**Problem : 09709/12/F/M/23/Q1**

A line has equation $y = 3x - 2k$ and a curve has equation $y = x^2 - kx + 2$, where $k$ is a constant.

Show that the line and the curve meet for all values of $k$.

Sol

$$3x - 2k = x^2 - kx + 2$$

$$x^2 - kx - 3x + 2k + 2 = 0$$

$$x^2 - (k+3)x + 2k + 2 = 0$$

As line and curve meet $b^2 - 4ac = 0$

$$[-(k+3)]^2 - 4 \times 1 \times (2k+2) = 0$$

$$k^2 + 6k + 9 - 8k - 8 = 0$$

$$k^2 - 2k + 1 = 0$$

$$(k-1)^2 = 0$$

$$(k-1)^2 > 0$$

**Problem : 09709/12/F/M/23/Q2**

A function f is defined by $f(x) = x^2 - 2x + 5$ for $x \in \mathbb{R}$. A sequence of transformations is applied in the following order to the graph of $y = f(x)$ to give the graph of $y = g(x)$.

Stretch parallel to the $x$-axis with scale factor $\frac{1}{2}$

Reflection in the $y$-axis

Stretch parallel to the $y$-axis with scale factor 3

Find $g(x)$, giving your answer in the form $ax^2 + bx + c$, where $a$, $b$ and $c$ are constants. [4]

Sol

$$f(x) = x^2 - 2x + 5$$

$$g(x) = 3[(-2x)^2 - 2(-2x) + 5]$$

$$g(x) = 3[4x^2 + 4x + 5]$$

$$g(x) = 12x^2 + 12x + 15$$

**Problem : 09709/12/F/M/23/Q3**

A curve has equation $y = \frac{1}{60}(3x + 1)^2$ and a point is moving along the curve.

Find the $x$-coordinate of the point on the curve at which the $x$- and $y$-coordinates are increasing at the same rate. [4]

Sol

$$\frac{dy}{dt} = \frac{dx}{dt} = K \quad (\text{say})$$

$$y = \frac{1}{60}(3x+1)^2$$

$$\frac{dy}{dx} = \frac{1}{\cancel{60}_{10}} \times \cancel{2}(3x+1) \times \cancel{3}$$

$$\frac{dy}{dx} = \frac{1}{10}(3x+1)$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$\cancel{K} = \frac{1}{10} \times (3x+1) \times 1\cancel{K}$$

$$1 = \frac{1}{10}(3x+1)$$

$$10 = 3x+1$$

$$x = 3$$

**Problem : 09709/12/F/M/23/Q4**

The circumference round the trunk of a large tree is measured and found to be 5.00 m. After one year the circumference is measured again and found to be 5.02 m.

**(a)** Given that the circumferences at yearly intervals form an arithmetic progression, find the circumference 20 years after the first measurement. [2]

**(b)** Given instead that the circumferences at yearly intervals form a geometric progression, find the circumference 20 years after the first measurement. [3]

Sol $a = 5 \quad n = 20 \quad d = 5.02 - 5 = 0.02$

(a) $= 5 + 20 \times 0.02 = 5.40$

(b) $a = 5 \quad n = 20 \quad r = \dfrac{5.02}{5} = 1.004$

$= 5(1.004)^{20} = 5.42$

**Problem : 09709/12/F/M/23/Q5**

Points $A\,(7,\ 12)$ and $B$ lie on a circle with centre $(-2,\ 5)$. The line $AB$ has equation $y = -2x + 26$.

Find the coordinates of $B$.

Sol

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x+2)^2 + (y-5)^2 = r^2$$

$$(7+2)^2 + (12-5)^2 = r^2$$

$$81 + 49 = r^2$$

$$r^2 = 130$$

$$(x+2)^2 + (y-5)^2 = 130$$

$$(x+2)^2 + (-2x+26-5)^2 = 130$$

$$x^2 + 4x + 4 + 4x^2 - 84x + 441 = 130$$

$$5x^2 - 80x + 445 - 130 = 0$$

$$5x^2 - 80x + 315 = 0$$

$$x^2 - 16x + 63 = 0$$

$$x^2 - 9x - 7x + 63 = 0$$

$$x(x-9) - 7(x-9) = 0$$

$$(x-9)(x-7) = 0$$

$$\underline{x = 9}\ ,\ x = 7$$

$$y = -2x + 26$$

$$= -2 \times 9 + 26 = 8$$

$$B\,(9, 8)$$

**Problem : 09709/12/F/M/23/Q6**

In the expansion of $\left(\frac{x}{a} + \frac{a}{x^2}\right)^7$, it is given that

$$\frac{\text{the coefficient of } x^4}{\text{the coefficient of } x} = 3.$$

Find the possible values of the constant $a$. [6]

