Problem : 09709/31/M/J/23/Q1

Solve the equation

$$3e^{2x} - 4e^{-2x} = 5.$$

Give the answer correct to 3 decimal places. [3]

Sol let $e^{2x} = t$

$$3t - \frac{4}{t} = 5$$

$$3t^2 - 4 = 5t$$

$$3t^2 - 5t - 4 = 0$$

$t = 2.257 \qquad t = -0.5906$

$e^{2x} = 2.257 \qquad e^{2x} = -0.5906$ ×

$$2x = \ln 2.257$$

$$x = \frac{1}{2} \ln 2.257$$

$$= 0.407$$

Problem : 09709/31/M/J/23/Q2

(a) Sketch the graph of $y = |2x + 3|$. [1]

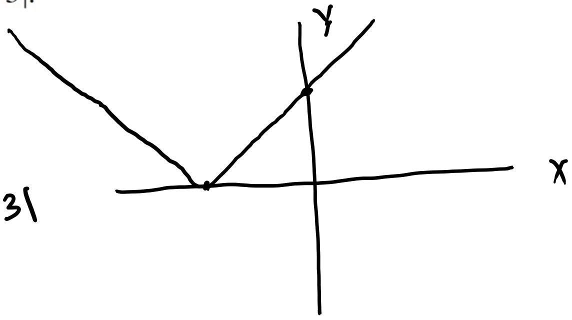
(b) Solve the inequality $3x + 8 > |2x + 3|$. [3]

Sol (a) $0 = 2x+3$

$x = -3/2$

$y = |2\times 0+3|$

$= 3$



(b) By squaring both side

$(3x+8)^2 > (2x+3)^2$

$9x^2 + 48x + 64 > 4x^2 + 12x + 9$

$5x^2 + 36x + 55 > 0$

$(5x+11)(x+5) > 0$

$-\frac{11}{5}$ -5 ✓

$x > -\frac{11}{5}$

Problem : 09709/31/M/J/23/Q3

Find the coefficient of $\underline{x^3}$ in the binomial expansion of $(3 + x)\sqrt{1 + 4x}$. [4]

Sol $(3+x)(1+4x)^{\frac{1}{2}}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$$

$$= \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(4x)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(4x)^3$$

$$= -2x^2 + \frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{\cancel{3}}{2}}{\cancel{6}\,2} 64x^3$$

$$= -2x^2 + 4x^3$$

$$(3+x)(1+4x)^{1/2} = (3+x)(1+4x)^{1/2}$$

$$= (3+x)(-2x^2+4x^3)$$

$$= -2x^3 + 12x^3 = 10x^3$$

$$= 10$$

Problem : 09709/31/M/J/23/Q4

(a) Show that the equation $\sin 2\theta + \cos 2\theta = 2\sin^2\theta$ can be expressed in the form

$$\cos^2\theta + 2\sin\theta\cos\theta - 3\sin^2\theta = 0. \qquad [2]$$

(b) Hence solve the equation $\sin 2\theta + \cos 2\theta = 2\sin^2\theta$ for $0° < \theta < 180°$. [4]

Sol (a) $\sin 2\theta + \cos 2\theta = 2\sin^2\theta$

$2\sin\theta\cos\theta + \cos^2\theta - \sin^2\theta = 2\sin^2\theta$

$\cos^2\theta + 2\sin\theta\cos\theta - 3\sin^2\theta$

(b) $\cos^2\theta + 2\sin\theta\cos\theta - 3\sin^2\theta = 0$

$\cos^2\theta + 3\sin\theta\cos\theta - \sin\theta\cos\theta - 3\sin^2\theta = 0$

$\cos\theta(\cos\theta + 3\sin\theta) - \sin\theta(\cos\theta + 3\sin\theta) = 0$

$(\cos\theta - \sin\theta)(\cos\theta + 3\sin\theta) = 0$

$\cos\theta - \sin\theta = 0$

$\tan\theta = 1$

$\theta = 45°$

$\cos\theta + 3\sin\theta = 0$

$\tan\theta = -\frac{1}{3}$

consider

$\tan\theta = \frac{1}{3}$

$\theta = \tan^{-1}\frac{1}{3}$

$= 18.43°$

$\theta = 180 - 18.43$

$= 161.6°$

Problem : 09709/31/M/J/23/Q5

The equation of a curve is $x^2y - ay^2 = 4a^3$, where a is a non-zero constant.

