Problem : 09709/11/M/J/23/Q1

Solve the equation $4 \sin \theta + \tan \theta = 0$ for $0° < \theta < 180°$. [3]

Sol

$$4 \sin\theta + \frac{\sin\theta}{\cos\theta} = 0$$

$$4 \sin\theta \cos\theta + \sin\theta = 0$$

$$\sin\theta \, (4\cos\theta + 1) = 0$$

$$\sin\theta = 0 \qquad 4\cos\theta + 1 = 0$$

$$\theta = 0^\circ \; \text{x} \qquad \cos\theta = -\frac{1}{4}$$

$$\theta = \cos^{-1}\left(\frac{1}{4}\right)$$

$$= 75.5^\circ$$

$$\theta = 180 - 75.5$$

$$= 104.5^\circ$$

Problem : 09709/11/M/J/23/Q2

(a) Find the first three terms in the expansion, in ascending powers of x, of $(2+3x)^4$. [2]

(b) Find the first three terms in the expansion, in ascending powers of x, of $(1-2x)^5$. [2]

(c) Hence find the coefficient of x^2 in the expansion of $(2+3x)^4(1-2x)^5$. [2]

Sol (a) $(2+3x)^4 = \binom{4}{0} 2^{4-0} (3x)^0 + \binom{4}{1} 2^{4-1} (3x)^1 +$
$\binom{4}{2} 2^{4-2} (3x)^2$

$= 16 + 96x + 216x^2$

(b) $(1-2x)^5 = \binom{5}{0} (-2x)^0 + \binom{5}{1} (-2x)^1 + \binom{5}{2} (-2x)^2$

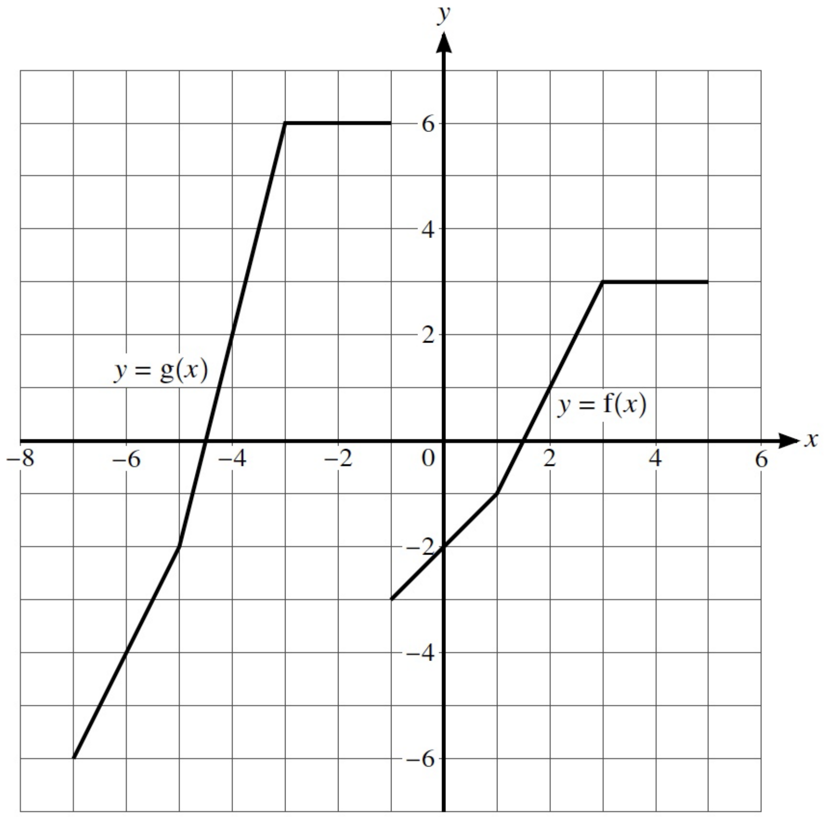
$= 1 - 10x + 40x^2$

(c) $(2+3x)^4(1-2x)^5 = 640x^2 - 960x^2 + 216x^2$

$= -104x^2$

$= \underline{\underline{-104}}$

Problem : 09709/11/M/J/23/Q4



The diagram shows graphs with equations $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$.

