Subject – Math AI(Higher Level)
Topic - Number and Algebra
Year - May 2021 – Nov 2022
Paper -2
Answers

Question 1

(a) $\begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}$ ***M1A1***

[2 marks]

(b) $\begin{vmatrix} 0.8-\lambda & 0.1 \\ 0.2 & 0.9-\lambda \end{vmatrix} = 0$ ***M1***

$\lambda = 1$ and 0.7 ***A1***

eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***(M1)A1***

Note: Accept any scalar multiple of the eigenvectors.

[4 marks]

(c) ***EITHER***

$\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ $\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix}$ ***A1A1***

OR

$\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ $\boldsymbol{D} = \begin{pmatrix} 0.7 & 0 \\ 0 & 1 \end{pmatrix}$ ***A1A1***

[2 marks]

(d) $\boldsymbol{P}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ ***A1***

$\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.7^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1200 \\ 1200 \end{pmatrix}$ ***M1A1***

attempt to multiply matrices ***M1***

so in company A, after n years, $400\left(2+0.7^n\right)$ ***A1***

[5 marks]

(e) $400 \times 2 = 800$ ***A1***

[1 mark]

Total [14 marks]

Question 2

(a) (i) $\mathrm{N} = 24$

$\mathrm{I\%} = 14$

$\mathrm{PV} = -14000$

$\mathrm{FV} = 0$

$\mathrm{P/Y} = 4$

$\mathrm{C/Y} = 4$ ***(M1)(A1)***

> **Note:** Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct. Accept $\mathrm{PV} = 14000$.

$(€)871.82$ ***A1***

(ii) $4 \times 6 \times 871.82$ ***(M1)***

$(€)20923.68$ ***A1***

(iii) $20923.68 - 14000$ ***(M1)***

$(€)6923.68$ ***A1***

[7 marks]

(b) (i) $0.9 \times 14000\ (= 14000 - 0.10 \times 14000)$ ***M1***

$(€)12600.00$ ***A1***

(ii) $\mathrm{N} = 72$

$\mathrm{PV} = 12600$

$\mathrm{PMT} = -250$

$\mathrm{FV} = 0$

$\mathrm{P/Y} = 12$

$\mathrm{C/Y} = 12$ ***(M1)(A1)***

> **Note:** Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct. Accept $\mathrm{PV} = -12600$ provided $\mathrm{PMT} = 250$.

$12.56(\%)$ ***A1***

[5 marks]

(c) ***EITHER***

Bryan should choose Option A ***A1***

no deposit is required ***R1***

Note: Award ***R1*** for stating that no deposit is required. Award ***A1*** for the correct choice from that fact. Do not award ***R0A1.***

OR

Bryan should choose Option B ***A1***

cost of Option A $(6923.69) >$ cost of Option B $(72 \times 250 - 12\,600 = 5400)$ ***R1***

Note: Award ***R1*** for a correct comparison of costs. Award ***A1*** for the correct choice from that comparison. Do not award ***R0A1***.

[2 marks]

(d) real interest rate is $0.4 - 0.1 = 0.3\%$ ***(M1)***

value of other payments $250 + 250 \times 1.003 + \ldots + 250 \times 1.003^{71}$
use of sum of geometric sequence formula or financial app on a GDC ***(M1)***

$= 20\,058.43$

value of deposit at the end of 6 years

$1400 \times (1.003)^{72} = 1736.98$ ***(A1)***

Total value is $(€)\,21\,795.41$ ***A1***

Note: Both ***M*** marks can awarded for a correct use of the GDC's financial app:

$\text{N} = 72\ (6 \times 12)$
$\text{I\%} = 3.6\ (0.3 \times 12)$
$\text{PV} = 0$
$\text{PMT} = -250$
$\text{FV} =$
$\text{P/Y} = 12$
$\text{C/Y} = 12$

