SATPREP

Assignment : Complex Number

1. (a) Express the complex number 8i in polar form.

(b) The cube root of 8i which lies in the first quadrant is denoted by z. Express z

(i) in polar form;

(ii) in cartesian form.

2. Let $P(z) = z^3 + az^2 + bz + c$, where a, b, and $c \in \mathbb{R}$. Two of the roots of $P(z) = 0$ are -2 and $(-3 + 2i)$. Find the value of a, of b and of c.

3. $(z + 2i)$ is a factor of $2z^3 - 3z^2 + 8z - 12$. Find the other two factors.

4. (a) Express $z^5 - 1$ as a product of two factors, one of which is linear.

(b) Find the zeros of $z^5 - 1$, giving your answers in the form $r(\cos\theta + i\sin\theta)$ where $r > 0$ and $-\pi < 6 \le \pi$.

(c) Express $z^4 + z^3 + z^2 + z + 1$ as a product of two real quadratic factors.

5. (a) Prove, using mathematical induction, that for a positive integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \text{where } i^2 = -1.$$

(b) The complex number z is defined by $z = \cos\theta + i\sin\theta$.

(i) Show that $\frac{1}{z} = \cos(-\theta) + i\sin(-\theta)$.

(ii) Deduce that $z^n + z^{-n} = 2\cos n\theta$.

(c) (i) Find the binomial expansion of $(z + z^{-1})^5$.

(ii) Hence show that $\cos^5\theta = \frac{1}{16}(a\cos 5\theta + b\cos 3\theta + c\cos\theta)$, where a, b, c are positive integers to be found.

6. Consider the complex number $z = \cos\theta + i\sin\theta$.

(a) Using De Moivre's theorem show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta.$$

(b) By expanding $\left(z + \frac{1}{z}\right)^4$ show that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3).$$