Subject – Math AI(Higher Level)
Topic - Number and Algebra
Year - May 2021 – Nov 2024
Paper -2
Answers

Question 1

(a) $\begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}$ ***M1A1***

[2 marks]

(b) $\begin{vmatrix} 0.8-\lambda & 0.1 \\ 0.2 & 0.9-\lambda \end{vmatrix} = 0$ ***M1***

$\lambda = 1$ and 0.7 ***A1***

eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***(M1)A1***

Note: Accept any scalar multiple of the eigenvectors.

[4 marks]

(c) ***EITHER***

$\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ $\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix}$ ***A1A1***

OR

$\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ $\boldsymbol{D} = \begin{pmatrix} 0.7 & 0 \\ 0 & 1 \end{pmatrix}$ ***A1A1***

[2 marks]

(d) $\boldsymbol{P}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ ***A1***

$\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.7^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1200 \\ 1200 \end{pmatrix}$ ***M1A1***

attempt to multiply matrices ***M1***

so in company A, after n years, $400\left(2+0.7^n\right)$ ***A1***

[5 marks]

(e) $400 \times 2 = 800$ ***A1***

[1 mark]

Total [14 marks]

Question 2

(a) (i) $\mathrm{N} = 24$

$\mathrm{I\%} = 14$

$\mathrm{PV} = -14000$

$\mathrm{FV} = 0$

$\mathrm{P/Y} = 4$

$\mathrm{C/Y} = 4$ ***(M1)(A1)***

> **Note:** Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct. Accept $\mathrm{PV} = 14000$.

$(€)871.82$ ***A1***

(ii) $4 \times 6 \times 871.82$ ***(M1)***

$(€)20923.68$ ***A1***

(iii) $20923.68 - 14000$ ***(M1)***

$(€)6923.68$ ***A1***

[7 marks]

(b) (i) $0.9 \times 14000 \ (= 14000 - 0.10 \times 14000)$ ***M1***

$(€)12600.00$ ***A1***

(ii) $\mathrm{N} = 72$

$\mathrm{PV} = 12600$

$\mathrm{PMT} = -250$

$\mathrm{FV} = 0$

$\mathrm{P/Y} = 12$

$\mathrm{C/Y} = 12$ ***(M1)(A1)***

> **Note:** Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct. Accept $\mathrm{PV} = -12600$ provided $\mathrm{PMT} = 250$.

$12.56(\%)$ ***A1***

[5 marks]

(c) ***EITHER***

Bryan should choose Option A ***A1***

no deposit is required ***R1***

Note: Award ***R1*** for stating that no deposit is required. Award ***A1*** for the correct choice from that fact. Do not award ***R0A1.***

OR

Bryan should choose Option B ***A1***

cost of Option A $(6923.69) >$ cost of Option B $(72 \times 250 - 12\,600 = 5400)$ ***R1***

Note: Award ***R1*** for a correct comparison of costs. Award ***A1*** for the correct choice from that comparison. Do not award ***R0A1***.

[2 marks]

(d) real interest rate is $0.4 - 0.1 = 0.3\%$ ***(M1)***

value of other payments $250 + 250 \times 1.003 + \ldots + 250 \times 1.003^{71}$
use of sum of geometric sequence formula or financial app on a GDC ***(M1)***

$= 20\,058.43$

value of deposit at the end of 6 years

$1400 \times (1.003)^{72} = 1736.98$ ***(A1)***

Total value is $(€)\,21\,795.41$ ***A1***

Note: Both ***M*** marks can awarded for a correct use of the GDC's financial app:

$N = 72\ (6 \times 12)$
$I\% = 3.6\ (0.3 \times 12)$
$PV = 0$
$PMT = -250$
$FV =$
$P/Y = 12$
$C/Y = 12$

OR

$N = 72\ (6 \times 12)$
$I\% = 0.3$
$PV = 0$
$PMT = -250$
$FV =$
$P/Y = 1$
$C/Y = 1$

[4 marks]

Total [18 marks]

Question 3

(a) finding T^3 **OR** use of tree diagram *(M1)*

$T^3 = \begin{pmatrix} 0.65 & 0.525 \\ 0.35 & 0.475 \end{pmatrix}$

the probability of sunny in three days' time is 0.65 ***A1***

[2 marks]

(b) attempt to find eigenvalues *(M1)*

Note: Any indication that $\det(T - \lambda I) = 0$ has been used is sufficient for the ***(M1)***.

$\begin{vmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{vmatrix} = (0.8-\lambda)(0.7-\lambda) - 0.06 = 0$

$(\lambda^2 - 1.5\lambda + 0.5 = 0)$

$\lambda = 1,\ \lambda = 0.5$ ***A1***

attempt to find either eigenvector *(M1)*

$0.8x + 0.3y = x \Rightarrow -0.2x + 0.3y = 0$ so an eigenvector is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ ***A1***

$0.8x + 0.3y = 0.5x \Rightarrow 0.3x + 0.3y = 0$ so an eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***A1***

Note: Accept multiples of the stated eigenvectors.

[5 marks]

(c) (i) $P = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$ **OR** $P = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$ ***A1***

Note: Examiners should be aware that different, correct, matrices P may be seen.

(ii) $D = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}$ **OR** $D = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$ ***A1***

Note: P and D must be consistent with each other.

[2 marks]

(d) $0.5^n \to 0$ *(M1)*

$D^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ **OR** $D^n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *(A1)*

Note: Award ***A1*** only if their D^n corresponds to their P.

$PD^nP^{-1} = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}$ *(M1)*

$60\ \%$ ***A1***

[4 marks]

Total [13 marks]

Question 4

(a) **EITHER**

$N = 2$

$PV = -37\,000$

$I\% = 6.4$

$P/Y = 1$

$C/Y = 4$ ***(M1)(A1)***

Note: Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct.

OR

$N = 8$

$PV = -37\,000$

$I\% = 6.4$

$P/Y = 4$

$C/Y = 4$ ***(M1)(A1)***

Note: Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct.

OR

$$FV = 37\,000 \times \left(1 + \frac{6.4}{100 \times 4}\right)^{4\times 2}$$ ***(M1)(A1)***

Note: Award ***M1*** for substitution into compound interest formula, ***(A1)*** for correct substitution.

$= 42\,010$ AUD ***A1***

Note: Award ***(M1)(A1)A0*** for unsupported 42009.87.