OR

$\text{N} = 72\ (6 \times 12)$
$\text{I\%} = 0.3$
$\text{PV} = 0$
$\text{PMT} = -250$
$\text{FV} =$
$\text{P/Y} = 1$
$\text{C/Y} = 1$

[4 marks]

Total [18 marks]

Question 3

(a) finding T^3 **OR** use of tree diagram ***(M1)***

$T^3 = \begin{pmatrix} 0.65 & 0.525 \\ 0.35 & 0.475 \end{pmatrix}$

the probability of sunny in three days' time is 0.65 ***A1***

[2 marks]

(b) attempt to find eigenvalues ***(M1)***

Note: Any indication that $\det(T - \lambda I) = 0$ has been used is sufficient for the ***(M1)***.

$$\begin{vmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{vmatrix} = (0.8-\lambda)(0.7-\lambda) - 0.06 = 0$$

$(\lambda^2 - 1.5\lambda + 0.5 = 0)$

$\lambda = 1,\ \lambda = 0.5$ ***A1***

attempt to find either eigenvector ***(M1)***

$0.8x + 0.3y = x \Rightarrow -0.2x + 0.3y = 0$ so an eigenvector is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ ***A1***

$0.8x + 0.3y = 0.5x \Rightarrow 0.3x + 0.3y = 0$ so an eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***A1***

Note: Accept multiples of the stated eigenvectors.

[5 marks]

(c) (i) $P = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$ **OR** $P = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$ ***A1***

Note: Examiners should be aware that different, correct, matrices P may be seen.

(ii) $D = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}$ **OR** $D = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$ ***A1***

Note: P and D must be consistent with each other.

[2 marks]

(d) $0.5^n \to 0$ ***(M1)***

$D^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ **OR** $D^n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ***(A1)***

Note: Award ***A1*** only if their D^n corresponds to their P.

$PD^nP^{-1} = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}$ ***(M1)***

$60\ \%$ ***A1***

[4 marks]

Total [13 marks]

Question 4

(a) **EITHER**

$N = 2$
$PV = -37\,000$
$I\% = 6.4$
$P/Y = 1$
$C/Y = 4$ ***(M1)(A1)***

Note:	Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct.

OR

$N = 8$
$PV = -37\,000$
$I\% = 6.4$
$P/Y = 4$
$C/Y = 4$ ***(M1)(A1)***

Note:	Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct.

OR

$$FV = 37\,000 \times \left(1 + \frac{6.4}{100 \times 4}\right)^{4\times2}$$ ***(M1)(A1)***

Note:	Award ***M1*** for substitution into compound interest formula, ***(A1)*** for correct substitution.

$= 42\,010 \text{ AUD}$ ***A1***

Note:	Award ***(M1)(A1)A0*** for unsupported 42009.87.

[3 marks]

(b) **EITHER**

$PV = -37\,000$
$FV = 50\,000$
$I\% = 6.4$
$P/Y = 1$
$C/Y = 4$ ***(M1)(A1)***

OR

$PV = -37\,000$

$FV = 50\,000$

$I\% = 6.4$

$P/Y = 4$

$C/Y = 4$ ***(M1)(A1)***

Note: Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct.

OR

$50000 < 37\,000 \times \left(1 + \frac{6.4}{100 \times 4}\right)^{4 \times n}$ **OR** $50\,000 < 37\,000 \times \left(1 + \frac{6.4}{100 \times 4}\right)^{n}$ ***(M1)(A1)***

Note: Award ***M1*** for the correct inequality, $50\,000$ and substituted compound interest formula. Allow an equation. Award ***A1*** for correct substitution.

THEN

$N = 4.74$ (years) $(4.74230...)$ **OR** $N = 18.9692...$ (quarters) ***(A1)***

$m = 57$ months ***A1***

Note: Award ***A1*** for rounding their m to the correct number of months. The final answer must be a multiple of 3. Follow through within this part.

