Problem : 09709/12/M/J/23/Q1

The equation of a curve is such that $\frac{dy}{dx} = \frac{4}{(x-3)^3}$ for $x > 3$. The curve passes through the point (4, 5).

Find the equation of the curve. [3]

Sol

$$\frac{dy}{dx} = 4(x-3)^{-3}$$

$$\int dy = \int 4(x-3)^{-3}\,dx$$

$$y = 4\,\frac{(x-3)^{-2}}{-2} + c$$

$(4,5)$

$$5 = -\frac{\cancel{4}^{\,2}}{\cancel{2}}(4-3)^{-2} + c$$

$$7 = c$$

$$y = \frac{-2}{(x-3)^2} + 7$$

Problem : 09709/12/M/J/23/Q2

The coefficient of x^4 in the expansion of $(x + a)^6$ is p and the coefficient of x^2 in the expansion of $(ax + 3)^4$ is q. It is given that $p + q = 276$.

Find the possible values of the constant a. [4]

Sol $(a+x)^6$ $\binom{6}{4} a^{6-4} x^4$

$p = \underline{15a^2}$

$(3+ax)^4 = \binom{4}{2} 3^{4-2} a^2 x^2$

$= 6 \times 9 \times a^2 \times x^2$

$q = \underline{54a^2}$

$p + q = 276$

$15a^2 + 54a^2 = 276$

$69a^2 = 276$

$a^2 = 4$

$a = \pm 2$

Problem : 09709/12/M/J/23/Q3

(a) Express $4x^2 - 24x + p$ in the form $a(x + b)^2 + c$, where a and b are integers and c is to be given in terms of the constant p. [2]

$$4(x^2 - 6x) + p$$
$$4(x^2 - 6x + 9 - 9) + p$$
$$4(x^2 - 6x + 3) - 36 + p$$
$$4(x-3)^2 - 36 + p$$
$$a = 4 \quad b = 3 \quad c = p - 36$$

(b) Hence or otherwise find the set of values of p for which the equation $4x^2 - 24x + p = 0$ has no real roots. [1]

$$b^2 - 4ac < 0$$
$$(-24)^2 - 4 \times 4 \times p < 0$$
$$576 - 16p < 0$$
$$16p > 576$$
$$p > 36$$

Problem : 09709/12/M/J/23/Q4

Solve the equation $8x^6 + 215x^3 - 27 = 0$. [3]

Sol Let $x^3 = k$

$8k^2 + 215k - 27 = 0$

$k = \dfrac{-215 \pm \sqrt{(215)^2 - 4 \times 8 \times -27}}{2 \times 8}$

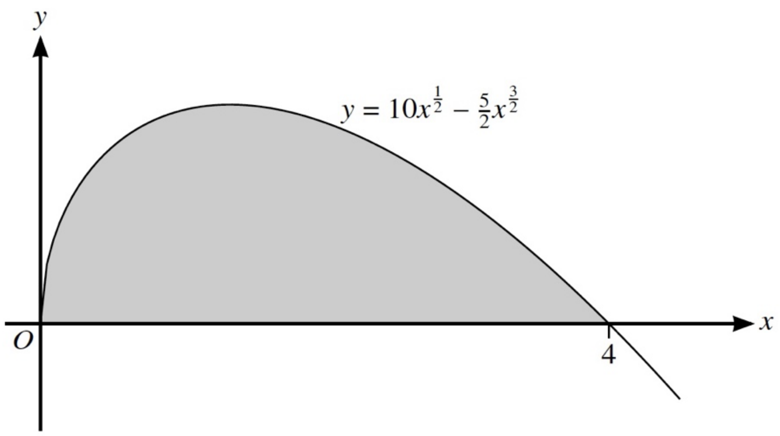
$= \dfrac{-215 + 217}{16}$

$k = \cancel{\dfrac{2}{16}}\ \dfrac{1}{8}$ $\qquad k = \dfrac{-215 - 217}{16}$

$x^3 = \dfrac{1}{8}$ $\qquad x^3 = -27$

$x = \dfrac{1}{2}$ $\qquad x = -3$

Problem : 09709/12/M/J/23/Q5



The diagram shows the curve with equation $y = 10x^{\frac{1}{2}} - \frac{5}{2}x^{\frac{3}{2}}$ for $x > 0$. The curve meets the x-axis at the points (0, 0) and (4, 0).