[3 marks]

(b) **EITHER**

$PV = -37\,000$

$FV = 50\,000$

$I\% = 6.4$

$P/Y = 1$

$C/Y = 4$ ***(M1)(A1)***

OR

$PV = -37\,000$

$FV = 50\,000$

$I\% = 6.4$

$P/Y = 4$

$C/Y = 4$ ***(M1)(A1)***

Note: Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct.

OR

$$50000 < 37\,000 \times \left(1 + \frac{6.4}{100 \times 4}\right)^{4 \times n} \textbf{ OR } 50000 < 37\,000 \times \left(1 + \frac{6.4}{100 \times 4}\right)^{n}$$ ***(M1)(A1)***

Note: Award ***M1*** for the correct inequality, $50\,000$ and substituted compound interest formula. Allow an equation. Award ***A1*** for correct substitution.

THEN

$N = 4.74$ (years) $(4.74230...)$ **OR** $N = 18.9692...$ (quarters) ***(A1)***

$m = 57$ months ***A1***

Note: Award ***A1*** for rounding their m to the correct number of months. The final answer must be a multiple of 3. Follow through within this part.

[4 marks]

(c) $150\,000$ AUD ***A1***

[1 mark]

(d) (i) $120 \times 1700 - 150000$ ***(M1)***

$= 54000$ AUD ***A1***

(ii) $N = 120$

$PV = -150000$

$PMT = 1700$

$FV = 0$

$P/Y = 12$

$C/Y = 12$ ***(M1)(A1)***

Note:	Award ***M1*** for an attempt to use a financial app in their technology or an attempt to use an annuity formula or $FV = 0$ seen. If a compound interest formula is equated to zero, award ***M1***, otherwise award ***M0*** for a substituted compound interest formula. Award ***A1*** for all entries correct in financial app or correct substitution in annuity formula, but award ***A0*** for a substituted compound interest formula. Follow through marks in part (d)(ii) are contingent on working seen.

$r = 6.46$ (%) (6.45779...) ***A1***

[5 marks]

(e) $N = 60$

$I = 6.46$ (6.45779...)

$PV = -150000$

$PMT = 1700$

$P/Y = 12$

$C/Y = 12$ ***(M1)(A1)***

Note:	Award ***M1*** for an attempt to use a financial app in their technology or an attempt to use an annuity formula. Award ***(M0)*** for a substituted compound interest formula. Award ***A1*** for all entries correct. Follow through marks in part (e) are contingent on working seen.

$FV = 86973$ AUD ***A1***

[3 marks]

(f) $204\,000-(60\times1700+86973)$ **OR** $204\,000-188973$ ***(M1)(M1)***

Note: Award ***M1*** for 60×1700. Award ***M1*** for subtracting their $(60\times1700+86973)$ from their $(204\,000)$. Award at most ***M1M0*** for their $204\,000-(60\times1700)$ or ***M0M0*** for their $204\,000-(86973)$. Follow through from parts (d)(i) and (e). Follow through marks in part (f) are contingent on working seen.

15027 AUD ***A1***

[3 marks]

Total [19 marks]

Question 5

(a) $\boldsymbol{T}=\begin{pmatrix}0.965 & 0.05\\ 0.035 & 0.95\end{pmatrix}$ ***M1A1***

> **Note:** Award ***M1A1*** for $\boldsymbol{T}=\begin{pmatrix}0.95 & 0.035\\ 0.05 & 0.965\end{pmatrix}$.
>
> Award the ***A1*** for a transposed T if used correctly in part (b) i.e. preceded by 1×2 matrix $(2100 \quad 3500)$ rather than followed by a 2×1 matrix.

[2 marks]

(b) $\begin{pmatrix}0.965 & 0.05\\ 0.035 & 0.95\end{pmatrix}^2\begin{pmatrix}2100\\ 3500\end{pmatrix}$ ***(M1)***

$=\begin{pmatrix}2294\\ 3306\end{pmatrix}$

so ratio is $2294:3306$ $(=1147:1653,\ 0.693889...)$ ***A1***

[2 marks]

(c) to solve $Ax=\lambda x$:

$\begin{vmatrix}0.965-\lambda & 0.05\\ 0.035 & 0.95-\lambda\end{vmatrix}=0$ ***(M1)***

$(0.965-\lambda)(0.95-\lambda)-0.05\times0.035=0$

$\lambda=0.915$ **OR** $\lambda=1$ ***(A1)***

attempt to find eigenvectors for at least one eigenvalue ***(M1)***

when $\lambda=0.915$, $x=\begin{pmatrix}1\\ -1\end{pmatrix}$ (or any real multiple) ***(A1)***

when $\lambda=1$, $x=\begin{pmatrix}10\\ 7\end{pmatrix}$ (or any real multiple) ***(A1)***

therefore $\boldsymbol{P}=\begin{pmatrix}1 & 10\\ -1 & 7\end{pmatrix}$ (accept integer valued multiples of their eigenvectors and columns in either order) ***A1***

[6 marks]

(d) $P^{-1} = \begin{pmatrix} 1 & 10 \\ -1 & 7 \end{pmatrix}^{-1} = \frac{1}{17}\begin{pmatrix} 7 & -10 \\ 1 & 1 \end{pmatrix}$ ***(A1)***

Note: This mark is independent, and may be seen anywhere in part (d).

$D = \begin{pmatrix} 0.915 & 0 \\ 0 & 1 \end{pmatrix}$ ***(A1)***

$T^n = PD^nP^{-1} = \begin{pmatrix} 1 & 10 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 0.915^n & 0 \\ 0 & 1^n \end{pmatrix}\frac{1}{17}\begin{pmatrix} 7 & -10 \\ 1 & 1 \end{pmatrix}$ ***(M1)A1***

Note: Award ***(M1)A0*** for finding $P^{-1}D^nP$ correctly.

as $n \to \infty$, $D^n = \begin{pmatrix} 0.915^n & 0 \\ 0 & 1^n \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ***R1***

so $T^n \to \frac{1}{17}\begin{pmatrix} 1 & 10 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 7 & -10 \\ 1 & 1 \end{pmatrix}$ ***A1***

$= \begin{pmatrix} \frac{10}{17} & \frac{10}{17} \\ \frac{7}{17} & \frac{7}{17} \end{pmatrix}$ ***AG***

Note: The ***AG*** line must be seen for the final ***A1*** to be awarded.

