Problem : 09709/32/F/M/23/Q1

It is given that $x = \ln(2y - 3) - \ln(y + 4)$.

Express y in terms of x.

Sol

$$x = \ln \frac{2y-3}{y+4}$$

$$\frac{2y-3}{y+4} = e^x$$

$$2y - 3 = e^x y + 4e^x$$

$$2y - e^x y = 3 + 4e^x$$

$$y(2 - e^x) = 3 + 4e^x$$

$$y = \frac{3 + 4e^x}{2 - e^x}$$

Problem : 09709/32/F/M/23/Q2

(a) On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $-\frac{1}{3}\pi \leqslant \arg(z - 1 - 2i) \leqslant \frac{1}{3}\pi$ and $\operatorname{Re} z \leqslant 3$. [3]

(b) Calculate the least value of $\arg z$ for points in the region from **(a)**. Give your answer in radians correct to 3 decimal places. [2]

Sol (a) $z - 1 - 2i = 0$

$z = 1 + 2i$

$\operatorname{Arg}(z-1-2i) \geq -\frac{1}{3}\pi$ $\quad$ $\operatorname{Arg}(z-1-2i) \leq \frac{\pi}{3}$ $\quad$ $\operatorname{Re}(z) \leq 3$

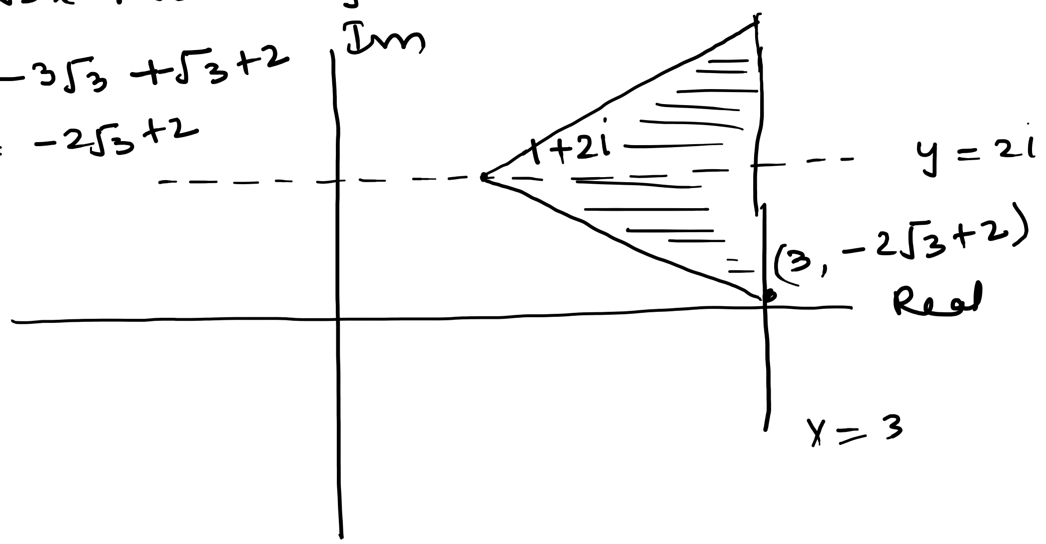
$x \leq 3$

$\frac{y-2}{x-1} = -\sqrt{3}$ $\quad$ $\frac{y-2}{x-1} \leq \sqrt{3}$

$y \geq -\sqrt{3}x + \sqrt{3} + 2$ $\quad$ $y \leq \sqrt{3}x - \sqrt{3} + 2$

$y = -3\sqrt{3} + \sqrt{3} + 2$

$= -2\sqrt{3} + 2$



(b)

$$\theta = \tan^{-1}\left(\frac{-2\sqrt{3}+2}{3}\right) = -0.454$$

Problem : 09709/32/F/M/23/Q1

It is given that $x = \ln(2y - 3) - \ln(y + 4)$.

Express y in terms of x.

