

groups first two columns as expected values in first column less than 5 *M1*

new observed table

	$(f-p)<0$	$0 \le (f-p) < 2$	$(f-p) \ge 2$
Male	14	10	9
Female	11	14	8

(A1)

p-value $= 0.581$ *A1*

no significant evidence that gender and improvement are dependent *R1*

[6 marks]

(h) *For example:*
larger samples / include data from whole school
take equal numbers of boys and girls in each sample
have a similar range of abilities in each sample
(if possible) have similar ranges of effort *R1R1*

Note: Award ***R1*** for each reasonable suggestion to improve the validity of the test.

[2 marks]

Total [27 marks]

Question 2

(a) Use of χ^2 test for independence **(M1)**

H_0 : Staying (or leaving) the firm and interview rating are independent.

H_1 : Staying (or leaving) the firm and interview rating are not independent **A1**

Note: For H_1 accept '…are dependent' in place of '…not independent'.

p-value $= 0.487$ $(0.487221...)$ **A2**

Note: Award ***A1*** for $\chi^2 = 1.438...$ if p-value is omitted or incorrect.

$0.487 > 0.05$ **R1**

(the result is not significant at the 5% level)
insufficient evidence to reject the H_0 (or "accept H_0") **A1**

Note: Do not award ***R0A1***. The final ***R1A1*** can follow through from their incorrect p-value

[6 marks]

(b) $\frac{55}{91} \times 18 = 10.9$ $(10.8791...)$ **M1A1**

Note: Award ***A1*** for anything that rounds to 10.9.

≈ 11 **AG**

[2 marks]

(c) (i) there seems to be a difference between the two departments ***(A1)***

the international department manager seems to be less generous than the national department manager ***R1***

Note: The ***A1*** is for commenting there is a difference between the two departments and the ***R1*** is for correctly commenting on the direction of the difference

[2 marks]

(ii)

	L	M	N	O	P	Q	R
Written assessment rank	1	2	3	4	5	6	7
Manager score rank	1	2.5	4.5	2.5	4.5	6	7

(M1)(A1)

Note: Award ***(M1)*** for an attempt to rank the data, and ***(A1)*** for correct ranks for both variables. Accept either set of rankings in reverse.

$r_s = 0.909 \ \ (0.909241.....)$ ***(M1)A1***

Note: The ***(M1)*** is for calculating the PMCC for their ranks.

Note: If a final answer of 0.9107 is seen, from use of $1-\frac{6\sum d^2}{n(n^2-1)}$, award ***(M1)(A1)A1***.

Accept -0.909 if one set of ranks has been ordered in reverse.

[4 marks]

(iii) **EITHER**

there is a (strong) association between the written assessment mark and the manager scores. ***A1***

OR

there is a (strong) agreement in the rank order of the written assessment marks and the rank order of the manager scores. ***A1***

OR

there is a (strong linear) correlation between the rank order of the written assessment marks and the rank order of the manager scores. ***A1***

Note: Follow through on a value for their value of r_s in c(ii).

THEN

the written assessment is likely to be a valid measure (of the level of employee performance) ***R1***

[2 marks]

(d) (i) test-retest **A1**

[1 mark]

(ii) p-value $= 0.00209$ $(0.0020939...)$ **A2**

$0.00209 < 0.05$ **R1**

(the result is significant at the 5% level)

(there is sufficient evidence to) reject H_0 **A1**

Note: Do not award ***R0A1***. Accept "accept H_1". The final ***R1A1*** can follow through from their incorrect p-value.

[4 marks]

(iii) the test seems reliable **A1**

Note: Follow through from their answer in part (d)(ii). Do not award if there is no conclusion in d(ii).

[1 mark]

(e) (i) 25 **A1**

[1 mark]

(ii) probability of significant result given no correlation is 0.05 ***(M1)***

probability of at least one significant result in 25 tests is

$1-0.95^{25}$ ***(M1)(A1)***

Note: Award ***(M1)*** for use of $1-\mathrm{P}(0)$ or the binomial distribution with any value of p.