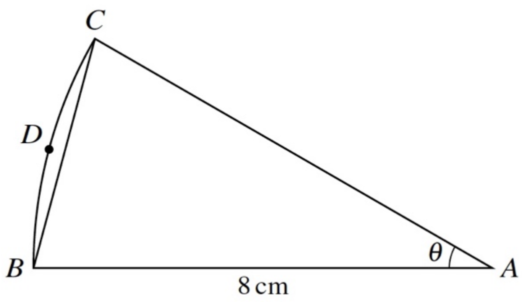
Describe fully a sequence of two transformations which transforms the graph of $y = \mathrm{f}(x)$ to $y = \mathrm{g}(x)$. [4]

Sol

(1) stretch by factor 2 vertically or along y-axis

(2) translate by $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$

Problem : 09709/11/M/J/23/Q4



The diagram shows a sector ABC of a circle with centre A and radius 8 cm. The area of the sector is $\frac{16}{3}\pi\,\text{cm}^2$. The point D lies on the arc BC.

Find the perimeter of the segment BCD. [4]

Sol Area of sector $= \frac{1}{2} r^2 \theta$

$$\frac{16\pi}{3} = \frac{1}{2} r^2\theta = \frac{1}{2} 8^2 \theta$$

$$\theta = \frac{16\pi \times 2}{3 \times 64} = \frac{\pi}{6}$$

Arc length $= r\theta = 8 \times \frac{\pi}{6}$

$$= \frac{4}{3}\pi$$

$$BC = \sqrt{8^2 + 8^2 - 2 \times 8^2 \times \cos\frac{\pi}{6}}$$

$$= 4.1411$$

Perimeter $= \frac{4}{3}\pi + 4.1411$

$$= 8.33 \text{ cm}$$

Problem : 09709/11/M/J/23/Q5

The line with equation $y = kx - k$, where k is a positive constant, is a tangent to the curve with equation $y = -\frac{1}{2x}$.

Find, in either order, the value of k and the coordinates of the point where the tangent meets the curve. [5]

Sol

$$y = -\frac{1}{2x} = -\frac{1}{2}x^{-1}$$

$$\frac{dy}{dx} = \frac{1}{2x^2}$$

$$k = \frac{1}{2x^2}$$

$$y = \frac{1}{2x^2}x - \frac{1}{2x^2}$$

$$-\frac{1}{2x} = \frac{1}{2x} - \frac{1}{2x^2}$$

$$\frac{1}{2x^2} = \frac{1}{2x} + \frac{1}{2x} = \frac{1}{x}$$

$$x = 2x^2$$

$$2x^2 - x = 0$$

$$x(2x-1) = 0$$

$$x = 0 \quad x = 1/2 \checkmark$$

$$k = \frac{1}{2 \times \frac{1}{4}} = 2$$

$$y = -\frac{1}{2 \times \frac{1}{2}} = -1$$

$$k = 2 \quad x = 1/2 \; y = -1$$

$$\left(\frac{1}{2}, -1\right)$$

Problem : 09709/11/M/J/23/Q6

The first three terms of an arithmetic progression are $\frac{p^2}{6}$, $2p - 6$ and p.

(a) Given that the common difference of the progression is not zero, find the value of p. [3]

(b) Using this value, find the sum to infinity of the geometric progression with first two terms $\frac{p^2}{6}$ and $2p - 6$. [2]