(a) Show that $\dfrac{dy}{dx} = \dfrac{2xy}{2ay - x^2}$. [4]

(b) Hence find the coordinates of the points where the tangent to the curve is parallel to the y-axis. [4]

Sol (a) $x^2y - ay^2 = 4a^3$

$$x^2\frac{dy}{dx} + 2xy - 2ay\frac{dy}{dx} = 0$$

$$(x^2 - 2ay)\frac{dy}{dx} = -2xy$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 - 2ay}$$

$$\frac{dy}{dx} = \frac{2xy}{2ay - x^2}$$

(b) $\dfrac{1}{0} = \dfrac{2xy}{2ay - x^2}$

$$2ay - x^2 = 0$$

$$x^2 = 2ay \text{ —}$$

$$2ay^2 - ay^2 = 4a^3$$

$$\cancel{a}y^2 = 4a^{\cancel{3}\,2}$$

$$y^2 = 4a^2$$

$$y = 2a$$

$$x^2 = 2a \times 2a$$

$$x^2 = 4a^2$$

$$x = \pm 2a$$

therefore

$(2a, 2a)$ $(-2a, 2a)$

Problem : 09709/31/M/J/23/Q6

Relative to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \qquad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}.$$

The quadrilateral $ABCD$ is a parallelogram.

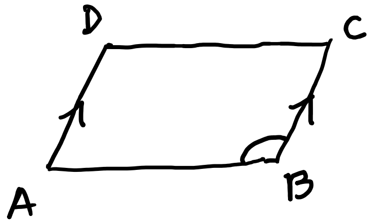
(a) Find the position vector of D. [3]

(b) The angle between BA and BC is θ.

Find the exact value of $\cos\theta$. [3]

(c) Hence find the area of $ABCD$, giving your answer in the form $p\sqrt{q}$, where p and q are integers. [4]

Sol (a) $\overrightarrow{AD} \parallel \overrightarrow{BC}$



$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$= \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \\ -6 \end{pmatrix} \checkmark$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$$

$$\overrightarrow{OD} = \overrightarrow{AD} + \overrightarrow{OA}$$

$$= \begin{pmatrix} -1 \\ -5 \\ -6 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix}$$

(b) $$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

$$\cos\theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| \cdot |\overrightarrow{BC}|} =$$

$$\cos\theta = \frac{\begin{pmatrix}-2\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\-5\\-6\end{pmatrix}}{3\times\sqrt{62}}$$

$$= \frac{2+10-6}{3\sqrt{62}}$$

$$= \frac{6}{3\sqrt{62}} = \frac{2}{\sqrt{62}}$$

(C) Area $= |\vec{BA}|\times|\vec{BC}|\times\sin\theta$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{4}{62}}$$

$$= \sqrt{\frac{58}{62}}$$

$$= 3\times\cancel{\sqrt{62}}\times\frac{\sqrt{58}}{\cancel{\sqrt{62}}}$$

$$= 3\sqrt{58}$$

Problem : 09709/31/M/J/23/Q7

The variables x and y satisfy the differential equation

$$\cos 2x \frac{dy}{dx} = \frac{4 \tan 2x}{\sin^2 3y},$$

where $0 \leqslant x < \frac{1}{4}\pi$. It is given that $y = 0$ when $x = \frac{1}{6}\pi$.

Solve the differential equation to obtain the value of x when $y = \frac{1}{6}\pi$. Give your answer correct to 3 decimal places. [8]

Sol $\int \sin^2 3y \, dy = 4\int \frac{\tan 2x}{\cos 2x} dx$

$\int \frac{1 - \cos 6y}{2} dy = 4 \int \frac{\sin 2x}{(\cos 2x)^2} dx$

let $\cos 2x = t$

$-\sin 2x = \frac{dt}{2}$

$\frac{1}{2}y - \frac{\sin 6y}{12} = -\frac{\cancel{4}^2}{\cancel{2}} \frac{-1}{t} + c$