[6 marks]

(e) **METHOD ONE**

$\begin{pmatrix} \frac{10}{17} & \frac{10}{17} \\ \frac{7}{17} & \frac{7}{17} \end{pmatrix}\begin{pmatrix} 2100 \\ 3500 \end{pmatrix} = \begin{pmatrix} 3294 \\ 2306 \end{pmatrix}$ ***(M1)***

so ratio is $3294 : 2306$ $(1647 : 1153,\ 1.42844...,\ 0.700060...)$

A1

METHOD TWO

long term ratio is the eigenvector associated with the largest eigenvalue ***(M1)***

$10 : 7$ ***A1***

[2 marks]

Total [18 marks]

Question 6

(a) (i) rotation anticlockwise $\frac{\pi}{6}$ is $\begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix}$ **OR** $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ ***(M1)A1***

reflection in $y = \frac{x}{\sqrt{3}}$

$\tan\theta = \frac{1}{\sqrt{3}}$ ***(M1)***

$\Rightarrow 2\theta = \frac{\pi}{3}$ ***(A1)***

matrix is $\begin{pmatrix} 0.5 & 0.866 \\ 0.866 & -0.5 \end{pmatrix}$ **OR** $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ ***A1***

rotation clockwise $\frac{\pi}{3}$ is $\begin{pmatrix} 0.5 & 0.866 \\ -0.866 & 0.5 \end{pmatrix}$ **OR** $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ ***A1***

(ii) an attempt to multiply three matrices ***(M1)***

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$ ***(A1)***

$$\boldsymbol{P} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \textbf{ OR } \begin{pmatrix} 0.866 & -0.5 \\ -0.5 & -0.866 \end{pmatrix}$$ ***A1***

(iii) $\left(P^2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ***A1***

Note: Do not award ***A1*** if final answer not resolved into the identity matrix I.

(iv) if the overall movement of the drone is repeated ***A1***
the drone would return to its original position ***A1***

[12 marks]

(b) **METHOD 1**

$|\det \boldsymbol{P}| = \left| \left(-\frac{3}{4}\right) - \left(\frac{1}{4}\right) \right| = 1$ ***A1***

area of triangle ABC = area of triangle $\mathrm{A'B'C'} \times |\det \boldsymbol{P}|$ ***R1***

area of triangle ABC = area of triangle $\mathrm{A'B'C'}$ ***AG***

Note: Award at most ***A1R0*** for responses that omit modulus sign.

METHOD 2

statement of fact that rotation leaves area unchanged ***R1***
statement of fact that reflection leaves area unchanged ***R1***
area of triangle ABC = area of triangle $\mathrm{A'B'C'}$ ***AG***

[2 marks]

(c) attempt to find angles associated with values of elements in matrix $\boldsymbol{P}$ ***(M1)***

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos\left(-\frac{\pi}{6}\right) & \sin\left(-\frac{\pi}{6}\right) \\ \sin\left(-\frac{\pi}{6}\right) & -\cos\left(-\frac{\pi}{6}\right) \end{pmatrix}$$

reflection (in $y = (\tan\theta)x$) ***(M1)***

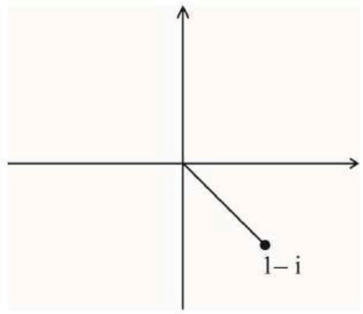
where $2\theta = -\frac{\pi}{6}$ ***A1***

reflection in $y = \tan\left(-\frac{\pi}{12}\right)x \ (= -0.268x)$ ***A1***

[4 marks]
Total: [18 marks]

Question 7

(a) (i)



A1

(ii) $z=\sqrt{2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}$ **A1A1**

Note: Accept an argument of $\frac{7\pi}{4}$. Do **NOT** accept answers that are not exact.

[3 marks]

(b) (i) $w_1+w_2=\mathrm{e}^{\mathrm{i}x}+\mathrm{e}^{\mathrm{i}\left(x-\frac{\pi}{2}\right)}$

$=\mathrm{e}^{\mathrm{i}x}\left(1+\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}}\right)$ ***(M1)***

$=\mathrm{e}^{\mathrm{i}x}(1-\mathrm{i})$ ***A1***

(ii) $w_1+w_2=\mathrm{e}^{\mathrm{i}x}\times\sqrt{2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}$ ***M1***

$=\sqrt{2}\mathrm{e}^{\mathrm{i}\left(x-\frac{\pi}{4}\right)}$ ***(A1)***

attempt extract real part using cis form ***(M1)***

$\mathrm{Re}(w_1+w_2)=\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$ **OR** $1.4142\ldots\cos(x-0.785398\ldots)$ ***A1***

[6 marks]

(c) (i) $I_t=12\cos(bt)+12\cos\left(bt-\frac{\pi}{2}\right)$ ***(M1)***

$I_t=12\,\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}bt}+\mathrm{e}^{\mathrm{i}\left(bt-\frac{\pi}{2}\right)}\right)$ ***(M1)***

$I_t=12\sqrt{2}\cos\left(bt-\frac{\pi}{4}\right)$

max $=12\sqrt{2}\ (=17.0)$ ***A1***

(ii) phase shift $=\frac{\pi}{4}(=0.785)$ ***A1***

[4 marks]

Total: [13 marks]

Question 8

(a) (i) $\boldsymbol{P}\begin{pmatrix}0\\0\end{pmatrix}+\boldsymbol{q}=\begin{pmatrix}0\\1\end{pmatrix}$ ***(M1)***

$\boldsymbol{q}=\begin{pmatrix}0\\1\end{pmatrix}$ ***A1***

(ii) **EITHER**

$\boldsymbol{P}\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{4}\\\frac{3}{4}\end{pmatrix}$ ***M1***

hence $\boldsymbol{P}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{4}\\-\frac{1}{4}\end{pmatrix}$ ***A1***

$\boldsymbol{P}\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{4}\\1+\frac{\sqrt{3}}{4}\end{pmatrix}$ ***M1***

hence $\boldsymbol{P}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{4}\\\frac{\sqrt{3}}{4}\end{pmatrix}$ ***A1***

OR

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{4} \\ \frac{3}{4} \end{pmatrix}$$ ***M1***

hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{4} \\ -\frac{1}{4} \end{pmatrix}$ ***A1***