[1 mark]

$= 0.723$ $(0.722610...)$ **A1**

[4 marks]

(iii) (though the result is significant) it is very likely that one significant result would be achieved by chance, so it should be disregarded or further evidence sought

R1

[1 mark]

[Total 28 marks]

Question 3

(a) (i) Any one from: **R1**

increase sample size / increase response rate / repeat process
check whether sample is representative
test-retest participants or do a parallel test
use a stratified sample
use a random sample

> **Note:** Do not condone:
> Ask different types of doctor
> Ask for proof of income
> Ask for proof of being a doctor
> Remove anonymity
> Remove response K.

[1 mark]

(ii) Any one from: **R1**

non-random sampling means a subset of population might be responding
self-reported happiness is not the same as happiness
happiness is not a constant / cannot be quantified / is difficult to measure
income might include external sources
Juliet is only sampling doctors in her city
correlation does not imply causation
sample might be biased

> **Note:** Do not condone the following common but vague responses unless they make a clear link to validity:
> Sample size is too small
> Result is not generalizable
> There may be other variables Juliet is ignoring
> Sample might not be representative

[1 mark]

(b) because the income is very different / implausible / clearly contrived **R1**

> **Note:** Answers must explicitly reference "income" to get credit.

[1 mark]

(c) (i) (\$) 90 200 *(M1)A1*

[2 marks]

(ii) $r = 0.558 \ (0.557723...)$ *A2*

[2 marks]

(d) (i) **EITHER**

only looking for change in one direction *R1*

OR

only looking for greater happiness with greater income *R1*

OR

only looking for evidence of positive correlation *R1*

[1 mark]

(ii) $H_0 : \rho = 0; H_1 : \rho > 0$ *A1A1*

Note: Award ***A1*** for ρ seen (do not accept r), ***A1*** for both correct hypotheses, using **their** ρ or r. Accept an equivalent statement in words, however reference to "correlation for the population" or "association for the population" must be explicit for the first ***A1*** to be awarded.

Watch out for a null hypothesis in words similar to "Annual income is not associated with greater happiness". This is effectively saying $\rho \leq 0$ and should not be condoned.

[2 marks]

(iii) **METHOD 1 – using critical value of *r***

$0.558 > 0.549$ $(0.557723... > 0.549)$ ***R1***

(therefore significant evidence of) a positive correlation ***A1***

Note: Do not award ***R0A1***.

METHOD 2 – using *p*-value

$0.0469 < 0.05$ $(0.0469463... < 0.05)$ ***A1***

Note: Follow through from their r-value from part (c)(ii).

(therefore significant evidence of) a positive correlation ***A1***

Note: Do not award ***A0A1***.

[2 marks]

(e) (i) $a = 0.000126$ $(0.000125842...)$, $b = 41.1$ $(41.1490...)$ ***A1***

[1 mark]

(ii) **EITHER**

the amount the happiness score increases for every $1 increase in (annual) income ***A1***

OR

rate of change of happiness with respect to (annual) income ***A1***

Note: Accept equivalent responses e.g. an increase of 1.26 in happiness for every $\$10000$ increase in salary.

[1 mark]

(iii) $c = -2.06 \times 10^{-9}$ $(-2.06191... \times 10^{-9})$,

$d = 7.05 \times 10^{-4}$ $(7.05272... \times 10^{-4})$,

$e = 12.6$ $(12.5878...)$ ***A1***

[1 mark]

(iv) for quadratic model: $R^2 = 0.659$ $(0.659145...)$ *A1*

for linear model: $R^2 = 0.311$ $(0.311056...)$ *A1*

Note: Follow through from their r value from part (c)(ii).

[2 marks]

(v) **EITHER**

quadratic model is a better fit to the data / more accurate *A1*

OR

quadratic model explains a higher proportion of the variance *A1*

[1 mark]

(vi) **EITHER**

not valid, R^2 not a useful measure to compare models with different numbers of parameters *A1*

OR

not valid, quadratic model will always have a better fit than a linear model *A1*

Note: Accept any other sensible critique of the validity of the method. Do not accept any answers which focus on the conclusion rather than the method of model selection.

