Problem : 09709/32/M/J/23/Q1

Solve the inequality $|5x - 3| < 2|3x - 7|$. [4]

Sol Squaring both side

$(5x-3)^2 < 2^2(3x-7)^2$

$25x^2 - 30x + 9 < 4(9x^2 - 42x + 49)$

$25x^2 - 30x + 9 < 36x^2 - 168x + 196$

$11x^2 - 138x + 187 > 0$

$(x-11)(11x-17) > 0$

$x < \frac{17}{11} \cup x > 11$

Problem : 09709/32/M/J/23/Q2

Solve the equation $\ln(2x^2 - 3) = 2\ln x - \ln 2$, giving your answer in an exact form. [3]

Sol

$\ln(2x^2 - 3) = \ln x^2 - \ln 2$

$\cancel{\ln}(2x^2 - 3) = \cancel{\ln} \frac{x^2}{2}$

$4x^2 - 6 = x^2$

$3x^2 = 6$

$x^2 = 2$

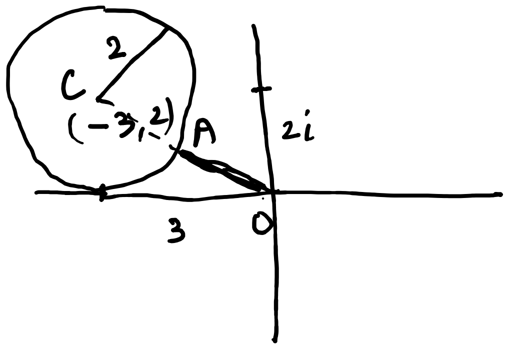
$x = \pm\sqrt{2}$

$x = \sqrt{2}$

Problem : 09709/32/M/J/23/Q3

(a) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying $|z + 3 - 2i| = 2$. [2]

$$(x+3)^2 + (y-2)^2 = 4$$



(b) Find the least value of $|z|$ for points on this locus, giving your answer in an exact form. [2]

$$\sqrt{3^2+2^2} = \sqrt{13}$$

least value of $|z|$ $OA = OC - \text{Radius}$

$$= \sqrt{13} - 2$$

Problem : 09709/32/M/J/23/Q4

Solve the equation $2\cos x - \cos\frac{1}{2}x = 1$ for $0 \leqslant x \leqslant 2\pi$. [5]

Sol let $\frac{x}{2} = t$

$2\cos 2t - \cos t - 1 = 0$

$2\left[2\cos^2 t - 1\right] - \cos t - 1 = 0$

$4\cos^2 t - 2 - \cos t - 1 = 0$

$4\cos^2 t - \cos t - 3 = 0$

let $\cos t = a$

$4a^2 - a - 3 = 0$

$a = 1 \qquad a = -\frac{3}{4}$

$\cos t = 1 \qquad \cos t = \frac{-3}{4}$

$t = 0 \qquad t = 2\pi \qquad t = \cos^{-1}\frac{3}{4} = 0.722$

$\frac{x}{2} = 0 \qquad \frac{x}{2} = 2\pi \qquad \frac{x}{2} = 2.42$

$x = 0 \qquad x = 4\pi \qquad x = 4.84$

$x = 7.73$

$x = 0$ and $x = 4.84$

Problem : 09709/32/M/J/23/Q5

The complex number $2 + yi$ is denoted by a, where y is a real number and $y < 0$. It is given that $f(a) = a^3 - a^2 - 2a$.

(a) Find a simplified expression for $f(a)$ in terms of y. [3]

(b) Given that $\mathrm{Re}(f(a)) = -20$, find $\arg a$. [3]

Sol (a) $f(a) = (2+yi)^3 - (2+yi)^2 - 2(2+yi)$

$= 8 + 3(2)^2 yi + 3(2)(yi)^2 + (yi)^3 - (4 + 4yi + (yi)^2)$
$- 2(2+yi)$

$= 8 + 12yi + 6y^2(-1) - y^3 i - 4 - 4yi + y^2 - 4 - 2yi$

$= 8 - 6y^2 - 8 + y^2 + i(12y - y^3 - 4y - 2y)$

$f(a) = -5y^2 + i(6y - y^3)$

(b) $-5y^2 = -20$

$y^2 = 4$

$y = -2$

$2 - 2i$

$\mathrm{Arg}(a) = \tan^{-1}(-1) = -\frac{\pi}{4}$

Problem : 09709/32/M/J/23/Q6

The equation $\cot \frac{1}{2}x = 3x$ has one root in the interval $0 < x < \pi$, denoted by α.