Find the area of the shaded region. [4]

Sol Area $= \int_a^b y\,dx$

$$= \int_0^4 \left(10x^{1/2} - \frac{5}{2}x^{3/2}\right) dx$$

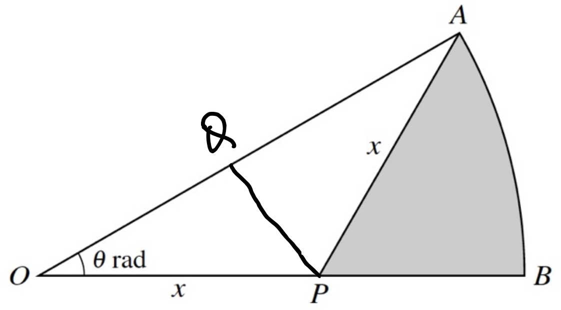
$$= \left[10\frac{x^{3/2}}{3/2} - \frac{5}{2}\frac{x^{5/2}}{5/2}\right]_0^4$$

$$= \left[\frac{20}{3}x^{3/2} - x^{5/2}\right]_0^4$$

$$= \left[\frac{20}{3}(4)^{3/2} - (4)^{5/2}\right]$$

$$= \frac{64}{3}$$

Problem : 09709/12/M/J/23/Q6

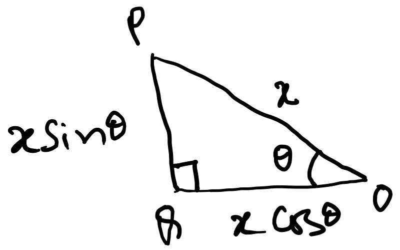


The diagram shows a sector OAB of a circle with centre O. Angle $AOB = \theta$ radians and $OP = AP = x$.

(a) Show that the arc length AB is $2x\theta\cos\theta$. [2]

(b) Find the area of the shaded region APB in terms of x and θ. [4]

Sol (a) Radius of sector $OA = 2QO$



$OQ = x\cos\theta$

$OA = 2x\cos\theta$

Arc length $= r\theta$

$= 2x\cos\theta \times \theta$

$= 2x\theta\cos\theta$

(b)

Area of shaded region

$= $ Area of sector $-$ Area of Δ OPA

$= \frac{1}{2}(2x\cos\theta)^2\theta - 2\times$ Area of ΔPQO

$= \frac{1}{\cancel{2}}\cancel{4}^{2}x^2\theta\cos^2\theta - \cancel{2}\times\frac{1}{\cancel{2}}\times x\cos\theta\times x\sin\theta$

$= 2x^2\theta\cos^2\theta - x^2\sin\theta\cos\theta$

$= x^2\cos\theta(2\theta\cos\theta - \sin\theta)$

Problem : 09709/12/M/J/23/Q7

(a) **(i)** By first expanding $(\cos\theta + \sin\theta)^2$, find the three solutions of the equation

$$(\cos\theta + \sin\theta)^2 = 1$$

for $0 \leqslant \theta \leqslant \pi$. [3]

(ii) Hence verify that the only solutions of the equation $\cos\theta + \sin\theta = 1$ for $0 \leqslant \theta \leqslant \pi$ are 0 and $\frac{1}{2}\pi$. [2]

(b) Prove the identity $\dfrac{\sin\theta}{\cos\theta + \sin\theta} + \dfrac{1 - \cos\theta}{\cos\theta - \sin\theta} \equiv \dfrac{\cos\theta + \sin\theta - 1}{1 - 2\sin^2\theta}$. [3]

(c) Using the results of **(a)(ii)** and **(b)**, solve the equation

$$\frac{\sin\theta}{\cos\theta + \sin\theta} + \frac{1 - \cos\theta}{\cos\theta - \sin\theta} = 2(\cos\theta + \sin\theta - 1)$$

for $0 \leqslant \theta \leqslant \pi$. [3]