Sol

$$^7C_r \left(\frac{x}{a}\right)^{7-r} \left(\frac{a}{x^2}\right)^r$$

$$x^{7-3r} = x^4$$

$$7 - 3r = 4$$

$$r = 1$$

$$7 - 3r = 1$$

$$r = 2$$

$$\frac{^7C_1\, a^{-5}}{^7C_2\, a^{-3}} = 3$$

$$\frac{\cancel{7}\, a^{-5}}{\cancel{21}\, a^{-3}} = 3$$

$$\frac{1}{a^2} = 9$$

$$a = \pm\frac{1}{3}$$

**Problem : 09709/12/F/M/23/Q7**

**(a)** By first obtaining a quadratic equation in $\cos\theta$, solve the equation

$$\tan\theta\sin\theta = 1$$

for $0° < \theta < 360°$.

**(b)** Show that $\dfrac{\tan\theta}{\sin\theta} - \dfrac{\sin\theta}{\tan\theta} \equiv \tan\theta\sin\theta$.

Sol (a)

$$\tan\theta\sin\theta = 1$$

$$\frac{\sin\theta}{\cos\theta} \times \sin\theta = 1$$

$$\sin^2\theta = \cos\theta$$

$$1 - \cos^2\theta = \cos\theta$$

$$\cos^2\theta + \cos\theta - 1 = 0$$

let $\cos\theta = t$ $\quad t^2 + t - 1 = 0$

$t = 0.618 \quad t = -1.618$ ✗

$$\cos\theta = 0.618$$

$$\theta = \cos^{-1}(0.618) = 51.8°$$

$$\theta = 360 - 51.8 = 308.2°$$

(b)

$$\frac{\tan\theta}{\sin\theta} - \frac{\sin\theta}{\tan\theta} = \tan\theta . \sin\theta$$

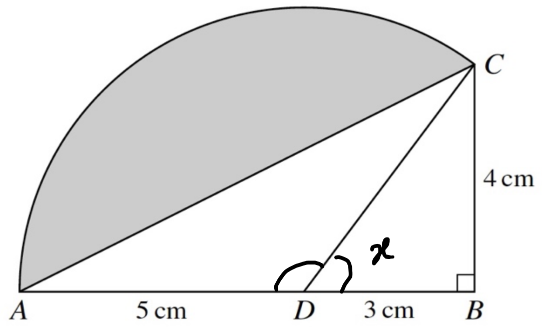
LHS $\quad \dfrac{\cancel{\sin\theta}}{\cancel{\sin\theta}\cos\theta} - \dfrac{\cancel{\sin\theta} \times \cos\theta}{\cancel{\sin\theta}}$

$$\frac{1}{\cos\theta} - \cos\theta = \frac{1-\cos^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta} . \sin\theta = \tan\theta\sin\theta$$

$= \text{RHL}$

**Problem : 09709/12/F/M/23/Q8**




The diagram shows triangle $ABC$ in which angle $B$ is a right angle. The length of $AB$ is 8 cm and the length of $BC$ is 4 cm. The point $D$ on $AB$ is such that $AD = 5$ cm. The sector $DAC$ is part of a circle with centre $D$.

**(a)** Find the perimeter of the shaded region. [5]

**(b)** Find the area of the shaded region. [3]

Sol (a) $\tan x = \frac{4}{3}$ $\quad x = \tan^{-1}\frac{4}{3} = 0.927$

$\angle ADC = \pi - 0.927 = 2.214$

Arc length $= r\theta = 5 \times 2.214 = 11.07$

$AC = \sqrt{8^2 + 4^2} = 8.94$

hence perimeter $= 11.07 + 8.94 = 20.01$ cm

(b) Area of shaded region

$= $ Area of sector $-$ Area of $\triangle ACD$

$= \frac{1}{2} 5^2 \times 2.214 - \frac{1}{2} \times 5 \times 5 \times \sin 2.214$

$= 17.7$ cm$^2$

**Problem : 09709/12/F/M/23/Q9**

The function f is defined by $f(x) = -3x^2 + 2$ for $x \leqslant -1$.

**(a)** State the range of f.

**(b)** Find an expression for $f^{-1}(x)$.

The function g is defined by $g(x) = -x^2 - 1$ for $x \leqslant -1$.

