SATPREP Name____________________

Assignment Mathmatical Induction (Series) Date____________

Use mathematical induction to prove that each statement is true for all positive integers.

1) $6 + 11 + 16 + \cdots + 5n + 1 = \frac{n(5n + 7)}{2}$

2) $0 + 5 + 10 + \cdots + 5n - 5 = \frac{n(5n - 5)}{2}$

3) $4 + 7 + 10 + \cdots + 3n + 1 = \frac{n(3n + 5)}{2}$

4) $25 + 225 + 625 + \cdots + (10n - 5)^2 = \frac{25n(4n^2 - 1)}{3}$

5) $9 + 81 + 729 + \cdots + 3^{2n} = \frac{3^2(3^{2n} - 1)}{3^2 - 1}$

6) $5 + 23 + 53 + \cdots + 6n^2 - 1 = n^2(2n + 3)$

7) $1 + 25 + 625 + \cdots + 5^{2n - 2} = \frac{5^{2n} - 1}{5^2 - 1}$

8) $35 + 161 + 371 + \cdots + 42n^2 - 7 = 7n^2(2n + 3)$

9) $10 + 46 + 106 + \cdots + 12n^2 - 2 = 2n^2(2n + 3)$

10) $20 + 92 + 212 + \cdots + 24n^2 - 4 = 4n^2(2n + 3)$

Answers to Assignment Mathmatical Induction (Series)

1) Let P_n be the statement $6 + 11 + 16 + \cdots + 5n + 1 = \frac{n(5n + 7)}{2}$

Anchor Step

P_1 is true since $5 + 1 = \frac{5 + 7}{2}$

Inductive Hypothesis

Assume that P_k is true: $6 + 11 + 16 + \cdots + 5k + 1 = \frac{k(5k + 7)}{2}$

Inductive Step

We now show that P_{k+1} is true:

$$
\begin{aligned}
6 + 11 + 16 + \cdots + 5k + 1 + 5(k + 1) + 1 &= \frac{k(5k + 7)}{2} + 5(k + 1) + 1 \\
&\frac{5k^2 + 7k}{2} + 5k + 5 + 1 \\
&\frac{5k^2 + 7k}{2} + \frac{10k + 12}{2} \\
&\frac{5k^2 + 17k + 12}{2} \\
&\frac{5k^2 + 10k + 5 + 7k + 7}{2} \\
&\frac{5(k^2 + 2k + 1) + 7(k + 1)}{2} \\
&\frac{5(k + 1)^2 + 7(k + 1)}{2} \\
&\frac{(k + 1)(5(k + 1) + 7)}{2}
\end{aligned}
$$

Conclusion

By induction P_n is true for all $n \geq 1$.

2) Let P_n be the statement $0 + 5 + 10 + \cdots + 5n - 5 = \dfrac{n(5n - 5)}{2}$

Anchor Step

P_1 is true since $5 - 5 = \dfrac{5 - 5}{2}$

Inductive Hypothesis

Assume that P_k is true: $5 + 10 + \cdots + 5k - 5 = \dfrac{k(5k - 5)}{2}$

Inductive Step

We now show that P_{k+1} is true:

$$
\begin{aligned}
5 + 10 + \cdots + 5k - 5 + 5(k + 1) - 5 &= \frac{k(5k - 5)}{2} + 5(k + 1) - 5 \\
&\frac{5k^2 - 5k}{2} + 5k + 5 - 5 \\
&\frac{5k^2 - 5k}{2} + \frac{10k}{2} \\
&\frac{5k^2 + 5k}{2} \\
&\frac{5k^2 + 10k + 5 - 5k - 5}{2} \\
&\frac{5(k^2 + 2k + 1) - 5(k + 1)}{2} \\
&\frac{5(k + 1)^2 - 5(k + 1)}{2} \\
&\frac{(k + 1)(5(k + 1) - 5)}{2}
\end{aligned}
$$

Conclusion

By induction P_n is true for all $n \geq 1$.

3) Let P_n be the statement $4 + 7 + 10 + \cdots + 3n + 1 = \dfrac{n(3n+5)}{2}$

Anchor Step

P_1 is true since $3 + 1 = \dfrac{3+5}{2}$

Inductive Hypothesis

Assume that P_k is true: $4 + 7 + 10 + \cdots + 3k + 1 = \dfrac{k(3k+5)}{2}$

Inductive Step

We now show that P_{k+1} is true:

$$
\begin{aligned}
4 + 7 + 10 + \cdots + 3k + 1 + 3(k+1) + 1 &= \frac{k(3k+5)}{2} + 3(k+1) + 1 \\
&\frac{3k^2+5k}{2} + 3k + 3 + 1 \\
&\frac{3k^2+5k}{2} + \frac{6k+8}{2} \\
&\frac{3k^2+11k+8}{2} \\
&\frac{3k^2+6k+3+5k+5}{2} \\
&\frac{3(k^2+2k+1)+5(k+1)}{2} \\
&\frac{3(k+1)^2+5(k+1)}{2} \\
&\frac{(k+1)(3(k+1)+5)}{2}
\end{aligned}
$$

Conclusion

By induction P_n is true for all $n \geq 1$.