[4 marks]

(c) $150\,000$ AUD ***A1***

[1 mark]

(d) (i) $120 \times 1700 - 150\,000$ ***(M1)***

$= 54\,000$ AUD ***A1***

(ii) $N = 120$

$PV = -150\,000$

$PMT = 1700$

$FV = 0$

$P/Y = 12$

$C/Y = 12$ ***(M1)(A1)***

Note: Award ***M1*** for an attempt to use a financial app in their technology or an attempt to use an annuity formula or $FV = 0$ seen. If a compound interest formula is equated to zero, award ***M1***, otherwise award ***M0*** for a substituted compound interest formula.
Award ***A1*** for all entries correct in financial app or correct substitution in annuity formula, but award ***A0*** for a substituted compound interest formula.
Follow through marks in part (d)(ii) are contingent on working seen.

$r = 6.46$ (%) (6.45779...) ***A1***

[5 marks]

(e) $N = 60$

$I = 6.46$ (6.45779...)

$PV = -150\,000$

$PMT = 1700$

$P/Y = 12$

$C/Y = 12$ ***(M1)(A1)***

Note: Award ***M1*** for an attempt to use a financial app in their technology or an attempt to use an annuity formula. Award ***(M0)*** for a substituted compound interest formula. Award ***A1*** for all entries correct. Follow through marks in part (e) are contingent on working seen.

$FV = 86973$ AUD ***A1***

[3 marks]

(f) $204\,000-(60\times1700+86973)$ **OR** $204\,000-188973$ ***(M1)(M1)***

Note: Award ***M1*** for 60×1700. Award ***M1*** for subtracting their $(60\times1700+86973)$ from their $(204\,000)$. Award at most ***M1M0*** for their $204\,000-(60\times1700)$ or ***M0M0*** for their $204\,000-(86973)$. Follow through from parts (d)(i) and (e). Follow through marks in part (f) are contingent on working seen.

$15\,027$ AUD ***A1***

[3 marks]

Total [19 marks]

Question 5

(a) $\boldsymbol{T}=\begin{pmatrix}0.965 & 0.05\\0.035 & 0.95\end{pmatrix}$ ***M1A1***

> **Note:** Award ***M1A1*** for $\boldsymbol{T}=\begin{pmatrix}0.95 & 0.035\\0.05 & 0.965\end{pmatrix}$.
>
> Award the ***A1*** for a transposed $\boldsymbol{T}$ if used correctly in part (b) i.e. preceded by 1×2 matrix $(2100 \quad 3500)$ rather than followed by a 2×1 matrix.

[2 marks]

(b) $\begin{pmatrix}0.965 & 0.05\\0.035 & 0.95\end{pmatrix}^2\begin{pmatrix}2100\\3500\end{pmatrix}$ ***(M1)***

$=\begin{pmatrix}2294\\3306\end{pmatrix}$

so ratio is $2294:3306$ $(=1147:1653,\ 0.693889...)$ ***A1***

[2 marks]

(c) to solve $Ax=\lambda x$:

$\begin{vmatrix}0.965-\lambda & 0.05\\0.035 & 0.95-\lambda\end{vmatrix}=0$ ***(M1)***

$(0.965-\lambda)(0.95-\lambda)-0.05\times0.035=0$

$\lambda=0.915$ **OR** $\lambda=1$ ***(A1)***

attempt to find eigenvectors for at least one eigenvalue ***(M1)***

when $\lambda=0.915$, $x=\begin{pmatrix}1\\-1\end{pmatrix}$ (or any real multiple) ***(A1)***

when $\lambda=1$, $x=\begin{pmatrix}10\\7\end{pmatrix}$ (or any real multiple) ***(A1)***

therefore $\boldsymbol{P}=\begin{pmatrix}1 & 10\\-1 & 7\end{pmatrix}$ (accept integer valued multiples of their eigenvectors and columns in either order) ***A1***

[6 marks]

(d) $P^{-1} = \begin{pmatrix} 1 & 10 \\ -1 & 7 \end{pmatrix}^{-1} = \frac{1}{17}\begin{pmatrix} 7 & -10 \\ 1 & 1 \end{pmatrix}$ ***(A1)***

Note: This mark is independent, and may be seen anywhere in part (d).