[1 mark]

(f) (i) (single sample) t-test *A1*

[1 mark]

(ii) **EITHER**

$H_0 : \mu = 80\,000$; $H_1 : \mu \neq 80\,000$ *A1*

OR

H_0 : (sample is drawn from a population where) the population mean is $\$80\,000$

H_1 : the population mean is not $\$80\,000$ *A1*

Note: Do not allow ***FT*** from an incorrect test in part (f)(i) other than a z-test.

[1 mark]

(iii) $p = 0.610$ $(0.610322...)$ *A1*

Note: For a z-test follow through from part (f)(i), either 0.578 (from biased estimate of variance) or 0.598 (from unbiased estimate of variance).

$0.610 > 0.05$ *R1*

EITHER

no (significant) evidence that mean differs from $\$80\,000$ *A1*

OR

the sample could plausibly have been drawn from the quoted population *A1*

Note: Allow ***R1FTA1FT*** from an incorrect p-value, but the final ***A1*** must still be in the context of the original research question.

[3 marks]

[Total 24 marks]

Question 4

(a) (i) mean $= 4.23 \ (4.23333\ldots)$ **A1**

variance $= 4.27 \ (4.26777...)$ **A1**

[2 marks]

(ii) mean is close to the variance **A1**

[1 mark]

(b) *One of the following:*
the number of bags sold each day is independent of any other day
the sale of one bag is independent of any other bag sold
the sales of bags of rice (each day) occur at a *constant mean* rate **A1**

Note: Award ***A1*** for a correct answer in context. Any statement referring to independence must refer to either the independence of each bag sold or the independence of the number of bags sold each day. If the third option is seen, the statement must refer to a "constant mean" or "constant average". Do not accept "the number of bags sold each day is constant".

[1 mark]

(c) attempt to find Poisson probabilities and multiply by 90 ***(M1)***
$a = 7.018$ **A1**
$b = 17.498$ **A1**

EITHER
$90 \times \mathrm{P}(X \geq 8) = 90 \times (1 - \mathrm{P}(X \leq 7))$ ***(M1)***
$c = 5.755$ **A1**

OR
$90 - 7.018 - 11.903 - 16.665 - 17.498 - 14.698 - 10.289 - 6.173$ ***(M1)***
$c = 5.756$ **A1**

Note: Do not penalize the omission of clear a, b and c labelling as this will be penalized later if correct values are interchanged.

[5 marks]

(d) (i) 7 ***A1***

[1 mark]

(ii) H_0: The number of bags of rice sold each day follows a Poisson distribution with mean 4.2. ***A1***

H_1: The number of bags of rice sold each day does not follow a Poisson distribution with mean 4.2. ***A1***

Note: Award ***A1A1*** for **both** hypotheses correctly stated and in correct order. Award ***A1A0*** if reference to the data and/or "mean 4.2" is not included in the hypotheses, but otherwise correct.

evidence of attempting to group data to obtain the observed frequencies for ≤ 1 and ≥ 8 ***(M1)***

$p\text{-value} = 0.728 \ (0.728100...)$ ***A2***

$0.728 \ (0.728100...) > 0.05$ ***R1***

the result is not significant so there is no reason to reject H_0
(the number of bags sold each day follows a Poisson distribution) ***A1***

Note: Do not award ***R0A1***. The conclusion MUST follow through from their hypotheses. If no hypotheses are stated, the final ***A1*** can still be awarded for a correct conclusion as long as it is in context (e.g. therefore the data follows a Poisson distribution).