Sol

$$x = \ln \frac{2y-3}{y+4}$$

$$\frac{2y-3}{y+4} = e^x$$

$$2y - 3 = e^x y + 4e^x$$

$$2y - e^x y = 3 + 4e^x$$

$$y(2 - e^x) = 3 + 4e^x$$

$$y = \frac{3 + 4e^x}{2 - e^x}$$

Problem : 09709/32/F/M/23/Q4

Solve the equation

$$\frac{5z}{1+2\mathrm{i}} - zz^* + 30 + 10\mathrm{i} = 0,$$

giving your answers in the form $x + \mathrm{i}y$, where x and y are real.

Sol

$$\frac{5(x+iy)}{1+2i} - (x+iy)(x-iy) + 30 + 10i = 0$$

$$\frac{5(x+iy)(1-2i)}{(1-2i)(1+2i)} - (x^2+y^2) + 30 + 10i = 0$$

$$\frac{\cancel{5}(x-2ix+iy+2y)}{1\cancel{+}4} - (x^2+y^2) + 30 + 10i = 0$$

$$x + 2y + i(y-2x) - (x^2+y^2) + 30 + 10i = 0$$

$$x + 2y - x^2 - y^2 + 30 = 0 \qquad y - 2x + 10 = 0$$

$$x + 4x - 20 + 30 - x^2 - (2x-10)^2 = 0 \qquad y = 2x - 10$$

$$5x + 10 - x^2 - 4x^2 + 40x - 100 = 0$$

$$5x + 10 - 5x^2 + 40x - 100 = 0$$

$$-5x^2 + 45x - 90 = 0$$

$$x^2 - 9x + 18 = 0$$

$$x = 6 \qquad x = 3$$

$$y = 2 \qquad y = -4$$

$$6 + 2i \qquad 3 - 4i$$

Problem : 09709/32/F/M/23/Q5

The parametric equations of a curve are

$$x = te^{2t}, \qquad y = t^2 + t + 3.$$

(a) Show that $\frac{dy}{dx} = e^{-2t}$. [3]

(b) Hence show that the normal to the curve, where $t = -1$, passes through the point $\left(\underline{0}, 3 - \frac{1}{e^4}\right)$. [3]

Sol (a) $\frac{dx}{dt} = t \cdot e^{2t}(2) + e^{2t}$ $\qquad \frac{dy}{dt} = 2t+1$

$= e^{2t}(2t+1)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cancel{2t+1}}{e^{2t}(\cancel{2t+1})} = e^{-2t}$$

(b) $t = -1 \qquad \frac{dy}{dx} = e^2$

gradient of Normal $= -\frac{1}{e^2}$

$x = -e^{-2} \qquad y = 3$

$$y - 3 = -\frac{1}{e^2}(x + e^{-2})$$

$$y = -\frac{x}{e^2} - \frac{1}{e^4} + 3$$

$x = 0 \qquad y = 0 - \frac{1}{e^4} + 3$

$$= 3 - \frac{1}{e^4}$$ ✓

Problem : 09709/32/F/M/23/Q6

(a) Express $5\sin\theta + 12\cos\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. [3]

(b) Hence solve the equation $5\sin 2x + 12\cos 2x = 6$ for $0 \leqslant x \leqslant \pi$. [4]