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{4} \\ -\frac{1}{4} \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ 1+\frac{\sqrt{3}}{4} \end{pmatrix}$$ ***M1***

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix}$$ ***A1***

$$\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix}$$

THEN

$$\Rightarrow \boldsymbol{P} = \begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix}$$ ***AG***

[6 marks]

(b) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ ***A1***

[1 mark]

(c) (i) **EITHER**

$S^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ ***(A1)***

$R = PS^{-1}$ ***(M1)***

Note: The ***M1*** is for an attempt at rearranging the matrix equation. Award even if the order of the product is reversed.

$$R = \begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$ ***(A1)***

OR

$$\begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix} = R \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

let $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

attempt to solve a system of equations ***M1***

$\frac{\sqrt{3}}{4} = 0.5a$, $\frac{1}{4} = 0.5b$

$-\frac{1}{4} = 0.5c$, $\frac{\sqrt{3}}{4} = 0.5d$ ***A2***

Note: Award ***A1*** for two correct equations, ***A2*** for all four equations correct.

THEN

$R = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ **OR** $\begin{pmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{pmatrix}$ **OR** $\left(\begin{pmatrix} 0.866025\ldots & 0.5 \\ -0.5 & 0.866025\ldots \end{pmatrix}\right)$ ***A1***

(ii) clockwise **A1**

arccosine or arcsine of value in matrix seen **(M1)**

$30°$ **A1**

Note: Both ***A1*** marks are dependent on the answer to part (c)(i) and should only be awarded for a valid rotation matrix.

[7 marks]

(d) **METHOD 1**

(i) $\begin{pmatrix} a \\ b \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} a \\ b \end{pmatrix} + \boldsymbol{q}$ ***A1***

(ii) solving $\begin{pmatrix} a \\ b \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} a \\ b \end{pmatrix} + \boldsymbol{q}$ using simultaneous equations or $\boldsymbol{a} = (\boldsymbol{I} - \boldsymbol{P})^{-1}\boldsymbol{q}$

(M1)

$a = 0.651\ (0.651084...),\ b = 1.48\ \ (1.47662...)$ ***A1A1***

$$\left(a = \frac{5+2\sqrt{3}}{13},\ b = \frac{14+3\sqrt{3}}{13}\right)$$

METHOD 2

(i) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} x-a \\ y-b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$ ***A1***

Note: Accept substitution of x and y (and x' and y') with particular points given in the question.

(ii) $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} 0-a \\ 0-b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$ ***(M1)***

Note: This line, with any of the points substituted, may be seen in part (d)(i) and if so the ***M1*** can be awarded there.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (\boldsymbol{I} - \boldsymbol{P})\begin{pmatrix} a \\ b \end{pmatrix}$$

$a = 0.651084...,\ b = 1.47662...$ ***A1A1***

$$\left(a = \frac{5+2\sqrt{3}}{13},\ b = \frac{14+3\sqrt{3}}{13}\right)$$

[4 marks]

[Total 18 marks]

Question 9

(a) (i) 0.02 **A1**

(ii) the probability of mutating from 'not normal state' to 'normal state' **A1**

Note: The ***A1*** can only be awarded if it is clear that transformation is **from** the mutated state.

[2 marks]

(b) $\det\begin{pmatrix} 0.94-\lambda & 0.02 \\ 0.06 & 0.98-\lambda \end{pmatrix} = 0$ ***(M1)***

Note: Award ***M1*** for an attempt to find eigenvalues. Any indication that $\det(\boldsymbol{M}-\lambda\boldsymbol{I})=0$ has been used is sufficient for the ***(M1)***.

$(0.94-\lambda)(0.98-\lambda)-0.0012=0$ **OR** $\lambda^2-1.92\lambda+0.92=0$ ***(A1)***

$\lambda=1,\ 0.92\ \left(\frac{23}{25}\right)$ ***A1***

[3 marks]

(c) $\begin{pmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ **OR** $\begin{pmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0.92\begin{pmatrix} x \\ y \end{pmatrix}$ ***(M1)***

Note: This ***M1*** can be awarded for attempting to find either eigenvector.

$0.02y-0.06x=0$ **OR** $0.02y+0.02x=0$

$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***A1A1***

Note: Accept any multiple of the given eigenvectors.

[3 marks]

(d) (i) $\begin{pmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{pmatrix}^5\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **OR** $\begin{pmatrix} 0.744 & 0.0852 \\ 0.256 & 0.915 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ***(M1)***

Note: Condone omission of the initial state vector for the ***M1***.

$0.744 \quad (0.744311\ldots)$ ***A1***

(ii) $\begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$ ***(A1)***

Note: Award ***A1*** for $\begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$ **OR** $\begin{pmatrix} 0.25 & 0.25 \\ 0.75 & 0.75 \end{pmatrix}$ seen.

0.25 ***A1***

[4 marks]

Total [12 marks]

Question 10

(a) (i) **EITHER**

$115.5 = u_1 + (3-1) \times d \quad (115.5 = u_1 + 2d)$

$108 = u_1 + (8-1) \times d \quad (108 = u_1 + 7d)$ ***(M1)(A1)***

Note: Award ***M1*** for attempting to use the arithmetic sequence term formula, ***A1*** for both equations correct. Working for ***M1*** and ***A1*** can be found in parts (i) or (ii).

$(d = -1.5)$

1.5 (cups/day) ***A1***

Note: Answer must be written as a positive value to award ***A1***.

OR

$(d =) \dfrac{115.5 - 108}{5}$ ***M1A1***

Note: Award ***M1*** for attempting a calculation using the difference between term 3 and term 8; ***A1*** for a correct substitution.

$(d =)$ 1.5 (cups/day) ***A1***

(ii) $(u_1 =)$ 118.5 (cups) ***A1***

[4 marks]

(b) attempting to substitute their values into the term formula for arithmetic sequence equated to zero ***(M1)***

$0 = 118.5 + (n-1) \times (-1.5)$

$(n =)$ 80 days ***A1***

Note: Follow through from part (a) only if their answer is positive.

[2 marks]

(c) $(t_5 =) \ 625 \times 1.064^{(5-1)}$ ***(M1)(A1)***

Note: Award ***M1*** for attempting to use the geometric sequence term formula; ***A1*** for a correct substitution.

$ 801 ***A1***

Note: The answer must be rounded to a whole number to award the final ***A1***.