$D = \begin{pmatrix} 0.915 & 0 \\ 0 & 1 \end{pmatrix}$ ***(A1)***

$T^n = PD^nP^{-1} = \begin{pmatrix} 1 & 10 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 0.915^n & 0 \\ 0 & 1^n \end{pmatrix}\frac{1}{17}\begin{pmatrix} 7 & -10 \\ 1 & 1 \end{pmatrix}$ ***(M1)A1***

Note: Award ***(M1)A0*** for finding $P^{-1}D^nP$ correctly.

as $n \to \infty$, $D^n = \begin{pmatrix} 0.915^n & 0 \\ 0 & 1^n \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ***R1***

so $T^n \to \frac{1}{17}\begin{pmatrix} 1 & 10 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 7 & -10 \\ 1 & 1 \end{pmatrix}$ ***A1***

$= \begin{pmatrix} \frac{10}{17} & \frac{10}{17} \\ \frac{7}{17} & \frac{7}{17} \end{pmatrix}$ ***AG***

Note: The ***AG*** line must be seen for the final ***A1*** to be awarded.

[6 marks]

(e) **METHOD ONE**

$\begin{pmatrix} \frac{10}{17} & \frac{10}{17} \\ \frac{7}{17} & \frac{7}{17} \end{pmatrix}\begin{pmatrix} 2100 \\ 3500 \end{pmatrix} = \begin{pmatrix} 3294 \\ 2306 \end{pmatrix}$ ***(M1)***

so ratio is $3294:2306$ $(1647:1153, 1.42844..., 0.700060...)$

A1

METHOD TWO

long term ratio is the eigenvector associated with the largest eigenvalue ***(M1)***

$10:7$ ***A1***

[2 marks]

Total [18 marks]

Question 6

(a) (i) rotation anticlockwise $\frac{\pi}{6}$ is $\begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix}$ **OR** $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ ***(M1)A1***

reflection in $y = \frac{x}{\sqrt{3}}$

$\tan\theta = \frac{1}{\sqrt{3}}$ ***(M1)***

$\Rightarrow 2\theta = \frac{\pi}{3}$ ***(A1)***

matrix is $\begin{pmatrix} 0.5 & 0.866 \\ 0.866 & -0.5 \end{pmatrix}$ **OR** $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ ***A1***

rotation clockwise $\frac{\pi}{3}$ is $\begin{pmatrix} 0.5 & 0.866 \\ -0.866 & 0.5 \end{pmatrix}$ **OR** $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ ***A1***

(ii) an attempt to multiply three matrices ***(M1)***

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$ ***(A1)***

$$\boldsymbol{P} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \textbf{ OR } \begin{pmatrix} 0.866 & -0.5 \\ -0.5 & -0.866 \end{pmatrix}$$ ***A1***

(iii) $\left(P^2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ***A1***

Note: Do not award ***A1*** if final answer not resolved into the identity matrix I.

(iv) if the overall movement of the drone is repeated ***A1***
the drone would return to its original position ***A1***

[12 marks]

(b) **METHOD 1**

$|\det \boldsymbol{P}| = \left| \left(-\frac{3}{4}\right) - \left(\frac{1}{4}\right) \right| = 1$ ***A1***

area of triangle ABC = area of triangle $\mathrm{A'B'C'} \times |\det \boldsymbol{P}|$ ***R1***

area of triangle ABC = area of triangle $\mathrm{A'B'C'}$ ***AG***

Note: Award at most ***A1R0*** for responses that omit modulus sign.