**(c)** Solve the equation $fg(x) - gf(x) + 8 = 0$.

Sol (a) $y \leqslant 2$

(b) $y = -3x^2 + 2$

$y - 2 = -3x^2$

$\frac{y-2}{-3} = x^2$

$x^2 = \frac{2-y}{3}$

$x = \pm\sqrt{\frac{2-y}{3}}$

$f^{-1}(x) = -\sqrt{\frac{2-x}{3}}$

(c) $f(x) = -3x^2 + 2 \qquad g(x) = -x^2 - 1$

$fg(x) = -3(-x^2-1)^2 + 2$

$= -3(x^4 + 2x^2 + 1) + 2$

$= -3x^4 - 6x^2 - 1$

$gf(x) = -(-3x^2+2)^2 - 1$

$= -(9x^4 - 12x^2 + 4) - 1$

$= -9x^4 + 12x^2 - 5$

$-3x^4 - 6x^2 - 1 + 9x^4 - 12x^2 + 5 + 8 = 0$

$6x^4 - 18x^2 + 12 = 0$

$x^4 - 3x^2 + 2 = 0$

let $x^2 = t$

$$t^2 - 3t + 2 = 0$$

$$t^2 - 2t - t + 2 = 0$$

$$t(t-2) - 1(t-2) = 0$$

$$(t-2)(t-1) = 0$$

$t = 2$ $\quad$ $t = 1$

$x^2 = 2$ $\quad$ $x^2 = 1$

$x = -\sqrt{2}$ $\quad$ $x = -1$

**Problem : 09709/12/F/M/23/Q10**

At the point $(4, -1)$ on a curve, the gradient of the curve is $-\frac{3}{2}$. It is given that $\frac{dy}{dx} = x^{-\frac{1}{2}} + k$, where $k$ is a constant.

**(a)** Show that $k = -2$. [1]

**(b)** Find the equation of the curve. [4]

**(c)** Find the coordinates of the stationary point. [3]

Sol (a) $-\frac{3}{2} = 4^{-\frac{1}{2}} + k$

$-\frac{3}{2} = \frac{1}{2} + k$

$k = -\frac{3}{2} - \frac{1}{2} = -2$

(b) $\int dy = \int (x^{-\frac{1}{2}} - 2)\, dx$

$y = \frac{x^{1/2}}{1/2} - 2x + c$

$y = 2x^{1/2} - 2x + c$

$-1 = 2(4)^{1/2} - 2(4) + c$

$c = -1 - 4 + 8 = 3$

$y = 2x^{1/2} - 2x + 3$

(c) $\frac{dy}{dx} = x^{-\frac{1}{2}} - 2$

$0 = \frac{1}{x^{1/2}} - 2$

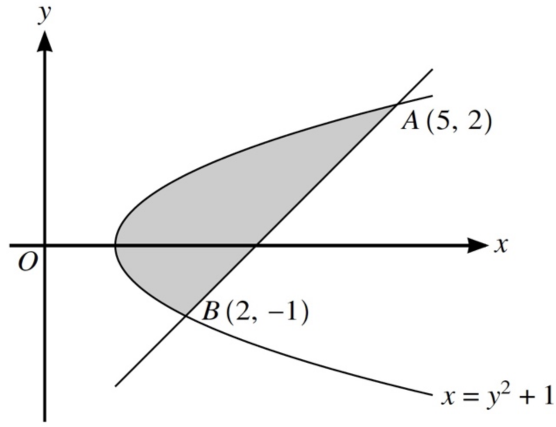
$x^{1/2} = \frac{1}{2}$

$x = \frac{1}{4}$

$y = 2\left(\frac{1}{4}\right)^{1/2} - 2\left(\frac{1}{4}\right) + 3 = \frac{7}{2}$

$\left(1/4, 7/2\right)$

**Problem : 09709/12/F/M/23/Q11**




The diagram shows the curve with equation $x = y^2 + 1$. The points $A\,(5, 2)$ and $B\,(2, -1)$ lie on the curve.

**(a)** Find an equation of the line $AB$. [2]

**(b)** Find the volume of revolution when the region between the curve and the line $AB$ is rotated through 360° about the **y-axis**. [9]

Sol (a) $A(5,2) \quad B(2,-1)$

$$m = \frac{2-(-1)}{5-2} = 1$$

eq. of line AB

$$y - 2 = 1(x-5)$$

$$y = x - 3$$

(b)

$$V_1 = \pi \int_{-1}^{2} (y^2+1)^2\, dy$$

$$= \pi \int_{-1}^{2} (y^4 + 2y^2 + 1)\, dy$$

$$= \pi \left[ \frac{y^5}{5} + 2\frac{y^3}{3} + y \right]_{-1}^{2}$$

$$= \pi \left[ \left( \frac{2^5}{5} + \frac{2}{3}(2)^3 + 2 \right) - \left( \frac{(-1)^5}{5} + \frac{2(-1)^3}{3} + (-1) \right) \right]$$

$$= \pi \left[ \frac{78}{5} \right]$$

$$V_2 = \pi \int_{-1}^{2} (y+3)^2 \, dy$$

$$= \pi \left[ \frac{(y+3)^3}{3} \right]_{-1}^{2}$$

$$= \frac{\pi}{3} \left[ 5^3 - 2^3 \right] = \frac{117}{3} \pi$$

$$V = \pi \left[ \frac{117}{3} - \frac{78}{5} \right] = \frac{117}{5} \pi$$