Sol (a) $p - (2p - 6) = 2p - 6 - \frac{p^2}{6}$

$p - 2p + 6 = 2p - 6 - \frac{p^2}{6}$

$\frac{p^2}{6} = 2p - 6 - 6 + 2p - p$

$\frac{p^2}{6} = 3p - 12$

$p^2 = 18p - 72$

$p^2 - 18p + 72 = 0$

$p = 12 \quad p = 6$

$p = 6 \quad 6 - (12 - 6) = 0 \times$

$\boxed{p = 12}$

(b) $\frac{p^2}{6} = \frac{12^2}{6} = 24$

$2p - 6 = 18$

$r = \frac{18}{24} = \frac{3}{4}$

$a = 24$

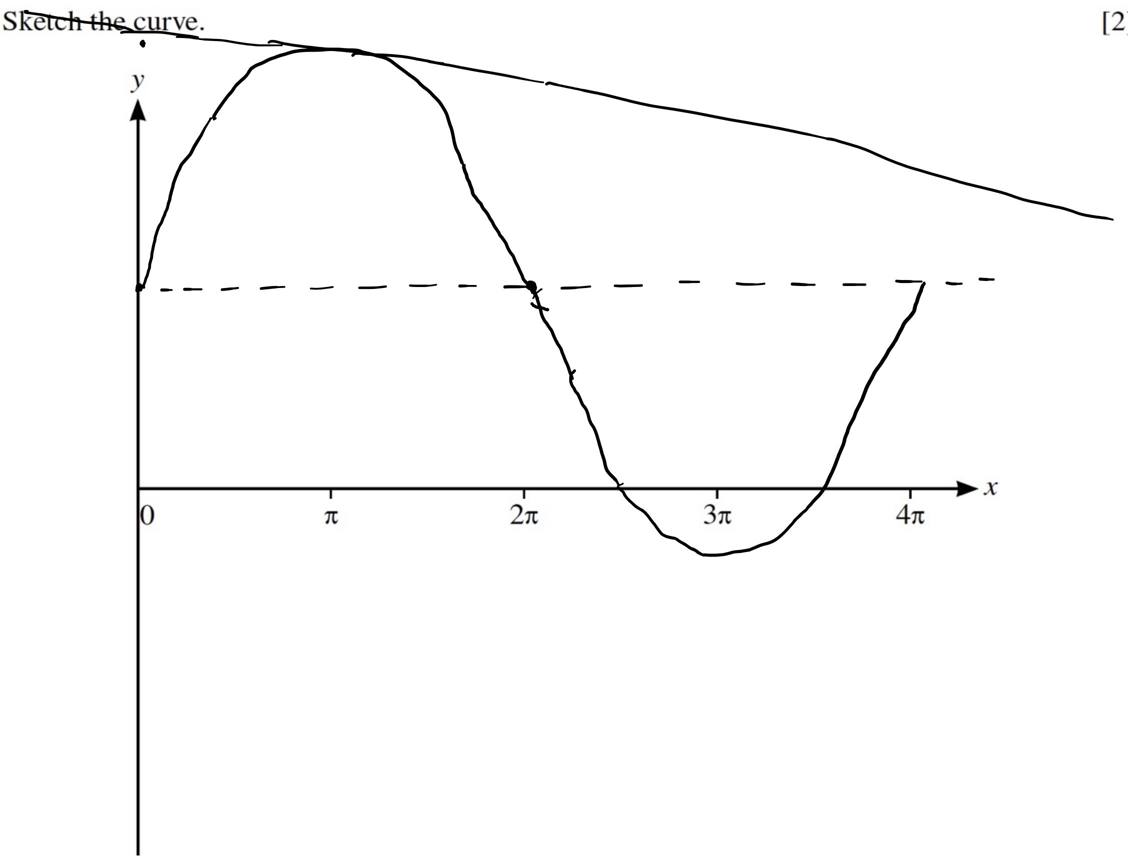
$S_\infty = \frac{a}{1 - r} = \frac{24}{1 - \frac{3}{4}} = 96$

Problem : 09709/11/M/J/23/Q7

A curve has equation $y = 2 + 3\sin\frac{1}{2}x$ for $0 \leqslant x \leqslant 4\pi$.