(a) Show by calculation that α lies between 0.5 and 1. [2]

(b) Show that, if a sequence of positive values given by the iterative formula

$$x_{n+1} = \frac{1}{3}\left(x_n + 4\tan^{-1}\left(\frac{1}{3x_n}\right)\right)$$

converges, then it converges to α. [2]

(c) Use this iterative formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Sol (a) $\cot \frac{1}{2}x = 3x$

$\tan \frac{x}{2} = \frac{1}{3x}$ ✓

$\tan \frac{x}{2} - \frac{1}{3x}$

$x = 0.5 \quad -0.4113$

$x = 1 \quad 0.2129$

Since sign of both values is different hence root lies between 0.5 and 1.

(b) $\frac{x}{2} = \tan^{-1}\frac{1}{3x}$ $\qquad x = 2\tan^{-1}\frac{1}{3x}$

$2x = 4\tan^{-1}\frac{1}{3x}$ $\qquad 3x - x = 4\tan^{-1}\frac{1}{3x}$

$3x = x + 4\tan^{-1}\frac{1}{3x}$ $\qquad x_{n+1} = \frac{1}{3}\left(x_n + 4\tan^{-1}\frac{1}{3x_n}\right)$

(c) $x_1 = 0.5$

$x_2 = \frac{1}{3}\left(0.5 + 4\tan^{-1}\frac{1}{3\times 0.5}\right) = 0.9506$

$x_3 = \frac{1}{3}\left(0.9506 + 4\tan^{-1}\frac{1}{3\times 0.9506}\right) = 0.7665$

$x_4 = 0.8024$

$x_5 = 0.7924$

$x_6 = 0.7950$

$x_7 = 0.7940$

$x_8 = 0.794$

$\therefore$ root value would be 0.79

Problem : 09709/32/M/J/23/Q7

The equation of a curve is $3x^2 + 4xy + 3y^2 = 5$.

(a) Show that $\frac{dy}{dx} = -\frac{3x+2y}{2x+3y}$. [4]

(b) Hence find the exact coordinates of the two points on the curve at which the tangent is parallel to $y + 2x = 0$. [5]

Sol (a) $3x^2 + 4xy + 3y^2 = 5$

$6x + 4\left[x\frac{dy}{dx} + y\right] + 6y\frac{dy}{dx} = 0$

$3x + 2\left[x\frac{dy}{dx} + y\right] + 3y\frac{dy}{dx} = 0$

$3x + 2x\frac{dy}{dx} + 2y + 3y\frac{dy}{dx} = 0$

$(2x+3y)\frac{dy}{dx} = -(3x+2y)$

$\frac{dy}{dx} = -\frac{(3x+2y)}{(2x+3y)}$ ✓

(b) $y = -2x$

$+2 = +\frac{3x+2y}{2x+3y}$

$4x + 6y = 3x + 2y$

$x = -4y$ ✓

$3(-4y)^2 + 4(-4y)y + 3y^2 = 5$

$48y^2 - 16y^2 + 3y^2 = 5$

$32y^2 + 3y^2 = 5$

$35y^2 = 5$

$y^2 = \frac{1}{7}$ $\quad y = \pm\frac{1}{\sqrt{7}}$

$y = \frac{1}{\sqrt{7}}$ $\quad x = \frac{-4}{\sqrt{7}}$

$y = -\frac{1}{\sqrt{7}}$ $\quad x = \frac{4}{\sqrt{7}}$

$\left(-\frac{4}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$ $\left(\frac{4}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right)$

Problem : 09709/32/M/J/23/Q8

(a) The variables x and y satisfy the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4+9y^2}{\mathrm{e}^{2x+1}}.$$

It is given that $y = 0$ when $x = 1$.