[3 marks]

(d) (i) $(S_{10} =)$ (\$) 8390 (8394.39...) ***A1***

(ii) **EITHER**

the total cost (of dog food) ***R1***
for 10 years beginning in 2021 **OR** 10 years before 2031 ***R1***

OR

the total cost (of dog food) ***R1***
from 2021 to 2030 (inclusive) **OR** from 2021 to (the start of) 2031 ***R1***

[3 marks]

(e) **EITHER**
According to the model, the cost of dog food per year will eventually be too high to keep a dog.
OR
The model does not necessarily consider changes in inflation rate.
OR
The model is appropriate as long as inflation increases at a similar rate.
OR
The model does not account for changes in the amount of food the dog eats as it ages/becomes ill/stops growing.
OR
The model is appropriate since dog food bags can only be bought in discrete quantities. ***R1***

Note: Accept reasonable answers commenting on the appropriateness of the model for the specific scenario. There should be a reference to the given context. A reference to the geometric model must be clear: either "model" is mentioned specifically, or other mathematical terms such as "increasing" or "discrete quantities" are seen. Do not accept a contextual argument in isolation, e.g. "The dog will eventually die".

[1 mark]
Total [13 marks]

Question 11

(a) $(\boldsymbol{T} =)$ $\begin{matrix} \text{(B)} & \text{(G)} & \text{(N)} \end{matrix}$ $\begin{pmatrix} 0.945 & 0.015 & 0.02 \\ 0.05 & 0.965 & 0.03 \\ 0.005 & 0.02 & 0.95 \end{pmatrix}$ ***M1A1A1***

Note:	Accept the columns in any order. Accept the transpose of this matrix. Award ***M1*** for a 3x3 matrix with all values between (but not including) 0 and 1, and all columns (or rows if transposed) adding up to 1, award ***A1*** for one correct row (or column if transposed) and ***A1*** for all rows (or columns if transposed) correct.

[3 marks]

(b) $(\boldsymbol{T}^6 =)$ $\begin{pmatrix} 0.72 & 0.077 & 0.098 \\ 0.24 & 0.83 & 0.16 \\ 0.035 & 0.098 & 0.74 \end{pmatrix}$ ***(M1)***

Note:	Accept a transposed matrix.

multiplying their $\boldsymbol{T}^6$ by a correct matrix of the initial populations ***(M1)***

$$\begin{pmatrix} 0.72 & 0.077 & 0.098 \\ 0.24 & 0.83 & 0.16 \\ 0.035 & 0.098 & 0.74 \end{pmatrix}\begin{pmatrix} 26000 \\ 240000 \\ 50000 \end{pmatrix}$$

Note:	Award this ***M1*** for a transposed $\boldsymbol{T}$ if used correctly in part (b) i.e. preceded by 1×3 matrix rather than followed by a 3×1 matrix.

$$= \begin{pmatrix} 42133 \\ 212205 \\ 61661 \end{pmatrix}$$ ***(A1)***

so the expected population of the German side would be 212000 (212205) ***A1***

Note:	Award ***M0M1A0A1*** for an answer of 174000 $(=174031)$. This is the case when $\boldsymbol{T}^{30}$ has been used.

[4 marks]

(c) (i) $$\begin{pmatrix} 0.945 & 0.015 & 0.02 \\ 0.05 & 0.965 & 0.03 \\ 0.005 & 0.02 & 0.95 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

at least two of these three:

$0.945u_1 + 0.015u_2 + 0.02u_3 = u_1$

$0.05u_1 + 0.965u_2 + 0.03u_3 = u_2$ ***A1***

$0.005u_1 + 0.02u_2 + 0.95u_3 = u_3$

and

$u_1 + u_2 + u_3 = 1$ (may be seen in part (c)(ii)) ***A1***

(ii) $(\boldsymbol{u} =) \begin{pmatrix} 0.231 \\ 0.533 \\ 0.236 \end{pmatrix} \left(\boldsymbol{u} = \begin{pmatrix} 0.231155\ldots \\ 0.532663\ldots \\ 0.236180\ldots \end{pmatrix} \right)$ ***A1***

Note:	The ***A1*** in part (c)(ii) can be awarded independently of the working in part (c)(i).

[3 marks]

(d) $0.532663\ldots \times (26000 + 240000 + 50000)$ ***(M1)***

$= 168000 \;\; (168321\ldots)$ ***A1***

Note:	Award ***(M1)A1*** for answers using T^n with n large that lead to a correct answer. Award ***(M0)A0*** for answers that use T^n that lead to an incorrect answer.

[2 marks]

(e) *Award **R1** for each appropriate reason. For example:*

Movement unlikely to be constant

Total population for entire region likely to grow over time

Each power of the transition matrix takes five years; a relatively long time in terms of population movement.

There may be other/new external factors such as wars in other adjoining countries, leading to an influx of economic migrants.

R1R1

Note:	Do not award ***R1*** for any response that shows a lack of understanding of the assumption that the total population remains constant.

[2 marks]

Total [14 marks]

Question 12

(a) using area of trapezoid formula **M1**

$\sin(15^\circ)\times\frac{1+2}{2}$ **A1**

$=\frac{3}{2}\sin(15^\circ)$ **AG**

[2 marks]

(b) (i) $\boldsymbol{M}_6=\begin{pmatrix}\frac{1}{2}\cos 90^\circ & -\frac{1}{2}\sin 90^\circ\\ \frac{1}{2}\sin 90^\circ & \frac{1}{2}\cos 90^\circ\end{pmatrix}$ **(M1)**

$=\begin{pmatrix}0 & -\frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}$ **A1**

(ii) multiplying their part (b)(i) and point $(0,\ -1)$ (in any order) **M1**

$\begin{pmatrix}0 & -\frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\times\begin{pmatrix}0\\ -1\end{pmatrix}$

$\left(\frac{1}{2},0\right)$ **A1**

[4 marks]

(c) (i) $\begin{pmatrix} \cos(k\times 15^\circ) & -\sin(k\times 15^\circ) \\ \sin(k\times 15^\circ) & \cos(k\times 15^\circ) \end{pmatrix}$ ***A1***

(ii) $\begin{pmatrix} 1-\frac{k}{12} & 0 \\ 0 & 1-\frac{k}{12} \end{pmatrix}$ ***A1***

(iii) $k\times 15^\circ$ ***A1***

(iv) $1-\frac{k}{12}$ ***A1***

[4 marks]