$\frac{1}{2}y - \frac{\sin 6y}{12} = \frac{1 \times 2}{\cos 2x} + c$

$y = 0 \quad x = \pi/6$

$0 - 0 = \frac{2}{\cos \cancel{2} \times \frac{\pi}{\cancel{6}\,3}} + c$

$0 = \frac{2}{\frac{1}{2}} + c$

$c = -4$

$\frac{1}{2}y - \frac{\sin 6y}{12} = \frac{2}{\cos 2x} - 4$

$\frac{1}{2} \times \frac{\pi}{6} - \frac{\sin \cancel{6} \times \frac{\pi}{\cancel{6}}}{12} = \frac{2}{\cos 2x} - 4$

$\frac{\pi}{12} - 0 = \frac{2}{\cos 2x} - 4$

$\frac{\pi}{12} + 4 = \frac{2}{\cos 2x}$

$\cos 2x = \frac{2 \times 12}{\pi + 48} = 0.47$

$2x = 1.081$

$x = 0.541$

Problem : 09709/31/M/J/23/Q8

Let $f(x) = \dfrac{3-3x^2}{(2x+1)(x+2)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence find the exact value of $\int_0^4 f(x)\,dx$, giving your answer in the form $a + b\ln c$, where a, b and c are integers. [5]

Sol (a) $\dfrac{3-3x^2}{(2x+1)(x+2)^2} = \dfrac{A}{(2x+1)} + \dfrac{B}{(x+2)} + \dfrac{C}{(x+2)^2}$

$3-3x^2 = A(x+2)^2 + B(2x+1)(x+2) + C(2x+1)$

Substitute $x=-2$

$3-3(-2)^2 = C(2\times -2+1)$

$-9 = -3C$

$C = 3$

Substitute $x = -1/2$

$3-3(-\frac{1}{2})^2 = A(-\frac{1}{2}+2)^2$

$3 - \frac{3}{4} = A\,\frac{9}{4}$

$\frac{9}{4} = A(9/4)$

$A = 1$

Const

$3 = 4A + 2B + C$

$\cancel{3} = 4 + 2B + \cancel{3}$

$2B = -4$

$B = -2$

$\dfrac{1}{2x+1} - \dfrac{2}{x+2} + \dfrac{3}{(x+2)^2}$

(b) $\int_0^4 \dfrac{1}{2x+1}dx - 2\int_0^4 \dfrac{1}{x+2}dx + 3\int_0^4 (x+2)^{-2}dx$

$\left[\dfrac{\ln(2x+1)}{2}\right]_0^4 - 2\left[\ln(x+2)\right]_0^4 + 3\left[\dfrac{-1}{(x+2)}\right]_0^4$

$\frac{1}{2}[\ln 9 - 0] - 2[\ln 6 - \ln 2] - 3\left[\frac{1}{6} - \frac{1}{2}\right]$

$\ln 3 - 2\ln 3 - \cancel{3}\left[\dfrac{-\cancel{2}^1}{\cancel{6}}\right]$

$-\ln 3 + 1$

$1 - \ln 3$

Problem : 09709/31/M/J/23/Q9

The constant a is such that $\int_0^a xe^{-2x}\,dx = \frac{1}{8}$.

(a) Show that $a = \frac{1}{2}\ln(4a+2)$. [5]

(b) Verify by calculation that a lies between 0.5 and 1. [2]

(c) Use an iterative formula based on the equation in **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Sol (a)

$$\int_0^a xe^{-2x}\,dx = \frac{1}{8}$$

So by integration By Parts

$$u = x \quad du = 1$$

$$v' = e^{-2x} \quad v = \frac{e^{-2x}}{-2}$$

$$\int uv' = u.v - \int u'v$$

$$\left[x.\frac{e^{-2x}}{-2}\right]_0^a - \int_0^a 1.\frac{e^{-2x}}{-2}\,dx = \frac{1}{8}$$

$$-\frac{ae^{-2a}}{2} + \frac{1}{2}\left[\frac{e^{-2x}}{-2}\right]_0^a = \frac{1}{8}$$

