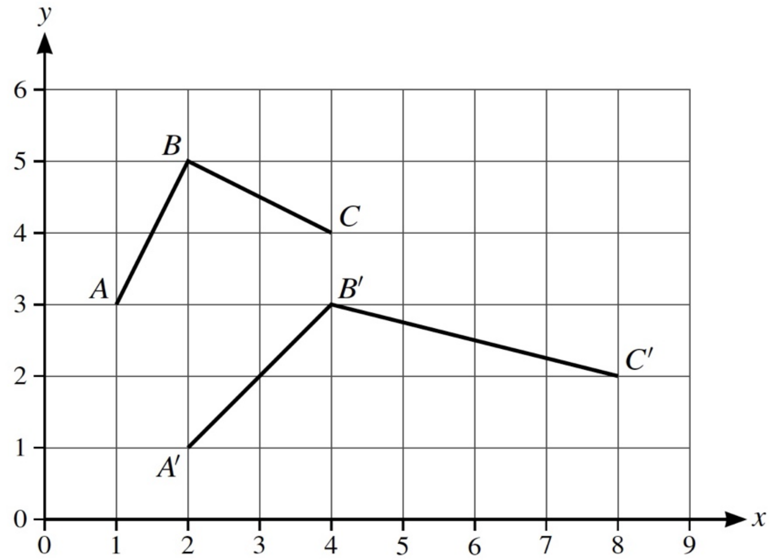
Problem : 09709/13/M/J/23/Q1



The diagram shows the graph of $y = f(x)$, which consists of the two straight lines AB and BC. The lines $A'B'$ and $B'C'$ form the graph of $y = g(x)$, which is the result of applying a sequence of two transformations, in either order, to $y = f(x)$.

State fully the two transformations. [4]

Sol 1. translation by vector $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

2. stretch by factor 2 along x-axis.

Problem : 09709/13/M/J/23/Q2

The function f is defined for $x \in \mathbb{R}$ by $f(x) = x^2 - 6x + c$, where c is a constant. It is given that $f(x) > 2$ for all values of x.

Find the set of possible values of c. [4]

Sol

$f(x) = x^2 - 6x + c \quad f(x) > 2$

$x^2 - 6x + c > 2$

$x^2 - 6x + c - 2 > 0$

$b^2 - 4ac > 0$

$(-6)^2 - 4 \times 1 \times (c-2) > 0$

$36 - 4c + 8 > 0$

$44 - 4c > 0$

$4c < 44$

$c < 11$

Problem : 09709/13/M/J/23/Q3

(a) Give the complete expansion of $\left(x + \frac{2}{x}\right)^5$. [2]

(b) In the expansion of $(a + bx^2)\left(x + \frac{2}{x}\right)^5$, the coefficient of x is zero and the coefficient of $\frac{1}{x}$ is 80.

Find the values of the constants a and b. [4]

Sol (a) $\left(x+\frac{2}{x}\right)^5 = \binom{5}{0} x^{5-0}\left(\frac{2}{x}\right)^0 + \binom{5}{1} x^{5-1}\left(\frac{2}{x}\right)^1 + \binom{5}{2} x^{5-2}\left(\frac{2}{x}\right)^2$

$+ \binom{5}{3} x^{5-3}\left(\frac{2}{x}\right)^3 + \binom{5}{4} x^{5-4}\left(\frac{2}{x}\right)^4$

$+ \binom{5}{5} x^{5-5}\left(\frac{2}{x}\right)^5$

$= 1.x^5 + 5x^4\frac{2}{x} + 10x^3\frac{4}{x^2} + 10x^2\frac{8}{x^3} + 5x\frac{16}{x^4}$

$+ 1.\frac{32}{x^5}$

$= x^5 + 10x^3 + 40x + \frac{80}{x} + \frac{80}{x^3} + \frac{32}{x^5}$

(b) $(a+bx^2)\left(x+\frac{2}{x}\right)^5$

$= (a + bx^2)\left(x^5 + 10x^3 + 40x + \frac{80}{x} + \frac{80}{x^3} + \frac{32}{x^5}\right)$

$40a + 80b = 0$ — (i) $\quad (x)$

$80a + 80b = 80$ — (ii) $\quad \left(\frac{1}{x}\right)$

$a = 2 \quad b = -1$

Problem : 09709/13/M/J/23/Q4

(a) Show that the equation

$$3\tan^2 x - 3\sin^2 x - 4 = 0$$

may be expressed in the form $a\cos^4 x + b\cos^2 x + c = 0$, where a, b and c are constants to be found. [3]

(b) Hence solve the equation $3\tan^2 x - 3\sin^2 x - 4 = 0$ for $0° \leqslant x \leqslant 180°$. [4]