Sol (a) $12\cos\theta + 5\sin\theta = R\cos(\theta - \alpha)$

$R = \sqrt{12^2 + 5^2} = \sqrt{144+25} = \sqrt{169} = 13$

$\alpha = \tan^{-1}\frac{5}{12} = 0.3947$

$13\cos(\theta - 0.395)$

(b) $13\cos(2x - 0.395) = 6$

$\cos(2x - 0.395) = \frac{6}{13}$

$2x - 0.395 = \cos^{-1}\frac{6}{13} = 1.091$

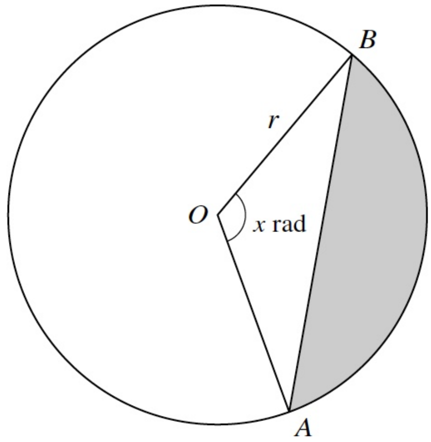
$2x = 0.395 + 1.091$

$x = \underline{\underline{0.743}}$

$2x - 0.395 = 5.192$

$x = \frac{5.192 + 0.395}{2} = \underline{\underline{2.79}}$

Problem : 09709/32/F/M/23/Q7



The diagram shows a circle with centre O and radius r. The angle of the **minor** sector AOB of the circle is x radians. The area of the **major** sector of the circle is 3 times the area of the shaded region.

(a) Show that $x = \frac{3}{4}\sin x + \frac{1}{2}\pi$. [4]

(b) Show by calculation that the root of the equation in **(a)** lies between 2 and 2.5. [2]

(c) Use an iterative formula based on the equation in **(a)** to calculate this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Sol (a) $\frac{1}{2}r^2(2\pi - x) = 3\left[\frac{1}{2}r^2 x - \frac{1}{2}r^2\sin x\right]$

$\cancel{\frac{1}{2}}\cancel{r^2}(2\pi - x) = \cancel{\frac{1}{2}}\cancel{r^2}\,3x - \cancel{\frac{1}{2}}\cancel{r^2}\,3\sin x$

$2\pi - x = 3x - 3\sin x$

$2\pi - 4x = -3\sin x$

$2\pi + 3\sin x = 4x$

$x = \frac{\pi}{2} + \frac{3}{4}\sin x$

(b)

$x = 2 \qquad 2 - \frac{\pi}{2} - \frac{3}{4}\sin 2$

-0.2527 ✓

$x = 2.5 \qquad 2.5 - \frac{\pi}{2} - \frac{3}{4}\sin 2.5$

0.4803 ✓

Since sign are different hence root lies between 2 and 2.5

(c) $x_{n+1} = \frac{\pi}{2} + \frac{3}{4}\sin x_n$

$x_1 = 2$

$x_2 = \frac{\pi}{2} + \frac{3}{4}\sin 2 = 2.2527$

$x_3 = \frac{\pi}{2} + \frac{3}{4}\sin 2.2527 = 2.1530$

$x_4 = \frac{\pi}{2} + \frac{3}{4}\sin(2.1530) = 2.1972$

$x_5 = \frac{\pi}{2} + \frac{3}{4}\sin(2.1972) = 2.1783$

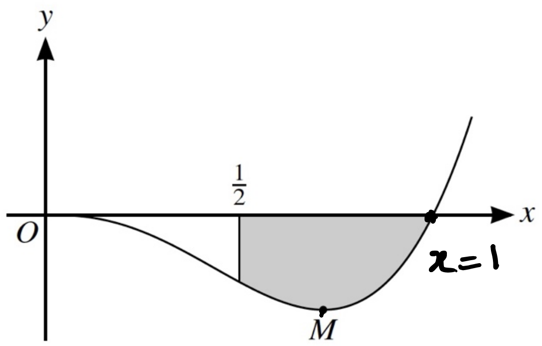
$x_6 = \frac{\pi}{2} + \frac{3}{4}\sin(2.1783) = 2.1865$

$x_7 = \frac{\pi}{2} + \frac{3}{4}\sin(2.1865) = 2.1830$

$x_8 = \frac{\pi}{2} + \frac{3}{4}\sin(2.1830) = 2.1845$

hence root of equation will be 2.18

Problem : 09709/32/F/M/23/Q8



The diagram shows the curve $y = x^3 \ln x$, for $x > 0$, and its minimum point M.