(a) State greatest and least values of y. [2]

(b) ~~Sketch the curve.~~ [2]



(c) State the number of solutions of the equation

$$2 + 3\sin\frac{1}{2}x = 5 - 2x$$

for $0 \leqslant x \leqslant 4\pi$. [1]

<u>Sol</u> (a) $y = 2 + 3\sin\frac{1}{2}x$

$-1 \leq \sin\theta \leq 1$

$y = 2 - 3 = -1$ (least)

$y = 2 + 3 = 5$ (greastat

(c) only one solution

Problem 09709/11/M/J/23/Q8

The functions f and g are defined as follows, where a and b are constants.

$$f(x) = 1 + \frac{2a}{x-a} \text{ for } x > a$$

$$g(x) = bx - 2 \text{ for } x \in \mathbb{R}$$

(a) Given that $f(7) = \frac{5}{2}$ and $gf(5) = 4$, find the values of a and b. [4]

$$f(7) = 1 + \frac{2a}{7-a}$$

$$\frac{5}{2} = 1 + \frac{2a}{7-a}$$

$$\frac{3}{2} = \frac{2a}{7-a}$$

$$21 - 3a = 4a$$

$$21 = 7a \qquad \boxed{a = 3}$$

$$g(f(x)) = b\left(1 + \frac{2 \times 3}{x-3}\right) - 2$$

$$g(f(5)) = b\left(1 + \frac{6}{2}\right) - 2$$

$$4 = b(4) - 2$$

$$b = 6/4 = 3/2$$

For the rest of this question, you should use the value of a which you found in **(a)**.

(b) Find the domain of f^{-1}. [1]

$$x > 1$$

(c) Find an expression for $f^{-1}(x)$. [3]

$$f(x) = 1 + \frac{6}{x-3}$$

let

$$y = 1 + \frac{6}{x-3}$$

$$y - 1 = \frac{6}{x-3}$$

$$x - 3 = \frac{6}{y-1}$$

$$x = \frac{6}{y-1} + 3$$

$$f^{-1}(x) = \frac{6}{x-1} + 3$$

Problem 09709/11/M/J/23/Q9

Water is poured into a tank at a constant rate of $500\,\text{cm}^3$ per second. The depth of water in the tank, t seconds after filling starts, is h cm. When the depth of water in the tank is h cm, the volume, $V\,\text{cm}^3$, of water in the tank is given by the formula $V = \frac{4}{3}(25+h)^3 - \frac{62500}{3}$.

(a) Find the rate at which h is increasing at the instant when $h = 10$ cm. [3]

Sol

$$\frac{dv}{dh} = \frac{4}{3} \times 3(25+h)^2$$

$$\frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt}$$

$$500 = \frac{4}{\cancel{3}} \times \cancel{3}(25+10)^2 \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{500}{4(25+10)^2} = 0.102$$

(b) At another instant, the rate at which h is increasing is 0.075 cm per second.

Find the value of V at this instant. [3]

$$\frac{dv}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$500 = 4(25+h)^2 \times 0.075$$

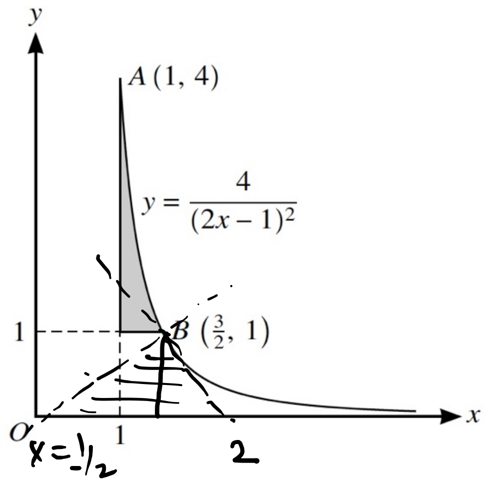
$$h = \sqrt{\frac{500}{4 \times 0.075}} - 25$$

$$= 15.8$$

$$V = \frac{4}{3}(25+15.8)^3 - \frac{62500}{3}$$

$$= 69723 \approx 69900$$

Problem : 09709/11/M/J/23/Q10



The diagram shows part of the curve with equation $y = \dfrac{4}{(2x-1)^2}$ and parts of the lines $x = 1$ and $y = 1$. The curve passes through the points $A(1, 4)$ and $B, (\frac{3}{2}, 1)$.