Sol (a) $3\tan^2 x - 3\sin^2 x - 4 = 0$

$$\frac{3(1-\cos^2 x)}{\cos^2 x} - 3(1-\cos^2 x) - 4 = 0$$

$$3(1-\cos^2 x) - 3\cos^2 x(1-\cos^2 x) - 4\cos^2 x = 0$$

$$3 - 3\cos^2 x - 3\cos^2 x + 3\cos^4 x - 4\cos^2 x = 0$$

$$3\cos^4 x - 10\cos^2 x + 3 = 0$$

(b) by part (a) result

let $\cos^2 x = t$

$3t^2 - 10t + 3 = 0$

$t = 3 \quad t = \frac{1}{3}$

$\cos^2 x = 3$

$\cos x = \pm\sqrt{3}$ ✗

$\cos x = \frac{1}{\sqrt{3}} \quad \cos x = -\frac{1}{\sqrt{3}}$

$x = 54.7°$

$x = 180 - 54.7$

$= 125.3°$

Problem : 09709/13/M/J/23/Q5

A circle has equation $(x-1)^2 + (y+4)^2 = 40$. A line with equation $y = x - 9$ intersects the circle at points A and B.

(a) Find the coordinates of the two points of intersection. [4]

(b) Find an equation of the circle with diameter AB. [3]

Sol (a)

$$(x-1)^2 + (x-9+4)^2 = 40$$

$$(x-1)^2 + (x-5)^2 = 40$$

$$x^2 - 2x + 1 + x^2 - 10x + 25 - 40 = 0$$

$$2x^2 - 12x - 14 = 0$$

$$x^2 - 6x - 7 = 0$$

$$x^2 - 7x + x - 7 = 0$$

$$x(x-7) + 1(x-7) = 0$$

$$(x-7)(x+1) = 0$$

$$x = -1, 7$$

$$y = -10, -2$$

$$A(-1, -10) \quad B(7, -2)$$

(b)

$$C\left(\frac{-1+7}{2}, \frac{-10-2}{2}\right)$$

$$C(3, -6)$$

$$AB = \sqrt{(7-(-1))^2 + (-10+2)^2}$$

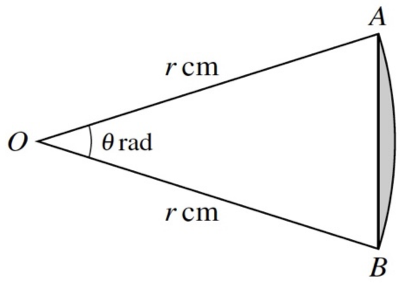
$$= \sqrt{64 + 64}$$

$$AB = \sqrt{128}$$

$$\text{Radius} = \frac{\sqrt{128}}{2} = \sqrt{32}$$

$$(x-3)^2 + (y+6)^2 = 32$$

Problem : 09709/13/M/J/23/Q6



The diagram shows a sector OAB of a circle with centre O and radius r cm. Angle $AOB = \theta$ radians. It is given that the length of the arc AB is 9.6 cm and that the area of the sector OAB is 76.8 cm^2.

(a) Find the area of the shaded region. [5]

(b) Find the perimeter of the shaded region. [2]

Sol (a) Area of shaded region

$= \text{Area of Sector} - \text{Area of } \Delta OAB$

$= 76.8 - \frac{1}{2} r^2 \sin\theta$

$\frac{76.8}{9.6} = \frac{\frac{1}{2} r^{\cancel{2}} \cancel{\theta}}{\cancel{r}\cancel{\theta}} \quad \therefore r = 16$

$9.6 = r\theta = 16\theta \therefore \theta = 0.6$

Area of shaded region

$= 76.8 - \frac{1}{2} 16^2 \sin 0.6$

$= 4.53 \text{ cm}^2$

(b)

$AB = \sqrt{r^2 + r^2 - 2r^2 \sin\theta}$

$= \sqrt{2r^2 - 2r^2 \sin\theta}$

$= \sqrt{2 \times 16^2 - 2 \times 16^2 \times \sin 0.6}$

$= 14.9 \text{ cm}$

Perimeter of shaded region

$= 14.9 + 9.6$

$= 24.5 \text{ cm}$

Problem : 09709/13/M/J/23/Q7

The function f is defined by $f(x) = 2 - \dfrac{5}{x+2}$ for $x > -2$.

(a) State the range of f. [1]

(b) Obtain an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [4]

The function g is defined by $g(x) = x + 3$ for $x > 0$.