(a) Find the exact coordinates of M. [4]

(b) Find the exact area of the shaded region bounded by the curve, the x-axis and the line $x = \frac{1}{2}$. [5]

Sol (a) $\frac{dy}{dx} = x^3 \cdot \frac{1}{x} + 3x^2 \ln x$

$\frac{dy}{dx} = x^2(1 + 3\ln x)$

Since M is stationary point. Hence $\frac{dy}{dx} = 0$

$0 = x^2(1 + 3\ln x)$

$x^2 = 0 \qquad 1 + 3\ln x = 0$

$x = 0$ ✗ $\qquad \ln x = -\frac{1}{3}$

$x = e^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{e}}$ ✓

$y = x^3 \ln x$

$= (e^{-\frac{1}{3}})^3 \ln e^{-1/3}$

$= \frac{1}{e} \cdot -\frac{1}{3} \ln e$

$= -\frac{1}{3e}$

$M\left(\frac{1}{\sqrt[3]{e}}, -\frac{1}{3e}\right)$

(b) $0 = x^3 \ln x$

$x^3 = 0 \qquad \ln x = 0$

$x = 0 \times \qquad x = e^0 = 1$

Area of shaded region

$$= -\int_{1/2}^{1} x^3 \ln x \, dx$$

$$= -\left[\left(\ln x \cdot \frac{x^4}{4} \right)_{1/2}^{1} - \int \left(\frac{1}{\cancel{x}} \times \frac{x^{\cancel{4}\,3}}{4} \right) dx \right]$$

$$= -\left[\left[\ln x \cdot \frac{x^4}{4} \right]_{1/2}^{1} - \left[\frac{1}{4} \, \frac{x^4}{4} \right]_{1/2}^{1} \right]$$

$$= -\left[\ln x \cdot \frac{x^4}{4} - \frac{x^4}{16} \right]_{1/2}^{1}$$

$$= -\left[\underbrace{\ln 1 \cdot \frac{1}{4}} - \frac{1}{16} - \ln \frac{1}{2} \cdot \frac{1}{64} + \frac{1}{256} \right]$$

$$= -\left[-\frac{1}{16} + \frac{1}{256} + \frac{1}{64} \ln 2 \right]$$

$$= -\left[\frac{-15}{256} + \frac{1}{64} \ln 2 \right]$$

$$= \frac{15}{256} - \frac{1}{64} \ln 2$$

Problem : 09709/32/F/M/23/Q9

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = e^{3y} \sin^2 2x.$$

It is given that $y = 0$ when $x = 0$.

Solve the differential equation and find the value of y when $x = \frac{1}{2}$. [7]

Sol. By Variable seperable method

$$\int \frac{dy}{e^{3y}} = \int \sin^2 2x \, dx$$

$\cos 2x = 1 - 2\sin^2 x$

$\cos 4x = 1 - 2\sin^2 2x$

$\sin^2 2x = \frac{1 - \cos 4x}{2}$

$$\int e^{-3y} dy = \int \left(\frac{1}{2} - \frac{\cos 4x}{2}\right) dx$$

$$\frac{e^{-3y}}{-3} = \frac{1}{2}x - \frac{\sin 4x}{8} + c$$

$x = 0 \quad y = 0$

$$-\frac{1}{3} = c$$

$$\frac{e^{-3y}}{-3} = \frac{1}{2}x - \frac{\sin 4x}{8} - \frac{1}{3}$$

$$e^{-3y} = -\frac{3}{2}x + \frac{3}{8}\sin 4x + 1$$

$$-3y = \ln\left[-\frac{3}{2}x + \frac{3}{8}\sin 4x + 1\right]$$

$$y = -\frac{1}{3}\ln\left[-\frac{3}{2}x + \frac{3}{8}\sin 4x + 1\right]$$

$x = 1/2$

$$y = -\frac{1}{3}\ln\left[-\frac{3}{4} + \frac{3}{8}\sin 2 + 1\right]$$

$$y = -\frac{1}{3}\ln\left[\frac{1}{4} + \frac{3}{8}\sin 2\right]$$

Problem : 09709/32/F/M/23/Q10

With respect to the origin O, the points A, B, C and D have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OD} = \begin{pmatrix} 5 \\ -6 \\ 11 \end{pmatrix}.$$

(a) Find the obtuse angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$. [3]

The line l passes through the points A and B.