Solve the differential equation, obtaining an expression for y in terms of x. [7]

(b) State what happens to the value of y as x tends to infinity. Give your answer in an exact form. [1]

Sol (a) $\displaystyle\int \frac{dy}{4+9y^2} = \int e^{-(2x+1)}\,dx$

$$\frac{1}{4}\int \frac{dy}{1+\left(\frac{3}{2}y\right)^2} = \int e^{-2x-1}\,dx$$

$$\frac{1}{4}\tan^{-1}\frac{3}{2}y = \frac{e^{-2x-1}}{-2} + c$$

$y=0 \quad x=1$

$$\frac{1}{4}\tan^{-1}0 = -\frac{e^{-3}}{2} + c$$

$$c = \frac{e^{-3}}{2}$$

$$\frac{1}{4}\tan^{-1}\frac{3}{2}y = -e^{-(2x+1)} + \frac{e^{-3}}{2}$$

$$\tan^{-1}\frac{3}{2}y = -4e^{-(2x+1)} + 2e^{-3}$$

$$\frac{3}{2}y = \tan\left[2e^{-3} - 4e^{-(2x+1)}\right]$$

$$y = \frac{2}{3}\tan\left[2e^{-3} - 4e^{-(2x+1)}\right]$$

(b) $x \to \infty \qquad y = \frac{2}{3}\tan\left[2e^{-3}\right]$

Problem : 09709/32/M/J/23/Q9

Let $f(x) = \dfrac{2x^2 + 17x - 17}{(1+2x)(2-x)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

(b) Hence show that $\int_0^1 f(x)\,dx = \frac{5}{2} - \ln 72$. [5]

Sol (a) $\dfrac{2x^2+17x-17}{(1+2x)(2-x)^2} = \dfrac{A}{(1+2x)} + \dfrac{B}{(2-x)} + \dfrac{C}{(2-x)^2}$

$2x^2+17x-17 = A(2-x)^2 + B(1+2x)(2-x) + C(1+2x)$

Substitute $x=2$

$2(2)^2+17(2)-17 = C(1+2\times 2)$

$25 = 5C \quad \therefore C=5$

Substitute $x=-1/2$

$2(-\frac{1}{2})^2 + 17(-\frac{1}{2}) - 17 = A(2+\frac{1}{2})^2$

$-25 = A\frac{25}{4}$

$A=-4$

by constant term

$-17 = 4A + 2B + C$

$-17 = -16 + 2B + 5$

$-17+16-5 = 2B$

$-6 = 2B$

$B = -3$

$\dfrac{-4}{1+2x} + \dfrac{-3}{(2-x)} + \dfrac{5}{(2-x)^2}$

(b) $-4\int_0^1 \dfrac{1}{1+2x}\,dx - 3\int_0^1 \dfrac{1}{2-x}\,dx + 5\int_0^1 (2-x)^{-2}\,dx$

$-\frac{4}{2}\left[\ln(1+2x)\right]_0^1 - 3\left[\dfrac{\ln(2-x)}{-1}\right]_0^1 + 5\left[\dfrac{(2-x)^{-1}}{1}\right]_0^1$

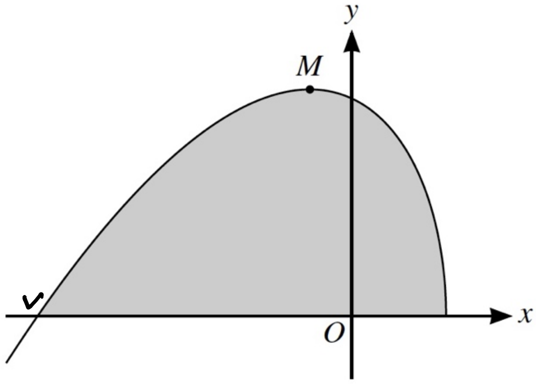
$-2[\ln 3] + 3[\ln 1 - \ln 2] + 5[1^{-1} - \frac{1}{2}]$

$$-2\ln 3 - 3\ln 2 + \frac{5}{2}$$

$$\frac{5}{2} - \left[\ln 9 + \ln 8\right]^{2}$$

$$\frac{5}{2} - \ln 72$$

Problem : 09709/32/M/J/23/Q10



The diagram shows the curve $y = (x + 5)\sqrt{3 - 2x}$ and its maximum point M.