[7 marks]

(e) (i) **METHOD 1**

evidence of multiplying 4.2×60 (seen anywhere) ***M1***

$H_0 : \mu = 252$

$H_1 : \mu > 252$ ***A1***

Note: Accept $H_0 : \mu = 4.2$ and $H_1 : \mu > 4.2$ for the ***A1***.

evidence of finding probabilities around critical region ***(M1)***

Note: Award ***(M1)*** for any of these values seen:
$P(X \geq 277) = 0.0630518\ldots$ **OR** $P(X \leq 276) = 0.936948\ldots$
$P(X \geq 278) = 0.0558415\ldots$ **OR** $P(X \leq 277) = 0.944158\ldots$
$P(X \geq 279) = 0.0493055\ldots$ **OR** $P(X \leq 278) = 0.950694\ldots$.

critical value $= 279$ ***A1***

$282 \geq 279$, ***R1***

the null hypothesis is rejected ***A1***

(the advertising increased the number of bags sold during the 60 days)

Note: Do not award ***R0A1***. Accept statements referring to the advertising being effective for ***A1*** as long as the ***R*** mark is satisfied. For the ***R1A1***, follow through within the part from their critical value.

METHOD 2

evidence of dividing 282 by 60 (or 4.7 seen anywhere) ***M1***

$H_0 : \mu = 4.2$

$H_1 : \mu > 4.2$ ***A1***

attempt to find critical value using central limit theorem ***(M1)***

(e.g. sample standard deviation $= \sqrt{\frac{4.2}{60}}$, $\bar{X} \sim N\left(4.2, \sqrt{\frac{4.2}{60}}\right)$, etc.)

Note: Award ***(M1)*** for a p-value of $0.0293907\ldots$ seen.

critical value $= 4.63518\ldots$ ***A1***

$4.7 > 4.63518\ldots$ ***R1***

the null hypothesis is rejected ***A1***

(the advertising increased the number of bags sold during the 60 days)

Note: Do not award ***R0A1***. Accept statements referring to the advertising being effective for ***A1*** as long as the ***R*** mark is satisfied. For the ***R1A1***, follow through within the part from their critical value.

[6 marks]

(ii) $(P(X \geq 279 \mid \mu = 252) =)$ 0.0493 $(0.0493055\ldots)$ ***A1***

Note: If a candidate uses **METHOD 2** in part (e)(i), allow an ***FT*** answer of 0.05 for this part but only if the candidate has attempted to find a p-value.

[1 mark]

(f) attempt to compare profit *difference* with cost of advertising ***(M1)***

Note: Award ***(M1)*** for evidence of candidate mathematically comparing a profit difference with the cost of the advertising.

EITHER
(comparing profit from 30 extra bags of rice with cost of advertising)
$14850 < 18000$ ***A1***

OR
(comparing total profit with and without advertising)
$121590 < 124740$ ***A1***

OR
(comparing increase of average daily profit with daily advertising cost)
$247.50 < 300$ ***A1***

THEN

EITHER
Even though the number of bags of rice increased, the advertising is not worth it as the overall profit did not increase. ***R1***

OR
The advertising is worth it even though the cost is less than the increased profit, since the number of customers increased (possibly buying other products and/or returning in the future after advertising stops) ***R1***

Note: Follow through within the part for correct reasoning consistent with their comparison.

[3 marks]

[Total 27 marks]

Question 5

(a) $\frac{12}{5}$ $\left(\frac{144}{60}, 2.4\right)$ ***(M1)A1***

[2 marks]

(b) $\frac{3}{5}$ $\left(\frac{144}{240}, 0.6\right)$ ***A1***

[1 mark]

(c) (i) wins $\sim \text{B}\left(4, \frac{3}{5}\right)$

$\text{P}(\text{wins} = 0) = 0.0256$ ***A1***

Note: Allow ***FT*** from use of their probability in part (b) but only when used with $n = 4$.

(ii) expected frequency $= 60 \times 0.0256$ ***(M1)***

$= 1.536$ ***A1***

[3 marks]

(d) (i) H_0: data follows a Binomial distribution with $n = 4$ ***A1***

(ii) $(\text{df} = 4 - 1 - 1 =)\ 2$ ***A1***

(iii) p-value $= 0.954$ $(0.953872...)$ ***(M1)A1***

(iv) $0.954 > 0.05$ ***R1***

insufficient evidence to reject H_0 ***A1***

Note: Condone "accept H_0". Follow through from their p-value in part (d)(iii) if the reasoning is correct and correct conclusions are made. Do NOT award ***R0A1***.