4) Let P_n be the statement $25 + 225 + 625 + \cdots + (10n - 5)^2 = \frac{25n(4n^2 - 1)}{3}$

Anchor Step

P_1 is true since $(10 - 5)^2 = \frac{25(4 \cdot 1^2 - 1)}{3}$

Inductive Hypothesis

Assume that P_k is true: $25 + 225 + 625 + \cdots + (10k - 5)^2 = \frac{25k(4k^2 - 1)}{3}$

Inductive Step

We now show that P_{k+1} is true:

$$
\begin{aligned}
25 + 225 + 625 + \cdots + (10k - 5)^2 + (10(k + 1) - 5)^2 &= \frac{25k(4k^2 - 1)}{3} + (10(k + 1) - 5)^2 \\
&\frac{25k(4k^2 - 1)}{3} + (10k + 5)^2 \\
&\frac{100k^3 - 25k}{3} + \frac{3(100k^2 + 100k + 25)}{3} \\
&\frac{100k^3 - 25k}{3} + \frac{300k^2 + 300k + 75}{3} \\
&\frac{100k^3 + 300k^2 + 275k + 75}{3} \\
&\frac{25(4k^3 + 12k^2 + 12k + 4 - k - 1)}{3} \\
&\frac{25(4(k^3 + 3k^2 + 3k + 1) - (k + 1))}{3} \\
&\frac{25(4(k + 1)^3 - (k + 1))}{3} \\
&\frac{25(k + 1)(4(k + 1)^2 - 1)}{3}
\end{aligned}
$$

Conclusion

By induction P_n is true for all $n \geq 1$.

5) Let P_n be the statement $9 + 81 + 729 + \cdots + 3^{2n} = \dfrac{3^2(3^{2n} - 1)}{3^2 - 1}$

Anchor Step

P_1 is true since $3^2 = \dfrac{3^2(3^2 - 1)}{3^2 - 1}$

Inductive Hypothesis

Assume that P_k is true: $9 + 81 + 729 + \cdots + 3^{2k} = \dfrac{3^2(3^{2k} - 1)}{3^2 - 1}$

Inductive Step

We now show that P_{k+1} is true:

$$
\begin{aligned}
9 + 81 + 729 + \cdots + 3^{2k} + 3^{2(k+1)} &= \frac{9(3^{2k} - 1)}{8} + 3^{2(k+1)} \\
& \frac{3^2(3^{2k} - 1)}{8} + 3^{2k+2} \\
& \frac{3^2(3^{2k} - 1)}{8} + \frac{3^{2k} \cdot 3^2(3^2 - 1)}{8} \\
& \frac{3^2(3^{2k} - 1 + 3^{2k+2} - 3^{2k})}{8} \\
& \frac{9(3^{2(k+1)} - 1)}{8}
\end{aligned}
$$

Conclusion

By induction P_n is true for all $n \geq 1$.

6) Let P_n be the statement $5 + 23 + 53 + \cdots + 6n^2 - 1 = n^2(2n + 3)$

Anchor Step

P_1 is true since $6 \cdot 1^2 - 1 = 1^2(2 + 3)$

Inductive Hypothesis

Assume that P_k is true: $5 + 23 + 53 + \cdots + 6k^2 - 1 = k^2(2k + 3)$

Inductive Step

We now show that P_{k+1} is true:

$$
\begin{aligned}
5 + 23 + 53 + \cdots + 6k^2 - 1 + 6(k + 1)^2 - 1 &= k^2(2k + 3) + 6(k + 1)^2 - 1 \\
& 2k^3 + 3k^2 + 6(k^2 + 2k + 1) - 1 \\
& 2k^3 + 3k^2 + 6k^2 + 12k + 6 - 1 \\
& 2k^3 + 9k^2 + 12k + 5 \\
& 2k^3 + 6k^2 + 6k + 2 + 3k^2 + 6k + 3 \\
& 2(k^3 + 3k^2 + 3k + 1) + 3(k^2 + 2k + 1) \\
& 2(k + 1)^3 + 3(k + 1)^2 \\
& (k + 1)^2(2(k + 1) + 3)
\end{aligned}
$$

Conclusion

By induction P_n is true for all $n \geq 1$.