Sol a(i) $(\cos\theta + \sin\theta)^2 = 1$

$\cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta = 1$

$\cancel{1} + 2\sin\theta\cos\theta = \cancel{1}$

$\sin\theta = 0 \qquad \cos\theta = 0$

$\theta = 0^\circ, \pi, \qquad \theta = \pi/2$

(ii) $\cos\theta + \sin\theta = 1$

✓ $\theta = 0 \qquad \cos 0 + \sin 0 = 1$

✓ $\theta = \dfrac{\pi}{2} \qquad \cos\dfrac{\pi}{2} + \sin\dfrac{\pi}{2} = 1$

✗ $\theta = \pi \qquad \cos\pi + \sin\pi = -1$

(b) $\dfrac{\sin\theta}{\cos\theta + \sin\theta} + \dfrac{1 - \cos\theta}{\cos\theta - \sin\theta}$

$\dfrac{\cancel{\sin\theta\cos\theta} - \sin^2\theta + \cos\theta + \sin\theta - \cos^2\theta - \cancel{\sin\theta\cos\theta}}{\cos^2\theta - \sin^2\theta}$

$\dfrac{\cos\theta + \sin\theta - 1}{1 - \sin^2\theta - \sin^2\theta}$

$$\frac{\cos\theta+\sin\theta-1}{1-2\sin^2\theta}$$

(c) $$\frac{\cos\theta+\sin\theta-1}{1-2\sin^2\theta} = 2(\cos\theta+\sin\theta-1)$$

$$\frac{\cos\theta+\sin\theta-1}{1-2\sin^2\theta} - 2(\cos\theta+\sin\theta-1) = 0$$

$$(\cos\theta+\sin\theta-1)\left(\frac{1}{1-2\sin^2\theta}-2\right)=0$$

$\cos\theta+\sin\theta-1=0$

$$\frac{1}{1-2\sin^2\theta} = 2$$

$$1 = 2(1-2\sin^2\theta)$$

$$\frac{1}{2} = 1-2\sin^2\theta$$

$$2\sin^2\theta = \frac{1}{2}$$

$$\sin^2\theta = \frac{1}{4}$$

$\sin\theta = \frac{1}{2}$ $\quad \sin\theta = -\frac{1}{2}$ x

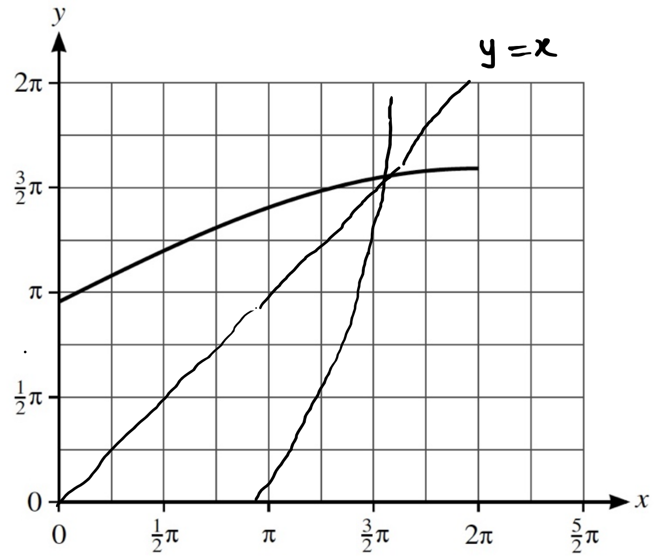
$$\theta = \sin^{-1}\frac{1}{2}$$

$$\theta = \pi/6$$

$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

Problem : 09709/12/M/J/23/Q8



The diagram shows the graph of $y = \mathrm{f}(x)$ where the function f is defined by

$$\mathrm{f}(x) = 3 + 2\sin\tfrac{1}{4}x \text{ for } 0 \leqslant x \leqslant 2\pi.$$

(a) On the diagram above, sketch the graph of $y = \mathrm{f}^{-1}(x)$. [2]

(b) Find an expression for $\mathrm{f}^{-1}(x)$. [2]