METHOD 2

statement of fact that rotation leaves area unchanged ***R1***
statement of fact that reflection leaves area unchanged ***R1***
area of triangle ABC = area of triangle $\mathrm{A'B'C'}$ ***AG***

[2 marks]

(c) attempt to find angles associated with values of elements in matrix $\boldsymbol{P}$ ***(M1)***

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos\left(-\frac{\pi}{6}\right) & \sin\left(-\frac{\pi}{6}\right) \\ \sin\left(-\frac{\pi}{6}\right) & -\cos\left(-\frac{\pi}{6}\right) \end{pmatrix}$$

reflection (in $y = (\tan\theta)x$) ***(M1)***

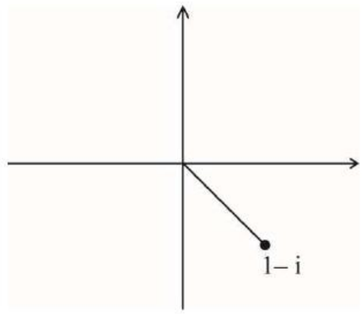
where $2\theta = -\frac{\pi}{6}$ ***A1***

reflection in $y = \tan\left(-\frac{\pi}{12}\right)x \ (= -0.268x)$ ***A1***

[4 marks]
Total: [18 marks]

Question 7

(a) (i)



_A1_

(ii) $z = \sqrt{2}e^{-\frac{i\pi}{4}}$ **_A1A1_**

Note: Accept an argument of $\frac{7\pi}{4}$. Do **NOT** accept answers that are not exact.

[3 marks]

(b) (i) $w_1 + w_2 = e^{ix} + e^{i\left(x - \frac{\pi}{2}\right)}$

$= e^{ix}\left(1 + e^{-\frac{i\pi}{2}}\right)$ **_(M1)_**

$= e^{ix}(1 - i)$ **_A1_**

(ii) $w_1 + w_2 = e^{ix} \times \sqrt{2}e^{-\frac{i\pi}{4}}$ **_M1_**

$= \sqrt{2}e^{i\left(x - \frac{\pi}{4}\right)}$ **_(A1)_**

attempt extract real part using cis form **_(M1)_**

$\text{Re}(w_1 + w_2) = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$ **OR** $1.4142\ldots\cos(x - 0.785398\ldots)$ **_A1_**

[6 marks]

(c) (i) $I_t = 12\cos(bt) + 12\cos\left(bt - \frac{\pi}{2}\right)$ **_(M1)_**

$I_t = 12\,\text{Re}\left(e^{ibt} + e^{i\left(bt - \frac{\pi}{2}\right)}\right)$ **_(M1)_**

$I_t = 12\sqrt{2}\cos\left(bt - \frac{\pi}{4}\right)$

max $= 12\sqrt{2}$ $(= 17.0)$ **_A1_**

(ii) phase shift $= \frac{\pi}{4}(= 0.785)$ **_A1_**

[4 marks]
Total: [13 marks]

Question 8

(a) (i) $\boldsymbol{P}\begin{pmatrix}0\\0\end{pmatrix}+\boldsymbol{q}=\begin{pmatrix}0\\1\end{pmatrix}$ ***(M1)***

$\boldsymbol{q}=\begin{pmatrix}0\\1\end{pmatrix}$ ***A1***

(ii) **EITHER**

$\boldsymbol{P}\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{4}\\\frac{3}{4}\end{pmatrix}$ ***M1***

hence $\boldsymbol{P}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{4}\\-\frac{1}{4}\end{pmatrix}$ ***A1***

$\boldsymbol{P}\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{4}\\1+\frac{\sqrt{3}}{4}\end{pmatrix}$ ***M1***

hence $\boldsymbol{P}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{4}\\\frac{\sqrt{3}}{4}\end{pmatrix}$ ***A1***

OR

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{4} \\ \frac{3}{4} \end{pmatrix}$$ ***M1***

hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{4} \\ -\frac{1}{4} \end{pmatrix}$ ***A1***