7) Let P_n be the statement $1 + 25 + 625 + \cdots + 5^{2n-2} = \dfrac{5^{2n} - 1}{5^2 - 1}$

Anchor Step

P_1 is true since $5^{2-2} = \dfrac{5^2 - 1}{5^2 - 1}$

Inductive Hypothesis

Assume that P_k is true: $1 + 25 + 625 + \cdots + 5^{2k-2} = \dfrac{5^{2k} - 1}{5^2 - 1}$

Inductive Step

We now show that P_{k+1} is true:

$$1 + 25 + 625 + \cdots + 5^{2k-2} + 5^{2(k+1)-2} = \frac{5^{2k} - 1}{24} + 5^{2(k+1)-2}$$

$$\frac{5^0\left(5^{2k} - 1\right)}{24} + 5^{2k+2-2}$$

$$\frac{5^0\left(5^{2k} - 1\right)}{24} + \frac{5^{2k} \cdot 5^0\left(5^2 - 1\right)}{24}$$

$$\frac{5^0\left(5^{2k} - 1 + 5^{2k+2} - 5^{2k}\right)}{24}$$

$$\frac{5^{2(k+1)} - 1}{24}$$

Conclusion

By induction P_n is true for all $n \ge 1$.

8) Let P_n be the statement $35 + 161 + 371 + \cdots + 42n^2 - 7 = 7n^2(2n + 3)$

Anchor Step

P_1 is true since $42 \cdot 1^2 - 7 = 7 \cdot 1^2(2 + 3)$

Inductive Hypothesis

Assume that P_k is true: $35 + 161 + 371 + \cdots + 42k^2 - 7 = 7k^2(2k + 3)$

Inductive Step

We now show that P_{k+1} is true:

$$35 + 161 + 371 + \cdots + 42k^2 - 7 + 42(k + 1)^2 - 7 = 7k^2(2k + 3) + 42(k + 1)^2 - 7$$

$$14k^3 + 21k^2 + 42\left(k^2 + 2k + 1\right) - 7$$

$$14k^3 + 21k^2 + 42k^2 + 84k + 42 - 7$$

$$14k^3 + 63k^2 + 84k + 35$$

$$7\left(2k^3 + 6k^2 + 6k + 2 + 3k^2 + 6k + 3\right)$$

$$7\left(2\left(k^3 + 3k^2 + 3k + 1\right) + 3\left(k^2 + 2k + 1\right)\right)$$

$$7\left(2(k + 1)^3 + 3(k + 1)^2\right)$$

$$7(k + 1)^2\left(2(k + 1) + 3\right)$$

Conclusion

By induction P_n is true for all $n \ge 1$.

9) Let P_n be the statement $10 + 46 + 106 + \cdots + 12n^2 - 2 = 2n^2(2n + 3)$

Anchor Step

P_1 is true since $12 \cdot 1^2 - 2 = 2 \cdot 1^2(2 + 3)$

Inductive Hypothesis

Assume that P_k is true: $10 + 46 + 106 + \cdots + 12k^2 - 2 = 2k^2(2k + 3)$

Inductive Step

We now show that P_{k+1} is true:

$$
\begin{aligned}
10 + 46 + 106 + \cdots + 12k^2 - 2 + 12(k + 1)^2 - 2 = {} & 2k^2(2k + 3) + 12(k + 1)^2 - 2 \\
& 4k^3 + 6k^2 + 12(k^2 + 2k + 1) - 2 \\
& 4k^3 + 6k^2 + 12k^2 + 24k + 12 - 2 \\
& 4k^3 + 18k^2 + 24k + 10 \\
& 2(2k^3 + 6k^2 + 6k + 2 + 3k^2 + 6k + 3) \\
& 2(2(k^3 + 3k^2 + 3k + 1) + 3(k^2 + 2k + 1)) \\
& 2(2(k + 1)^3 + 3(k + 1)^2) \\
& 2(k + 1)^2(2(k + 1) + 3)
\end{aligned}
$$

Conclusion

By induction P_n is true for all $n \geq 1$.

10) Let P_n be the statement $20 + 92 + 212 + \cdots + 24n^2 - 4 = 4n^2(2n + 3)$

Anchor Step

P_1 is true since $24 \cdot 1^2 - 4 = 4 \cdot 1^2(2 + 3)$

Inductive Hypothesis

Assume that P_k is true: $20 + 92 + 212 + \cdots + 24k^2 - 4 = 4k^2(2k + 3)$

Inductive Step

We now show that P_{k+1} is true:

$$
\begin{aligned}
20 + 92 + 212 + \cdots + 24k^2 - 4 + 24(k + 1)^2 - 4 = {} & 4k^2(2k + 3) + 24(k + 1)^2 - 4 \\
& 8k^3 + 12k^2 + 24(k^2 + 2k + 1) - 4 \\
& 8k^3 + 12k^2 + 24k^2 + 48k + 24 - 4 \\
& 8k^3 + 36k^2 + 48k + 20 \\
& 4(2k^3 + 6k^2 + 6k + 2 + 3k^2 + 6k + 3) \\
& 4(2(k^3 + 3k^2 + 3k + 1) + 3(k^2 + 2k + 1)) \\
& 4(2(k + 1)^3 + 3(k + 1)^2) \\
& 4(k + 1)^2(2(k + 1) + 3)
\end{aligned}
$$

Conclusion

By induction P_n is true for all $n \geq 1$.