Let $y = 3 + 2\sin\frac{1}{4}x$

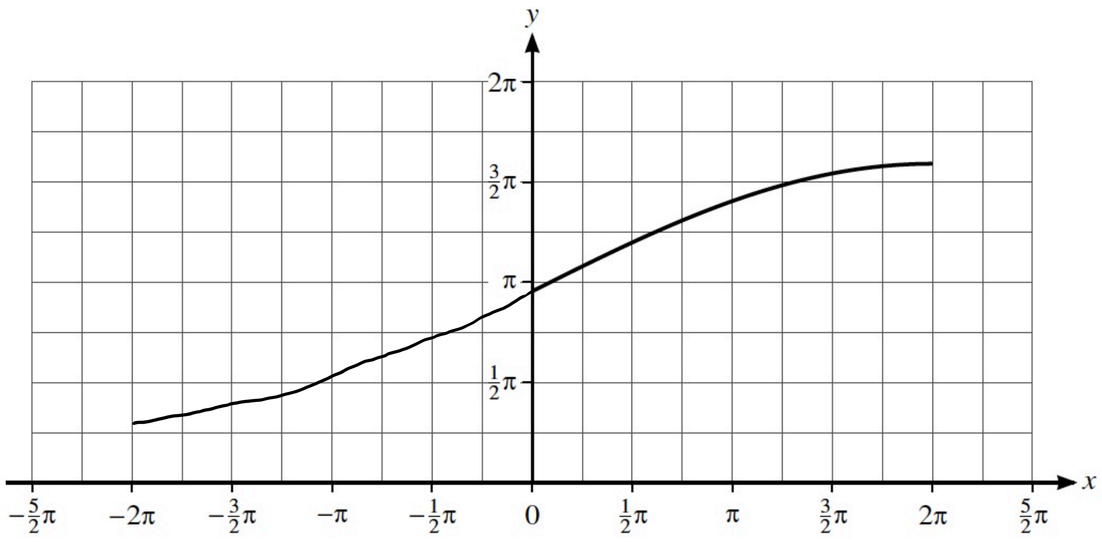
$$\left(\frac{y-3}{2}\right) = \sin\frac{1}{4}x$$

$$\frac{1}{4}x = \sin^{-1}\left(\frac{y-3}{2}\right)$$

$$x = 4\sin^{-1}\left(\frac{y-3}{2}\right)$$

$$f^{-1}(x) = 4\sin^{-1}\left(\frac{x-3}{2}\right)$$

(c)



The diagram above shows part of the graph of the function $g(x) = 3 + 2\sin\frac{1}{4}x$ for $-2\pi \leqslant x \leqslant 2\pi$.

Complete the sketch of the graph of $g(x)$ on the diagram above and hence explain whether the function g has an inverse. [2]

Since g(x) is one-one function hence inverse is possible

(d) Describe fully a sequence of three transformations which can be combined to transform the graph of $y = \sin x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ to the graph of $y = f(x)$, making clear the order in which the transformations are applied. [6]

(1) Stretch by SF 4 in x-direction

Stretch by SF 2 in y-direction

(2) Translation by vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

Problem : 09709/12/M/J/23/Q9

The second term of a geometric progression is 16 and the sum to infinity is 100.

(a) Find the two possible values of the first term. [4]

(b) Show that the nth term of one of the two possible geometric progressions is equal to 4^{n-2} multiplied by the nth term of the other geometric progression. [4]

Sol (a) $ar = 16$

$$\frac{a}{1-r} = 100$$

$$a = 100 - 100r$$

$$\frac{16}{r} = 100 - 100r$$

$$16 = 100r - 100r^2$$

$$100r^2 - 100r + 16 = 0$$

$$r = \frac{100 \pm \sqrt{(100)^2 - 4 \times 100 \times 16}}{200}$$

$$= \frac{100 \pm \sqrt{10000 - 6400}}{200}$$

$$= \frac{100 \pm \sqrt{3600}}{200}$$

$$= \frac{100 \pm 60}{200}$$

$r = \frac{16\cancel{0}}{20\cancel{0}} = \frac{4}{5}$ $\quad r = \frac{4\cancel{0}}{2\cancel{00}} = \frac{1}{5}$

$a = \frac{\cancel{16}^{4}}{\cancel{4}} \times 5$ $\quad a = \frac{16}{1/5} = 80$

$= 20$

(b)

$$u_n = a r^{n-1}$$

$$a_n = 20\left(\frac{4}{5}\right)^{n-1} = 20\left(\frac{1}{5}\right)^{n-1} 4^{n-1}$$

$$b_n = 80\left(\frac{1}{5}\right)^{n-1}$$

$$b_n = 4 \times 20 \times \left(\frac{1}{5}\right)^{n-1}$$

$$b_n = 4 \times \frac{a_n}{4^{n-1}}$$

$$a_n = \frac{4^{n-1}}{4} b_n$$

$$a_n = 4^{n-2} b_n$$

Problem : 09709/12/M/J/23/Q10

The equation of a circle is $(x-a)^2+(y-3)^2=20$. The line $y=\frac{1}{2}x+6$ is a tangent to the circle at the point P.