(d) **METHOD 1 (using part (c)(iv))**

$\left(1-\frac{k}{12}\right)^2$ ***A2***

METHOD 2 (using full matrix M_k)

$$\left|\begin{pmatrix} \left(1-\frac{k}{12}\right)\cos(k\times 15^\circ) & -\left(1-\frac{k}{12}\right)\sin(k\times 15^\circ) \\ \left(1-\frac{k}{12}\right)\sin(k\times 15^\circ) & \left(1-\frac{k}{12}\right)\cos(k\times 15^\circ) \end{pmatrix}\right|$$

$=\left(1-\frac{k}{12}\right)^2\cos^2(k\times 15^\circ)+\left(1-\frac{k}{12}\right)^2\sin^2(k\times 15^\circ)$ ***(M1)***

$=\left(1-\frac{k}{12}\right)^2\left(\cos^2(k\times 15^\circ)+\sin^2(k\times 15^\circ)\right)$

$=\left(1-\frac{k}{12}\right)^2$ ***A1***

[2 marks]

(e) recognizing to multiply by 2 and by original area ***(M1)***

attempt to sum their answer to part (d), $k = 0, 1, ..., 11$ ***(M1)***

a correct expression ***(A1)***

e.g. $0.776457...\left(1^2 + \left(\frac{11}{12}\right)^2 + ... + \left(\frac{1}{12}\right)^2\right)$ **OR** $2\sum_{k=0}^{11}\left(1-\frac{k}{12}\right)^2 \times \frac{3}{2}\sin 15^\circ$

OR $\sum_{k=0}^{11}\left(1-\frac{k}{12}\right)^2 \times 0.776457...$ **OR** $2\sum_{k=1}^{12}\left(\frac{k}{12}\right)^2 \times \frac{3}{2}\sin\left(15^\circ\right)$

3.50 $(3.50484...)$ (square units)

Note: Award at most ***M0(M1)(A1)A0*** for an unsupported final answer of "1.75242..."

(A1)

[4 marks]

(f) $N_k = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \times M_k$ ***A1A1***

Note: Award ***A1A0*** if correct matrices are written in the wrong order.

[2 marks]

[Total: 18 marks]

Question 13

(a) $x = -1 + 2\lambda,\ y = 1 - \lambda$ ***A1***

[1 mark]

(b) $\begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix}\begin{pmatrix} -1+2\lambda \\ 1-\lambda \end{pmatrix} = \begin{pmatrix} 6-5\lambda \\ -8+15\lambda \end{pmatrix}$ ***(M1)(A1)***

$\boldsymbol{r} = \begin{pmatrix} 6 \\ -8 \end{pmatrix} + \lambda\begin{pmatrix} -5 \\ 15 \end{pmatrix}$ (or equivalent) ***(M1)A1***

Note: Award ***(M1)*** for the correct format of a vector equation of a line, ***A1*** for the line being completely correct.

[4 marks]

(c) (i) $\begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix}$ **OR** $\begin{pmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{pmatrix}$ **OR** $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ ***A1***

(ii) $\begin{pmatrix} 5\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{pmatrix}$ ***A1***

[2 marks]

(d) $(\boldsymbol{R} =) \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$ ***A1***

[1 mark]

(e) (i) attempt to multiply matrices from part (c) (in any order) ***(M1)***

e.g. $X = \begin{pmatrix} 5\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix}$

$X = \begin{pmatrix} 5 & -5 \\ 5 & 5 \end{pmatrix}$ ***A1***

(ii) substituting $\boldsymbol{T}$, $\boldsymbol{R}$ and $\boldsymbol{X}$ ***(M1)***

$\begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \begin{pmatrix} 5 & -5 \\ 5 & 5 \end{pmatrix}$

multiplying by inverse (in any order) ***(M1)***

$\begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix} \begin{pmatrix} 5 & -5 \\ 5 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$

$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$ ***A1***

$\cos 2\alpha = -\frac{3}{5}$ **AND** $\sin 2\alpha = \frac{4}{5}$

$\alpha = 1.11 \quad (=1.107148...)$ **OR** $63.4^\circ \quad (63.4349\ldots^\circ)$ ***A1***

[6 marks]
[Total 14 marks]

Question 14

(a) (i) attempt to find 15% or 85% of 285000 ***(M1)***

285000×0.85

$242\,250$ (USD) ***A1***

Note: Do not award ***A1*** if answer is not given exact.

(ii) $N = 360$

$I\% = 4$

$PV = (\pm)242\,250$

$FV = 0$

$P/Y = 12$

$C/Y = 12$ ***(M1)(A1)***

Note: Award ***M1*** for an attempt to use a financial app in their technology with at least two entries seen, award ***A1*** for all entries correct.

$(PMT =)\ 1156.54$ (USD) ***A1***

Note: Do not award final ***A1*** if answer is not given to 2 dp.

[5 marks]

(b) 1156.54×360 ***(M1)***

416354 (USD) ***A1***

Note: Do not award ***A1*** if answer is not given to the nearest dollar, unless already penalized in part (a)(ii).

[2 marks]

(c) $I\% = 4$

$PV = (\pm)242\,250$

$PMT = (\mp)1300$

$FV = 0$

$P/Y = 12$

$C/Y = 12$ ***(A1)***

Note: Award ***A1*** for $PMT = (\mp)1300$ seen.

$(N =)\ 292$ ***A1***

[2 marks]

(d) **METHOD 1**

$N = 291$
$I\% = 4$
$PV = (\pm)242\,250$
$PMT = (\mp)1300$
$P/Y = 12$
$F/Y = 12$ ***(A1)***

Note: Award ***A1*** for $N = 291$ seen.

$(FV =)$ 871.91 $(871.908\ldots)$ ***A1***

valid attempt to find interest in final month (e.g. $N = 1$ **OR** $PV = 871.91$) ***(M1)***

$N = 1$
$I\% = 4$
$PV = 871.91$ $(871.908\ldots)$
$FV = 0$
$P/Y = 12$
$F/Y = 12$

$(PMT =)$ 874.82 (USD) ***A1***

Note: Do not award ***A1*** if answer is not given correct to 2dp, unless already penalized previously.

METHOD 2

$N = 292$
$I\% = 4$
$PV = (\pm)242\,250$
$PMT = (\mp)1300$
$P/Y = 12$
$F/Y = 12$ ***(A1)***

Note: Award ***A1*** for $N = 292$ seen.