[6 marks]

(e) wins $\sim \text{B}\left(4, \frac{3}{5}\right)$ **OR** $1 - 0.0256$ ***(M1)***

$\text{P}(\text{wins} \geq 1) = 0.974$ (0.9744) ***A1***

[2 marks]

(f) (i) there are 145 transitions that start with Argentina, ***(M1)***

(of which 85 lead to Argentina winning,) so the probability is $\frac{85}{145}$ ***A1***

$=\frac{17}{29}$ ***AG***

(ii) $\begin{pmatrix} \frac{17}{29} & \frac{31}{47} \\ \frac{12}{29} & \frac{16}{47} \end{pmatrix} \left(=\begin{pmatrix} 0.586 & 0.660 \\ 0.414 & 0.340 \end{pmatrix}\right)$ ***A1A1***

Note: Accept the transposed matrix as correct.
Award ***A1*** for $\frac{17}{29}$ placed in a leading diagonal.
Award ***A1*** for all other values correct and in correct position in the matrix.

[4 marks]

(g) (i) write their matrix with λ subtracted from the leading diagonal ***(M1)***
equate determinant to zero ***(M1)***

$$\det\begin{pmatrix} \frac{17}{29}-\lambda & \frac{31}{47} \\ \frac{12}{29} & \frac{16}{47}-\lambda \end{pmatrix}=0$$

$\left(\frac{17}{29}-\lambda\right)\left(\frac{16}{47}-\lambda\right)-\frac{12}{29}\times\frac{31}{47}=0$ ***A1***

correct intermediate step
$1363\lambda^2-1263\lambda-100=0$ ***AG***

Note: Do not award ***A1*** if there is no intermediate step leading from determinant to given answer.
Solving $\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix}=\lambda\begin{pmatrix} x \\ y \end{pmatrix}$ for x and y may be seen and is a valid alternative method.
Accept working in the form $\det(\lambda I-T)=0$.

(ii) $\lambda=1,\ -\frac{100}{1363}\ (-0.0733675...)$ ***A1***

(iii) attempt to solve $T\begin{pmatrix}x\\y\end{pmatrix}=\lambda\begin{pmatrix}x\\y\end{pmatrix}$ ***(M1)***

$y=\frac{564}{899}x$

eigenvector for $\lambda=1$ is $\begin{pmatrix}1\\\frac{564}{899}\end{pmatrix}\left(=\begin{pmatrix}1\\0.627\end{pmatrix}\right)$ ***A1***

eigenvector for $\lambda=-\frac{100}{1363}$ is $\begin{pmatrix}1\\-1\end{pmatrix}$ ***A1***

Note: Allow correct multiples of the eigenvectors.
If eigenvector $\begin{pmatrix}1\\-1\end{pmatrix}$ is stated without the second eigenvector, or any other working, then award ***M0A0A1***.

[7 marks]

(h) **EITHER**

solution found using $\begin{pmatrix}1\\\frac{564}{899}\end{pmatrix}\left(=\begin{pmatrix}1\\0.627\end{pmatrix}\right)$

$x+\frac{564}{899}x=1$ ***(M1)***

$x=0.614\ \left(0.614490\ldots,\ \frac{899}{1463}\right)$ ***(A1)***

OR

solution can be found from high power of transition matrix

$\begin{pmatrix}\frac{17}{29} & \frac{31}{47}\\ \frac{12}{29} & \frac{16}{47}\end{pmatrix}^{50}=\begin{pmatrix}0.614 & 0.614\\ 0.386 & 0.386\end{pmatrix}$ ***(M1)***

Note: Accept the transposed matrix if consistent with their answer to part (f)(ii).

probability $=0.614\ \left(0.614490\ldots,\ \frac{899}{1463}\right)$ ***(A1)***

THEN

$\mathrm{P}(3\text{ wins})=0.614\times0.586^2\ \left(=\frac{899}{1463}\times\left(\frac{17}{29}\right)^2\right)$ ***(M1)***

$=0.211\ \left(0.211162\ldots,\ \frac{8959}{42427}\right)$ ***A1***

[4 marks]
[Total: 29 marks]