(b) Find a vector equation for the line l. [2]

(c) Find the position vector of the point of intersection of the line l and the line passing through C and D. [4]

Sol (a)

$$\theta = \cos^{-1}\left[\frac{\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}}{\left|\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}\right|\cdot\left|\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}\right|}\right]$$

$$= \cos^{-1}\left[\frac{3-2-6}{\sqrt{14}\sqrt{14}}\right]$$

$$= \cos^{-1}\left[\frac{-5}{14}\right] = 110.9^\circ$$

(b) $l : r = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix}$

(c) $s : r = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + \mu\begin{pmatrix} 4 \\ -4 \\ 6 \end{pmatrix}$

$$3 - 2\lambda = 1 + 4\mu$$

$$2\lambda + 4\mu = 2$$

$$\lambda + 2\mu = 1 \quad \text{— (1)}$$

$$-2 - 4\mu = -1 + 3\lambda$$

$$3\lambda + 4\mu = -1 \quad \text{— (ii)}$$

Solve eq (1) & (ii)

$\lambda = -3 \quad \mu = 2$

$$l : r = \begin{pmatrix} 9 \\ -10 \\ 17 \end{pmatrix}$$

$$S : r = \begin{pmatrix} 9 \\ -10 \\ 17 \end{pmatrix}$$

Since point is same in both eq of line hence $\begin{pmatrix} 9 \\ -10 \\ 17 \end{pmatrix}$ would be point of intersection therefore position vecter would be

$$9i - 10j + 17K$$

Problem : 09709/32/F/M/23/Q11

Let $f(x) = \dfrac{5x^2 + x + 11}{(4 + x^2)(1 + x)}$.

(a) Express f(x) in partial fractions. [5]

(b) Hence show that $\int_0^2 f(x)\,dx = \ln 54 - \frac{1}{8}\pi$. [5]

Sol (a) $\dfrac{5x^2+x+11}{(4+x^2)(1+x)} = \dfrac{Ax+B}{4+x^2} + \dfrac{C}{1+x}$

$5x^2+x+11 = (Ax+B)(1+x) + C(4+x^2)$

$5x^2+x+11 = Ax + Ax^2 + B + Bx + 4C + Cx^2$

x^2 $\quad A+C=5 \quad A=2$

x $\quad A+B=1 \quad B=-1$

const $\quad B+4C=11 \quad C=3$

$$\frac{2x-1}{4+x^2} + \frac{3}{1+x}$$

(b) $\displaystyle\int_0^2 f(x)\,dx$

$$\int_0^2 \frac{2x}{4+x^2}dx - \int_0^2 \frac{1\,dx}{4+x^2} + \int_0^2 \frac{3}{1+x}dx$$

let $4+x^2 = t$

$2x\,dx = dt$

$x=0 \quad t=4$

$x=2 \quad t=8$

$$\int_4^8 \frac{dt}{t} - \int_0^2 \frac{1}{4+x^2}dx + \int_0^2 \frac{3}{1+x}dx$$

$$[\ln t]_4^8 - \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2 + 3\left[\ln(1+x)\right]_0^2$$

$$\ln 2 + \cancel{\ln 4} - \cancel{\ln 4} - \frac{1}{2}\times\frac{\pi}{4} + \ln 27$$

$$\ln 54 - \frac{\pi}{8}$$