(a) Show that one possible value of a is 4 and find the other possible value. [5]

(b) For $a=4$, find the equation of the normal to the circle at P. [4]

(c) For $a=4$, find the equations of the two tangents to the circle which are parallel to the normal found in **(b)**. [4]

Sol(a) $(x-a)^2+\left(\frac{1}{2}x+6-3\right)^2=20$

$x^2-2ax+a^2+\left(\frac{x+6}{2}\right)^2=20$

$4(x^2-2ax+a^2)+x^2+36+12x-80=0$

$4x^2-8ax+4a^2+x^2+36+12x-80=0$

$\underbrace{5}_{a}x^2+\underbrace{(12-8a)}_{b}x+\underbrace{4a^2-44}_{c}=0$ ✓

line is tangent to the circle hence

$b^2-4ac=0$

$(12-8a)^2-4\times5\times(4a^2-44)=0$

$144+64a^2-192a-80a^2+880=0$

$-16a^2-192a+1024=0$

$a^2+12a-64=0$

$a=4 \quad a=-16$

(b) $a=4$

$5x^2-20x+20=0$

$x^2-4x+4=0$

$(x-2)^2=0$

$x=2 \quad y=\frac{1}{2}\times2+6=7$

$P(2,7) \quad C(4,3)$

For finding eq of Normal at P

$$\text{gradient} = \frac{7-3}{2-4} = -2$$

$$y-7 = -2(x-2)$$

$$y-7 = -2x+4$$

$$y = -2x+11$$

(c) As tangents are parallel to Normal (b) part

let eq of tangents

$$y = -2x + c$$

$$(x-4)^2 + (y-3)^2 = 20$$

$$(x-4)^2 + (-2x+\underline{c-3})^2 = 20$$

$$x^2 - 8x + 16 + 4x^2 + (c-3)^2 - 4x(c-3) = 20$$

$$5x^2 - (8+4(c-3))x + (c-3)^2 - 4 = 0$$

$$b^2 - 4ac = 0$$

$$[8+4(c-3)]^2 - 4 \times 5 \times (c-3)^2 - 4 = 0$$

$$4c^2 - 88c + 84 = 0$$

$$c^2 - 22c + 21 = 0$$

$$c = 21 \quad c = 1$$

$$y = -2x+21 \quad y = -2x+1$$

Problem : 09709/12/M/J/23/Q11

The equation of a curve is

$$y = k\sqrt{4x+1} - x + 5,$$

where k is a positive constant.

(a) Find $\frac{dy}{dx}$. [2]

(b) Find the x-coordinate of the stationary point in terms of k. [2]

(c) Given that $k = 10.5$, find the equation of the normal to the curve at the point where the tangent to the curve makes an angle of $\tan^{-1}(2)$ with the positive x-axis. [4]

Sol (a)

$$y = k(4x+1)^{1/2} - x + 5$$

$$\frac{dy}{dx} = k \times \frac{1}{\cancel{2}}(4x+1)^{-1/2} \times \overset{2}{\cancel{4}} - 1 + 0$$

$$= \frac{2k}{(4x+1)^{1/2}} - 1$$

(b)

$$0 = \frac{2k}{(4x+1)^{1/2}} - 1$$

$$1 = \frac{2k}{(4x+1)^{1/2}}$$

$$(4x+1)^{1/2} = 2k$$

$$4x+1 = 4k^2$$

$$4x = 4k^2 - 1$$

$$x = \frac{4k^2-1}{4}$$

(c)

$$2 = \frac{2k}{(4x+1)^{1/2}} - 1$$

gradient of Normal $= -\frac{1}{2}$

Eq of Normal

$$3 = \frac{2(10.5)}{(4x+1)^{1/2}}$$

$$y - 66.5 = -\frac{1}{2}(x-12)$$

$$(4x+1)^{1/2} = \frac{2 \times 10.5}{3} = 7$$

$4x+1 = 49$ $\quad x = 12$

$$y = 10.5(4 \times 12 + 1)^{1/2} - 12 + 5 = 66.5$$