(a) Find the exact volume generated when the shaded region is rotated through 360° about the x-axis. [5]

(b) A triangle is formed from the tangent to the curve at B, the normal to the curve at B and the x-axis.

Find the area of this triangle. [6]

<u>Sol</u> (a)

$$V = \pi \int_a^b (f(x))^2 dx$$

$$V = \pi \int_1^{3/2} \left[\left[\frac{4}{(2x-1)^2} \right]^2 - 1 \right] dx$$

$$= \pi \int_1^{3/2} \left[\frac{16}{(2x-1)^4} - 1 \right] dx$$

$$= \pi \left[\frac{16}{-3(2x-1)^3 \times 2} - x \right]_1^{3/2}$$

$$= \pi \left[\frac{-16}{6 \times 8} - \frac{3}{2} + \frac{16}{6} + 1 \right]$$

$$= \frac{11}{6} \pi$$

(b) $y = \dfrac{4}{(2x-1)^2} = 4(2x-1)^{-2}$

$\dfrac{dy}{dx} = 4 \times -2(2x-1)^{-3} \times 2$

$= \dfrac{-16}{(2x-1)^3}$

$x = 3/2$ gradient tangent $= -2$

gradient of normal $= \frac{1}{2}$

Eq. of tangent $y - 1 = -2(x - 3/2)$

$y = 0$ $+1 = +2(x - 3/2)$

$\frac{1}{2} = x - \frac{3}{2}$

$\frac{1}{2} + \frac{3}{2} = x$ $\therefore x = 2$

Eq. of Normal $y - 1 = \frac{1}{2}(x - 3/2)$

$y = 0$ $-1 = \frac{1}{2}(x - 3/2)$

$-2 = x - 3/2$

$3/2 - 2 = x$

$-1/2 = x$

Area of $\Delta = \frac{1}{2} \times 1 \times (2 + \frac{1}{2})$

$= \frac{5}{4}$

Problem : 09709/11/M/J/23/Q11

The equation of a curve is such that $\frac{dy}{dx} = 6x^2 - 30x + 6a$, where a is a positive constant. The curve has a stationary point at $(a, -15)$.

(a) Find the value of a. [2]

(b) Determine the nature of this stationary point. [2]

(c) Find the equation of the curve. [3]

(d) Find the coordinates of any other stationary points on the curve. [2]

हल (a) $0 = 6a^2 - 30a + 6a$

$0 = 6a^2 - 24a$

$6a(a-4) = 0$

$a = 0 \quad \underline{a = 4}$

$\frac{dy}{dx} = 6x^2 - 30x + 24$

(b) $\frac{dy}{dx} = 6x^2 - 30x + 24$

$\frac{d^2y}{dx^2} = 12x - 30$

$x = 4 \quad \frac{d^2y}{dx^2} = 12 \times 4 - 30 > 0$

hence stationary point at $x = 4$ would be minimum.