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{4} \\ -\frac{1}{4} \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ 1+\frac{\sqrt{3}}{4} \end{pmatrix}$$ ***M1***

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix}$$ ***A1***

$$\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix}$$

THEN

$$\Rightarrow \boldsymbol{P} = \begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix}$$ ***AG***

[6 marks]

(b) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ ***A1***

[1 mark]

(c) (i) **EITHER**

$$S^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$ ***(A1)***

$$R = PS^{-1}$$ ***(M1)***

Note: The ***M1*** is for an attempt at rearranging the matrix equation. Award even if the order of the product is reversed.

$$R = \begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$ ***(A1)***

OR

$$\begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix} = R \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

let $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

attempt to solve a system of equations ***M1***

$$\frac{\sqrt{3}}{4} = 0.5a, \quad \frac{1}{4} = 0.5b$$

$$-\frac{1}{4} = 0.5c, \quad \frac{\sqrt{3}}{4} = 0.5d$$ ***A2***

Note: Award ***A1*** for two correct equations, ***A2*** for all four equations correct.

THEN

$$R = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \text{ OR } \begin{pmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{pmatrix} \text{ OR } \left(\begin{pmatrix} 0.866025... & 0.5 \\ -0.5 & 0.866025... \end{pmatrix}\right)$$ ***A1***

(ii) clockwise **A1**

arccosine or arcsine of value in matrix seen ***(M1)***

$30°$ **A1**

Note: Both ***A1*** marks are dependent on the answer to part (c)(i) and should only be awarded for a valid rotation matrix.

[7 marks]

(d) **METHOD 1**

(i) $\begin{pmatrix} a \\ b \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} a \\ b \end{pmatrix} + \boldsymbol{q}$ ***A1***

(ii) solving $\begin{pmatrix} a \\ b \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} a \\ b \end{pmatrix} + \boldsymbol{q}$ using simultaneous equations or $\boldsymbol{a} = (\boldsymbol{I} - \boldsymbol{P})^{-1}\boldsymbol{q}$

(M1)

$a = 0.651\ (0.651084...),\ b = 1.48\ \ (1.47662...)$ ***A1A1***

$\left(a = \frac{5 + 2\sqrt{3}}{13},\ b = \frac{14 + 3\sqrt{3}}{13}\right)$

METHOD 2

(i) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} x - a \\ y - b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$ ***A1***

Note: Accept substitution of x and y (and x' and y') with particular points given in the question.

(ii) $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} 0 - a \\ 0 - b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$ ***(M1)***

Note: This line, with any of the points substituted, may be seen in part (d)(i) and if so the ***M1*** can be awarded there.

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (\boldsymbol{I} - \boldsymbol{P})\begin{pmatrix} a \\ b \end{pmatrix}$

$a = 0.651084...,\ b = 1.47662...$ ***A1A1***

$\left(a = \frac{5 + 2\sqrt{3}}{13},\ b = \frac{14 + 3\sqrt{3}}{13}\right)$

[4 marks]

[Total 18 marks]

Question 9

(a) (i) 0.02 **A1**

(ii) the probability of mutating from 'not normal state' to 'normal state' **A1**

Note: The ***A1*** can only be awarded if it is clear that transformation is **from** the mutated state.

[2 marks]

(b) $\det\begin{pmatrix} 0.94-\lambda & 0.02 \\ 0.06 & 0.98-\lambda \end{pmatrix} = 0$ ***(M1)***

Note: Award ***M1*** for an attempt to find eigenvalues. Any indication that $\det(\boldsymbol{M}-\lambda\boldsymbol{I})=0$ has been used is sufficient for the ***(M1)***.