(c) Obtain an expression for $fg(x)$ giving your answer in the form $\dfrac{ax+b}{cx+d}$, where a, b, c and d are integers. [3]

Sol (a) $f(x) < 2$

(b) let $y = 2 - \dfrac{5}{x+2}$

$$\frac{5}{x+2} = 2-y$$

$$\frac{5}{2-y} = x+2$$

$$x = \frac{5}{2-y} - 2 = \frac{5-4+2y}{2-y}$$

$$x = \frac{2y+1}{2-y}$$

$$f^{-1}(x) = \frac{2x+1}{2-x}$$

domain of $f^{-1}(x)$ would be

$x < 2$

(c) $$fg(x) = 2 - \frac{5}{x+3+2}$$

$$= 2 - \frac{5}{x+5}$$

$$= \frac{2x+10-5}{x+5}$$

$$= \frac{2x+5}{x+5}$$

Problem : 09709/13/M/J/23/Q8

A progression has first term a and second term $\frac{a^2}{a+2}$, where a is a positive constant.

(a) For the case where the progression is geometric and the sum to infinity is 264, find the value of a. [5]

(b) For the case where the progression is arithmetic and $a = 6$, determine the least value of n required for the sum of the first n terms to be less than -480. [5]

Sol (a) geometric progression

1st a IInd $r = \frac{a^2}{a+2}$ ✓

$$\frac{a}{1-r} = 264 \checkmark$$

$$\frac{a}{1-\frac{a}{a+2}} = 264$$

$$\frac{a(a+2)}{\cancel{a}+2-\cancel{a}} = 264$$

$$a^2+2a-528=0$$

$$a=22 \checkmark \qquad a=-24$$

(b) Arithmetic progression

$a=6 \qquad S_n = -480$

$$a+d = \frac{a^2}{a+2}$$

$$6+d = \frac{36}{8}$$

$$d = \frac{36}{8} - 6 = \frac{36-48}{8}$$

$$= -\frac{3}{2}$$

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$-480 = \frac{n}{2}\left[12 + (n-1)\,-\frac{3}{2}\right]$$

$$-480 = 6n + \frac{n}{2}(n-1)\,-\frac{3}{2}$$

$$-480 = 6n - \frac{3n^2}{4} + \frac{3}{4}n$$

$$-1920 = 24n - 3n^2 + 3n$$

$$3n^2 - 27n - 1920 = 0$$

$$n = 30.195 \qquad n = -21.195$$

$$\underline{n = 31}$$

Problem : 09709/13/M/J/23/Q9

A curve which passes through (0, 3) has equation $y = f(x)$. It is given that $f'(x) = 1 - \frac{2}{(x-1)^3}$.

(a) Find the equation of the curve. [4]

The tangent to the curve at (0, 3) intersects the curve again at one other point, P.

(b) Show that the x-coordinate of P satisfies the equation $(2x+1)(x-1)^2 - 1 = 0$. ✓ [4]

(c) Verify that $x = \frac{3}{2}$ satisfies this equation and hence find the y-coordinate of P. [2]