(c) $\frac{dy}{dx} = 6x^2 - 30x + 24$

$\int dy = \int (6x^2 - 30x + 24)\,dx$

$y = \frac{6x^3}{3} - \frac{30x^2}{2} + 24x + c$

$(4, -15)$

$-15 = 2(4)^3 - 15(4)^2 + 24 \times 4 + c$

$-15 = 128 - 240 + 96 + c$

$c = 1$

$y = 2x^3 - 15x^2 + 24x + 1$

(d) $6x^2 - 30x + 24 = 0$

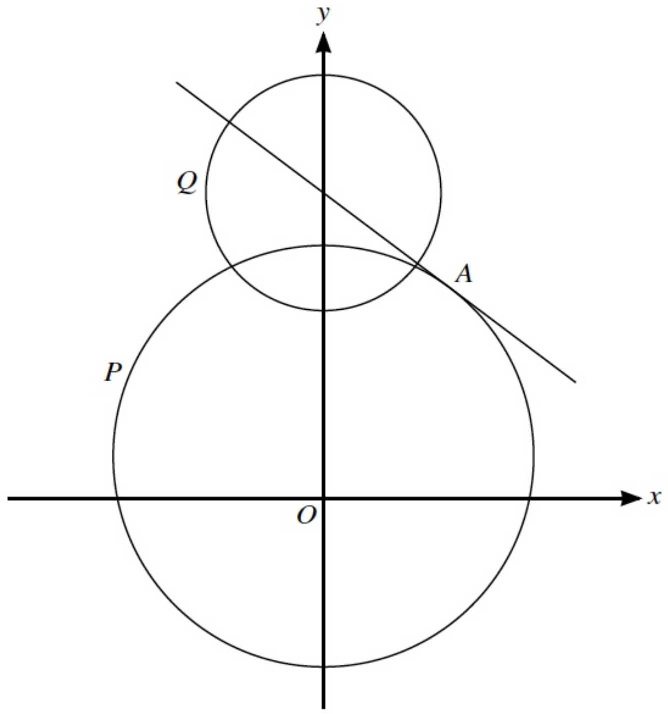
$x = 4 \quad x = 1$

$y = 2(1)^3 - 15(1)^2 + 24(1) + 1$

$= 12$

$(1, 12)$

Problem 09709/11/M/J/23/Q12



The diagram shows a circle P with centre (0, 2) and radius 10 and the tangent to the circle at the point A with coordinates (6, 10). It also shows a second circle Q with centre at the point where this tangent meets the y-axis and with radius $\frac{5}{2}\sqrt{5}$.

(a) Write down the equation of circle P. [1]

$$x^2 + (y-2)^2 = 100$$

(b) Find the equation of the tangent to the circle P at A. [2]

A (6,10) O(0,2)

gradient of Radius $= \frac{10-2}{6-0} = \frac{8}{6} = \frac{4}{3}$

gradient of tangent $= -3/4$

Equation of tangent $y - 10 = -\frac{3}{4}(x-6)$

$$y = -\frac{3}{4}x + \frac{29}{2}$$

(c) Find the equation of circle Q and hence verify that the y-coordinates of both of the points of intersection of the two circles are 11. [3]

$x = 0$ $\quad y = \frac{18}{4} + 10 = \frac{58}{4} = \frac{29}{2}$

centre $\left(0, \frac{29}{2}\right)$

Eq of circle P

$$x^2 + \left(y - \frac{29}{2}\right)^2 = \left(\frac{5\sqrt{5}}{2}\right)^2$$

$$x^2 + \left(y - \frac{29}{2}\right)^2 = \frac{125}{4}$$

$$100 - (y-2)^2 + \left(y - \frac{29}{2}\right)^2 = \frac{125}{4}$$

$$100 - (y^2 + 4 - 4y) + y^2 + \frac{841}{4} - 29y = \frac{125}{4}$$

$$-\cancel{y^2} - 4 + 4y + \cancel{y^2} + \frac{841}{4} - 29y = \frac{125}{4}$$

$$-25y = -275$$

$$y = 11$$

(d) Find the coordinates of the points of intersection of the tangent and circle Q, giving the answers in surd form. [3]

$$x^2 + \left(y - \frac{29}{2}\right)^2 = \frac{125}{4}$$

$$x^2 + \left(-\frac{3}{4}x + \cancel{\frac{29}{2}} - \cancel{\frac{29}{2}}\right)^2 = \frac{125}{4}$$

$$x^2 + \frac{9}{16}x^2 = \frac{125}{4}$$

$$\frac{\cancel{25}x^2}{\cancel{16}_4} = \frac{\cancel{125}^{5}}{\cancel{4}}$$

$$x^2 = 20$$

$$x = \pm\sqrt{20} = \pm 2\sqrt{5}$$

$$y = -\frac{3}{4}\left(\pm 2\sqrt{5}\right) - \frac{29}{2}$$

$$y = \frac{29 \mp 3\sqrt{5}}{2}$$