(a) Find the exact coordinates of M. [5]

(b) Using the substitution $u = 3 - 2x$, find by integration the area of the shaded region bounded by the curve and the x-axis. Give your answer in the form $a\sqrt{13}$, where a is a rational number. [5]

Sol (a) $y = (x+5)(3-2x)^{\frac{1}{2}}$

$$\frac{dy}{dx} = (x+5) \times \frac{1}{\cancel{2}}(3-2x)^{-1/2} \times -\cancel{2} + (3-2x)^{1/2} \cdot 1$$

$$\frac{dy}{dx} = -(x+5)(3-2x)^{-1/2} + (3-2x)^{1/2}$$

As M is Stationary point $\therefore \frac{dy}{dx} = 0$

$$\frac{(x+5)}{(3-2x)^{1/2}} = (3-2x)^{1/2}$$

$$x+5 = 3-2x$$

$$3x = -2$$

$$x = -\frac{2}{3}$$

$$y = \left(-\frac{2}{3}+5\right)\left(3-2 \times \frac{-2}{3}\right)^{1/2}$$

$$= \frac{13}{3}\left(\frac{13}{3}\right)^{1/2} = \frac{13}{3}\sqrt{\frac{13}{3} \times \frac{3}{3}}$$

$$= \frac{13\sqrt{39}}{9}$$

$$M\left(-\frac{2}{3}, \frac{13\sqrt{39}}{9}\right)$$

(b) $0 = (x+5)(3-2x)^{1/2}$

$x+5=0 \quad \therefore x=-5$

$(3-2x)^{1/2}=0$

$3-2x=0 \quad x = 3/2$

$$\int_{-5}^{3/2} (x+5)(3-2x)^{1/2}\,dx$$

$u = 3-2x \qquad du = -2dx$

$2x = 3-u \qquad dx = -\frac{1}{2}du$

$x = \frac{3-u}{2}$

$x=-5 \qquad u = 3-2\times-5 = 13$

$x = 3/2 \qquad u = 3-2\times\frac{3}{2}=0$

$$-\frac{1}{2}\int_{13}^{0}\left(\frac{3-u}{2}+5\right)\left(3-\cancel{2}\left(\frac{3-u}{\cancel{2}}\right)\right)^{1/2} du$$

$$\frac{1}{2}\int_0^{13} \frac{(13-u)}{2}\, u^{1/2}\, du$$

$$\frac{1}{4}\int_0^{13} 13u^{1/2} - u^{3/2}\, du$$

$$\frac{1}{4}\left[\frac{13\times2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^{13}$$

$$\frac{1}{4}\left[\frac{26}{3}(13)^{3/2} - \frac{2}{5}(13)^{5/2}\right]$$

$$\frac{1}{4}\sqrt{13}\left[\frac{26\times13}{3} - \frac{2}{5}(13)^2\right]$$

$$= \frac{169\sqrt{13}}{15} \qquad a = \frac{169}{15}$$

Problem : 09709/32/M/J/23/Q11

The points A and B have position vectors $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively. The line l has equation $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$.

(a) Show that l does not intersect the line passing through A and B. [5]

(b) Find the position vector of the foot of the perpendicular from A to l. [4]

Sol (a) $\vec{OA} = i + 2j - 2k$

$\vec{OB} = 2i - j + k$

$\vec{AB} = \vec{OB} - \vec{OA}$

$= i - 3j + 3k$

Vector eq of line passing through A and B

$r_1 = i + 2j - 2k + \lambda(i - 3j + 3k)$

$r = i - j + 3k + \mu(2i - 3j + 4k)$

x - comp: $1 + \lambda = 1 + 2\mu$

$\lambda = 2\mu$ — (i)

y - comp: $2 - 3\lambda = -1 - 3\mu$

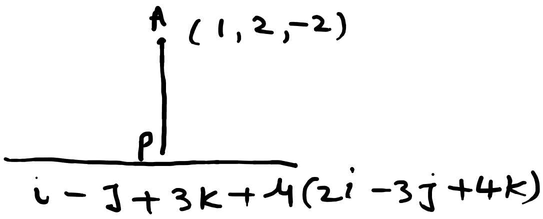
$-3\lambda + 3\mu = -3$ — (ii)

$\lambda = 2$, $\mu = 1$

$\lambda = 2$ $r_1 = 3i - 4j + 4k$

$\mu = 1$ $r = 3i - 4j + 7k$

Since there is no common point, hence lines do not intersect.



(b) $P(1 + 2\mu, -1 - 3\mu, 3 + 4\mu)$

$\vec{AP} = 2\mu i + (-3 - 3\mu)j + (5 + 4\mu)k$

$\vec{AP} \perp$ direction vector of line l.

$4\mu + 9 + 9\mu + 20 + 16\mu = 0$

$29\mu + 29 = 0$ $\therefore \mu = -1$

$P(-1, 2, -1)$

$-i + 2j - k$