$(0.94-\lambda)(0.98-\lambda)-0.0012=0$ **OR** $\lambda^2-1.92\lambda+0.92=0$ ***(A1)***

$\lambda=1,\ 0.92\ \left(\frac{23}{25}\right)$ ***A1***

[3 marks]

(c) $\begin{pmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ **OR** $\begin{pmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0.92\begin{pmatrix} x \\ y \end{pmatrix}$ ***(M1)***

Note: This ***M1*** can be awarded for attempting to find either eigenvector.

$0.02y-0.06x=0$ **OR** $0.02y+0.02x=0$

$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***A1A1***

Note: Accept any multiple of the given eigenvectors.

[3 marks]

(d) (i) $\begin{pmatrix} 0.94 & 0.02 \\ 0.06 & 0.98 \end{pmatrix}^5\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **OR** $\begin{pmatrix} 0.744 & 0.0852 \\ 0.256 & 0.915 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ***(M1)***

Note: Condone omission of the initial state vector for the ***M1***.

$0.744 \quad (0.744311\ldots)$ ***A1***

(ii) $\begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$ ***(A1)***

Note: Award ***A1*** for $\begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$ **OR** $\begin{pmatrix} 0.25 & 0.25 \\ 0.75 & 0.75 \end{pmatrix}$ seen.

0.25 ***A1***

[4 marks]

Total [12 marks]

Question 10

(a) (i) **EITHER**

$115.5 = u_1 + (3-1) \times d \quad (115.5 = u_1 + 2d)$

$108 = u_1 + (8-1) \times d \quad (108 = u_1 + 7d)$ ***(M1)(A1)***

Note:	Award ***M1*** for attempting to use the arithmetic sequence term formula, ***A1*** for both equations correct. Working for ***M1*** and ***A1*** can be found in parts (i) or (ii).

$(d = -1.5)$

1.5 (cups/day) ***A1***

Note:	Answer must be written as a positive value to award ***A1***.

OR

$(d =) \dfrac{115.5 - 108}{5}$ ***M1A1***

Note:	Award ***M1*** for attempting a calculation using the difference between term 3 and term 8; ***A1*** for a correct substitution.

$(d =)$ 1.5 (cups/day) ***A1***

(ii) $(u_1 =)$ 118.5 (cups) ***A1***

[4 marks]

(b) attempting to substitute their values into the term formula for arithmetic sequence equated to zero ***(M1)***

$0 = 118.5 + (n-1) \times (-1.5)$

$(n =)$ 80 days ***A1***

Note:	Follow through from part (a) only if their answer is positive.

[2 marks]

(c) $(t_5 =)\ 625 \times 1.064^{(5-1)}$ ***(M1)(A1)***

Note:	Award ***M1*** for attempting to use the geometric sequence term formula; ***A1*** for a correct substitution.

\$ 801 ***A1***

Note:	The answer must be rounded to a whole number to award the final ***A1***.

[3 marks]

(d) (i) $(S_{10} =)$ (\$) 8390 (8394.39...) ***A1***

(ii) **EITHER**

the total cost (of dog food) ***R1***
for 10 years beginning in 2021 **OR** 10 years before 2031 ***R1***

OR

the total cost (of dog food) ***R1***
from 2021 to 2030 (inclusive) **OR** from 2021 to (the start of) 2031 ***R1***

[3 marks]

(e) **EITHER**
According to the model, the cost of dog food per year will eventually be too high to keep a dog.
OR
The model does not necessarily consider changes in inflation rate.
OR
The model is appropriate as long as inflation increases at a similar rate.
OR
The model does not account for changes in the amount of food the dog eats as it ages/becomes ill/stops growing.
OR
The model is appropriate since dog food bags can only be bought in discrete quantities. ***R1***

Note: Accept reasonable answers commenting on the appropriateness of the model for the specific scenario. There should be a reference to the given context. A reference to the geometric model must be clear: either "model" is mentioned specifically, or other mathematical terms such as "increasing" or "discrete quantities" are seen. Do not accept a contextual argument in isolation, e.g. "The dog will eventually die".

[1 mark]
Total [13 marks]