$(FV =)$ $425.185\ldots$ ***A1***

$1300 - 425.185...$ ***(A1)***
$(PMT =)$ 874.82 (USD) ***A1***

Note: Accept 874.81. Do not award ***A1*** if answer is not given correct to 2dp, unless already penalized previously.

[4 marks]

(e) $291 \times 1300 + 874.82$ *(M1)*

379174.82

attempt to find difference between their value and their part (b) *(M1)*
$(416354 - 379174.82)$

37179 (USD) *A1*

Note: Accept 37180 (USD) from using the 2 dp. answer from part (b). Do not penalize for not rounding to nearest dollar if this has already been penalized in part (b).

[3 marks]
[Total 16 marks]

Question 15

(a) $\begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix}$ ***M1A1***

Note: Award ***M1*** for correct values used, ***A1*** if in correct positions.
Accept alternative consistent matrix (e.g. the transpose or diagonal elements exchanged) and follow through to eigenvectors and initial state vector.

[2 marks]

(b) 5 (seen) ***(A1)***

$\begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix}^5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.596608 \\ 0.403392 \end{pmatrix}$ **OR** $\begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix}^5 = \begin{pmatrix} 0.596608 & 0.268928 \\ 0.403392 & 0.731072 \end{pmatrix}$ ***(M1)***

$\text{P(Friday evening)} = 0.403 \ \ (0.403392)$ ***A1***

Note: Award ***A0M1A0*** for use of 4 (and resulting probability 0.354).

[3 marks]

(c) attempt to find $\det(\boldsymbol{A} - \lambda \boldsymbol{I})$ ***(M1)***

$\begin{vmatrix} 0.88-\lambda & 0.08 \\ 0.12 & 0.92-\lambda \end{vmatrix}$ **OR** $(0.88-\lambda)(0.92-\lambda)-(0.12)(0.08)$

$\lambda^2 - 1.8\lambda + 0.8$ ***A1***

[2 marks]

(d) eigenvalues are 0.8 and 1 ***(A1)***

Note: If no attempt is made to find eigenvectors, do not award ***A1*** for finding eigenvalues.

$\begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.8 \begin{pmatrix} x \\ y \end{pmatrix}$

$0.88x + 0.08y = 0.8x$

eigenvector = eg. $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***A1***

EITHER

$\begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$ ***(M1)***

$0.88x + 0.08y = x$

$0.08y = 0.12x$

OR

eigenvalue 1 gives

$\begin{pmatrix} -0.12 & 0.08 \\ 0.12 & -0.08 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ***(M1)***

$-0.12x + 0.08y = 0$

$0.08y = 0.12x$

Note: Award ***M1*** for an attempt to find the eigenvector with eigenvalue 1.

THEN

eigenvector = eg. $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ *A1*

Note: Award ***A0A1M0A0*** if only $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is seen and no eigenvalues are found.

[4 marks]

(e) $\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix}$, $\boldsymbol{P} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$ **OR** $\boldsymbol{D} = \begin{pmatrix} 0.8 & 0 \\ 0 & 1 \end{pmatrix}$, $\boldsymbol{P} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ *A1A1*

Note: Award ***A1*** for one of $\boldsymbol{P}$ or $\boldsymbol{D}$ correct. Do not award the second ***A1*** unless $\boldsymbol{P}$ and $\boldsymbol{D}$ are consistent.

[2 marks]

(f) **EITHER**

attempt to use $\boldsymbol{T}^n = (\boldsymbol{PDP}^{-1})^n = \boldsymbol{PD}^n\boldsymbol{P}^{-1}$ *M1*

Note: Award ***M1*** for their $\boldsymbol{D}^n$ seen.

limit of $\boldsymbol{D}^n$ calculated *A1*

$\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}^{-1}$

Note: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ must be seen to award ***A1***.

OR

attempt to expand their $\boldsymbol{PD}^n\boldsymbol{P}^{-1}$ using explicit $\boldsymbol{P}$, $\boldsymbol{P}^{-1}$ *M1*

$(\boldsymbol{T}^n =)\ \frac{1}{5}\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.8^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$

$(\boldsymbol{T}^n =)\ \frac{1}{5}\begin{pmatrix} 2+3(0.8^n) & 2-2(0.8^n) \\ 3-3(0.8^n) & 3+2(0.8^n) \end{pmatrix}$ *A1*

Note: Using this method, the limit of 0.8^n may be inferred and ***M1A1*** awarded.

THEN

0.6 *A1*

Note: Multiplication by initial condition $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ may be seen at any point as part of their method.

For an answer of 0.6 from incomplete methods award a maximum of ***M1A0A0***, or if no working is seen, award ***M0A0A1***.

[3 marks]
[Total: 16 marks]

Question 16

(a) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ ***M1A1***

: Award ***M1*** is for a 3x3 matrix with at least one column correct.
Column order is not explicit in question and may not be labelled in candidate response; accept their correct adjacency matrix.

[2 marks]

(b) **EITHER**

$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^5$ ***(M1)***

$= \begin{pmatrix} 11 & 11 & 10 \\ 11 & 10 & 11 \\ 10 & 11 & 11 \end{pmatrix}$ ***(A1)***

OR

Listing at least 8 possible walks ***(M1)***
AAAAAB, AAABAB, AAABCB, AABAAB, AABCCB,
ABABAB, ABACAB, ABAAAB, ABCCCB, ABCBAB, ABCBCB ***(A1)***

THEN

11 different routes ***A1***
[3 marks]

(c) (i) 0.5^5 $\left(\frac{1}{32}, 0.03125\right)$ ***A1***

(ii) **EITHER**
there are 11 possible walks so probability is 11×0.5^5 ***M1***

OR
total number of (equally likely) walks from A is 32, 11 end up at B ***M1***

THEN
$\frac{11}{32}$ **OR** 0.344 (0.34375) ***A1***

[3 marks]

(d) (i) $(1 \times 0.4 =)\ 0.4$ ***A1***

(ii) $(0.5 \times 0.5 =)\ 0.25$ ***A1***

(iii) $(0.5 \times 0.5 + 0.5 \times 0.5 =)\ 0.5$ ***A1***

[3 marks]

(e) transition matrix is $\begin{pmatrix} 0 & 0.25 & 0.4 \\ 0.6 & 0.5 & 0.6 \\ 0.4 & 0.25 & 0 \end{pmatrix}$ (with order AB, AC and BC) ***(M1)(A1)***