Sol (a) $f'(x) = 1 - 2(x-1)^{-3}$

$$\int f'(x)\,dx = \int 1 - 2(x-1)^{-3}\,dx$$

$$f(x) = x - 2\frac{(x-1)^{-2}}{-2} + c$$

$$f(x) = x + (x-1)^{-2} + c$$

passing through $(0,3)$

$$3 = 0 + (0-1)^{-2} + c$$

$$3 - 1 = c$$

$$c = 2$$

$$f(x) = x + \frac{1}{(x-1)^2} + 2$$

(b) $x = 0$ $f'(0) = 1 - \frac{2}{(0-1)^3} = 3$

eq. of tangent at $(0,3)$

$$y - 3 = 3(x - 0)$$

$$y = 3x + 3$$

$$3(x+1) = x + \frac{1}{(x-1)^2} + 2$$

$$(2x+1) = \frac{1}{(x-1)^2}$$

$$(2x+1)(x-1)^2 = 1$$

$$(2x+1)(x-1)^2 - 1 = 0$$

(c) $x = \frac{3}{2}$

$$(\cancel{2} \times \frac{3}{\cancel{2}} + 1)(\frac{3}{2} - 1)^2 - 1$$

$$= 4 \times \frac{1}{\cancel{4}} - 1 = 0$$

$y = 3x + 3$

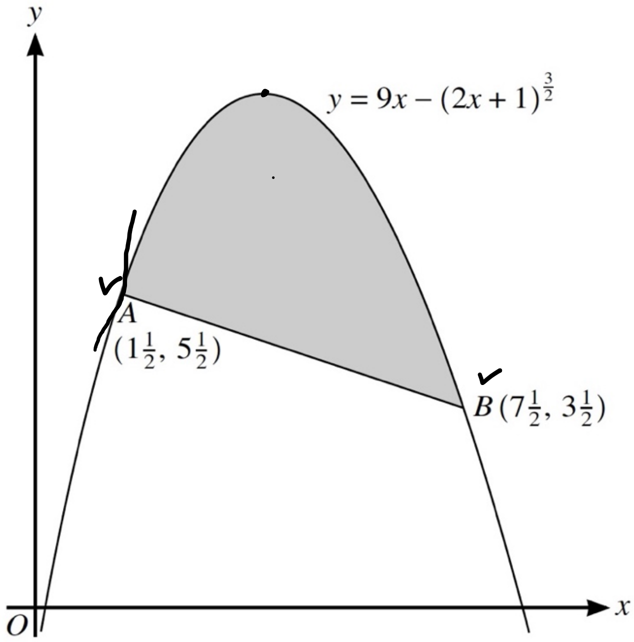
$x = \frac{3}{2}$

$$y = \frac{3 \times 3}{2} + 3 = \frac{9}{2} + 3$$

$$= \frac{9 + 6}{2} = \frac{15}{2}$$

$$= 7\frac{1}{2}$$

Problem : 09709/13/M/J/23/Q10



The diagram shows the points $A\left(1\frac{1}{2}, 5\frac{1}{2}\right)$ and $B\left(7\frac{1}{2}, 3\frac{1}{2}\right)$ lying on the curve with equation $y = 9x - (2x+1)^{\frac{3}{2}}$.

(a) Find the coordinates of the maximum point of the curve. [4]

(b) Verify that the line AB is the normal to the curve at A. [3]

(c) Find the area of the shaded region. [5]

Sol (a) As maximum point is stationary point

$\therefore \frac{dy}{dx} = 0$

$y = 9x - (2x+1)^{3/2}$ $\quad\quad y = 9\times4 - (2\times4+1)^{3/2}$

$\frac{dy}{dx} = 9 - \frac{3}{\cancel{2}}(2x+1)^{1/2} \times \cancel{2}$ $\quad\quad = 9$

$\frac{dy}{dx} = 9 - 3(2x+1)^{1/2}$

$0 = 9 - 3(2x+1)^{1/2}$

$3(2x+1)^{1/2} = 9$

$(2x+1)^{1/2} = 3$

$2x+1 = 9$

$2x = 8$

$x = 4$

Coordinate of Maximum point $(4, 9)$

(b) $A\left(\frac{3}{2}, \frac{11}{2}\right)$

$$\frac{dy}{dx} = 9 - 3(2x+1)^{1/2}$$

$x = \frac{3}{2}$

$$\frac{dy}{dx} = 9 - 3\left(\cancel{2} \times \frac{3}{\cancel{2}} + 1\right)^{1/2}$$

$$= 3$$

gradient of tangent at A on curve

$$= 3$$

gradient of line AB

$$= \frac{\frac{7}{2} - \frac{11}{2}}{\frac{15}{2} - \frac{3}{2}} = \frac{-\frac{4}{2}}{\frac{12}{2}} = -\frac{1}{3}$$

Since the product of both gradient is -1 hence line AB will be normal to the curve at point A.

(c) Equation of line AB

$A\left(\frac{3}{2}, \frac{11}{2}\right)$

$$y - \frac{11}{2} = -\frac{1}{3}\left(x - \frac{3}{2}\right)$$

$$y = -\frac{1}{3}x + \frac{1}{2} + \frac{11}{2}$$

$$y = -\frac{1}{3}x + 6$$

$$\text{Area} = \int_{3/2}^{15/2} \left[9x - (2x+1)^{3/2} - \left(-\frac{1}{3}x + 6\right)\right] dx$$

$$= \left[9\frac{x^2}{2} - \frac{2(2x+1)^{5/2}}{5} \times \frac{1}{2} + \frac{1}{3}\frac{x^2}{2} - 6x\right]_{3/2}^{15/2}$$

$$= \left[\frac{9}{2}\left(\frac{15}{2}\right)^2 - \frac{2}{5}\left(2 \times \frac{15}{2} + 1\right)^{5/2} \times \frac{1}{2} + \frac{1}{6}\left(\frac{15}{2}\right)^2 - 6\left(\frac{15}{2}\right)\right] - \left[\frac{9}{2}\left(\frac{3}{2}\right)^2 - \frac{2}{5}\left(2 \times \frac{3}{2} + 1\right)^{5/2} \times \frac{1}{2} + \frac{1}{6}\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right)\right]$$

$$= 17{\cdot}6$$