$$-\frac{ae^{-2a}}{2} - \frac{1}{4}\left[e^{-2a} - 1\right] = \frac{1}{8}$$

$$-\frac{ae^{-2a}}{2} - \frac{1}{4}e^{-2a} = -\frac{1}{8}$$

$$-4ae^{-2a} - 2e^{-2a} = -1$$

$$(4a+2)e^{-2a} = 1$$

$$e^{2a} = 4a+2$$

$$2a = \ln(4a+2)$$

$$a = \frac{1}{2}\ln(4a+2)$$

(b) $a - \frac{1}{2}\ln(4a+2)$

$a = 0.5$

$0.5 - \frac{1}{2}\ln(4\times0.5+2) = -0.193$

$a = 1$

$1 - \frac{1}{2}\ln(4\times1+2) = 0.104$

Since answer different for $a=0.5$ and $a=1$
hence a value lies between 0.5 and 1.

(c) $a_{n+1} = \frac{1}{2}\ln(4a_n+2)$

$a_1 = 0.5 \quad \frac{1}{2}\ln(4\times0.5+2)$

$= 0.6931$

$a_2 = \frac{1}{2}\ln(4\times0.6931+2)$

$= 0.7814$

$a_3 = \frac{1}{2}\ln(4\times0.7814+2)$

$= 0.8171$

$a_4 = 0.8308$

$a_5 = 0.8360$

$a_6 = 0.8380$

$a_7 = 0.8387$

$a_8 = 0.8390$

hence $a = 0.84$

Problem : 09709/31/M/J/23/Q10

The polynomial $x^3 + 5x^2 + 31x + 75$ is denoted by $p(x)$.

(a) Show that $(x + 3)$ is a factor of $p(x)$. [2]

(b) Show that $z = -1 + 2\sqrt{6}i$ is a root of $p(z) = 0$. [3]

(c) Hence find the complex numbers z which are roots of $p(z^2) = 0$. [7]

Sol (a) $p(x) = x^3 + 5x^2 + 31x + 75$

$x + 3 = 0$

$x = -3$

$p(-3) = (-3)^3 + 5(-3)^2 + 31 \times -3 + 75$

$= -27 + 45 - 93 + 75$

$= 0$

hence $x+3$ is factor of $P(x)$

(b) $P(z) = z^3 + 5z^2 + 31z + 75$

$z = -1 + 2\sqrt{6}i$

$P(-1 + 2\sqrt{6}i)$

$= (-1 + 2\sqrt{6}i)^3 + 5(-1 + 2\sqrt{6}i)^2 + 31(-1 + 2\sqrt{6}i) + 75$

$= 71 - 102.87i + (115 - 48.98i) - 31 + 151.86i + 75$

$= 0$

hence $z = -1 + 2\sqrt{6}i$ is factor of $P(z)$

(c) $P(z^2) = 0$

$z^2 = -3$

$z = \sqrt{3}i \quad z = -\sqrt{3}i$

$z^2 = -1 - 2\sqrt{6}i$

$z^2 = -1 + 2\sqrt{6}i$

$(x + iy)^2 = -1 + 2\sqrt{6}i$

$x^2 - y^2 = -1 \quad xy = \sqrt{6}$

$x^2 - \frac{6}{x^2} = -1 \quad y = \frac{\sqrt{6}}{x}$

$x^4 + x^2 - 6 = 0$

let $x^2 = t$

$t^2 + t - 6 = 0$

$\times\ t = -3 \quad t = 2$

$x^2 = 2 \quad x = \pm\sqrt{2}$

$y = \sqrt{3} \quad y = -\sqrt{3}$

$z = 2 + \sqrt{3}i \quad z = -\sqrt{2} - \sqrt{3}i$

$(x+iy)^2 = -1 - 2\sqrt{6}i$

$x^2 - y^2 = -1 \quad xy = -\sqrt{6}$

$x^2 - \frac{6}{x^2} = -1 \quad y = \frac{-\sqrt{6}}{x}$

$x^4 + x^2 - 6 = 0$

$x = \pm\sqrt{2}$

$x = \sqrt{2} \quad y = -\sqrt{3}$

$x = -\sqrt{2} \quad y = \sqrt{3}$

$z = \sqrt{2} - \sqrt{3}i$

$z = -\sqrt{2} + \sqrt{3}i$