> **Note:** Column order is not explicit in question and may not be labelled in candidate response; accept their correct transition matrix.
> Accept the transposed matrix.

$$\begin{pmatrix} 0 & 0.25 & 0.4 \\ 0.6 & 0.5 & 0.6 \\ 0.4 & 0.25 & 0 \end{pmatrix}^5$$ ***(M1)***

$$= \begin{pmatrix} 0.22215 & 0.227275 & 0.23239 \\ 0.54546 & 0.54545 & 0.54546 \\ 0.23239 & 0.227275 & 0.22215 \end{pmatrix}$$

$0.232\ \ (0.23239)$ ***A1***

[4 marks]

(f) (Taking a high power of a matrix)
long term probabilities are 0.227275, 0.545455 and 0.227275 ***(M1)***

B and 0.545 $\left(54.5\%\ \frac{6}{11}\right)$ ***A1A1***

> **Note:** Award ***(M0)A0A0*** for an unsupported answer of “B” (with either no probability or an incorrect probability).

[3 marks]
[Total 18 marks]

Question 17

(a) anticlockwise rotation of $15°$ about the origin **A1**

[1 mark]

(b) recognizing that *I* is equivalent to one rotation of $360°$ **(M1)**

e.g. $\frac{360}{15}$

$= 24$ **A1**

[2 marks]

(c) (i) $(\boldsymbol{B} =) \begin{pmatrix} 1.05 & 0 \\ 0 & 1.05 \end{pmatrix}$ **A1**

(ii) $(\boldsymbol{B}^{24} =) \begin{pmatrix} 3.23 & 0 \\ 0 & 3.23 \end{pmatrix}$ **(A1)**

Note: Award **A1** for 3.23 $(3.22509\ldots)$ **OR** 1.05^{24}.

enlargement, with a scale factor of 3.23 (1.05^{24}), (centre $(0, 0)$) **A1**

[3 marks]

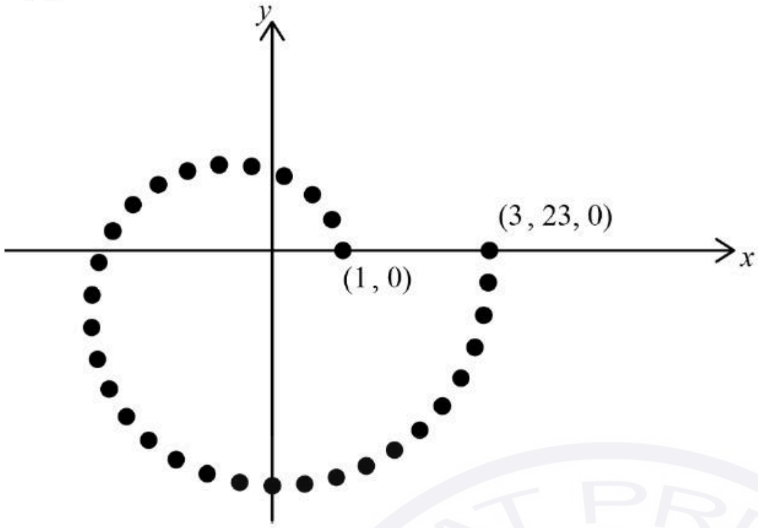
(d) $\boldsymbol{C} = \begin{pmatrix} 1.05\cos(15°) & -1.05\sin(15°) \\ 1.05\sin(15°) & 1.05\cos(15°) \end{pmatrix}$

$\left(= \begin{pmatrix} 1.01\ (1.01422...) & -0.272\ (-0.271759...) \\ 0.272\ (0.271759...) & 1.01\ (1.01422...) \end{pmatrix} = \frac{21}{80}\begin{pmatrix} \sqrt{6}+\sqrt{2} & -\sqrt{6}+\sqrt{2} \\ \sqrt{6}-\sqrt{2} & \sqrt{6}+\sqrt{2} \end{pmatrix} \right)$ **A2**

Note: Award **A1** for at least two correct elements in the matrix.

[2 marks]

(e)



anticlockwise spiral **A1**

starting at $(1, 0)$ and ending at $(3.23, 0)$ **A1**

Note: Accept a continuous curve instead of discrete points. The two ***A1*** marks can be awarded independently

[2 marks]

(f) $T\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$ ***(M1)***

$0.9p + 2 = p$ **OR** $0.8q + 1 = q$ ***(A1)***

$p = 20, q = 5$ **OR** $\begin{pmatrix} 20 \\ 5 \end{pmatrix}$ ***A1***

[3 marks]

[Total: 13 marks]

Question 18

(a) recognition of a geometric sequence **(M1)**

$3\left(\frac{2}{3}\right)^4$ **OR** 3, 2, 4/3…

= 0.593 (0.592592…, 16/27) (cm) **A1**

[2 marks]

(b) attempt to sum a geometric sequence **(M1)**

$\frac{3\left(1-\left(\frac{2}{3}\right)^5\right)}{1-\frac{2}{3}}$ **OR** $3+2+\frac{4}{3}\ldots$

= 7.81 (7.81481…, 211/27) (cm) **A1**

[2 marks]

(c) recognition of need to find sum to infinity **(M1)**

$\frac{3}{1-\frac{2}{3}}$ **(A1)**

= 9 (cm) **A1**

[3 marks]

(d) Comparing the sum of the widths greater than (or equal to) 8.5 **(M1)**

e.g. $\frac{3\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\frac{2}{3}} \geq 8.5$ **OR** sketch **OR** list of values with cross-over values

7.13 (7.1285338740543…) seen **(A1)**

$n \geq 7.13$

$n = 8$ **A1**

[3 marks]

(e) attempt to divide two adjacent areas **OR** $\left(\frac{2}{3}\right)^2$ **(M1)**

$\frac{4}{9}$ (0.444, 0.444444…) **A1**

[2 marks]

(f) $u_1 = 13.5$ (may be seen in part (e)) ***(A1)***

attempt to find the sum of their n terms, with their u_1 and their r ***M1***

$$S_8 = \frac{13.5\left(1-\left(\frac{4}{9}\right)^8\right)}{1-\frac{4}{9}}$$

Note: Do not award ***M1*** if 5 or infinity used for n or if $\frac{2}{3}$ used for r.

$= 24.3\ (24.2630\ldots)$ (cm²) ***A1***

Note: Accept $24.2\ (24.2439\ldots)$ from using 0.444.

[3 marks]

[Total: 15 marks]