Problem : 09709/33/M/J/23/Q1

Solve the equation $\ln(x + 5) = 5 + \ln x$. Give your answer correct to 3 decimal places. [4]

Sol

$$\ln(x+5) - \ln x = 5$$

$$\ln \frac{x+5}{x} = 5$$

$$\frac{x+5}{x} = e^5$$

$$x + 5 = xe^5$$

$$5 = x(e^5 - 1)$$

$$x = \frac{5}{e^5 - 1}$$

$$= 0.034$$

Problem : 09709/33/M/J/23/Q2

Find the quotient and remainder when $2x^4 - 27$ is divided by $x^2 + x + 3$. [3]

Sol

$$x^2 + x + 3 \,\big)\, \cancel{2x^4} - 27 \,\big(\, 2x^2 - 2x - 4$$

$$\cancel{2x^4} + 2x^3 + 6x^2$$

$$- \quad - \quad -$$

$$\cancel{-2x^3} - 6x^2 - 27$$

$$\cancel{-2x^3} - 2x^2 - 6x$$

$$+ \quad + \quad +$$

$$\cancel{-4x^2} + 6x - 27$$

$$\cancel{-4x^2} - 4x - 12$$

$$+ \quad + \quad +$$

$$10x - 15$$

Qustient : $2x^2 - 2x - 4$

Remainder: $10x - 15$

Problem : 09709/33/M/J/23/Q3

On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 3 - \mathrm{i}| \leqslant 3$ and $|z| \geqslant |z - 4\mathrm{i}|$. [4]

Sol $|z-3-i| \leq 3$ $\quad |z| \geq |z-4i|$

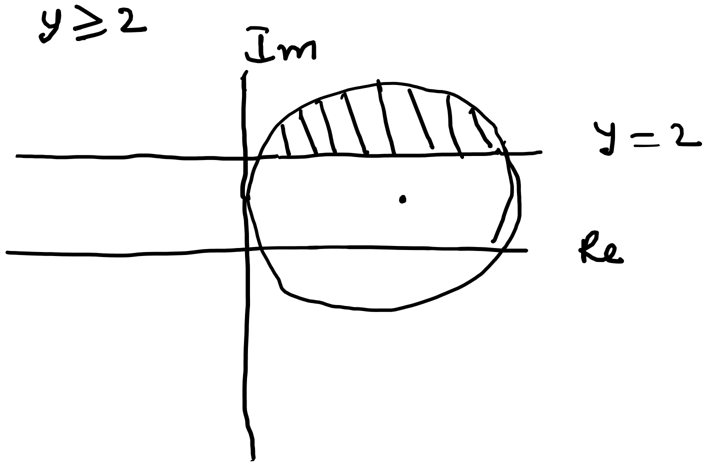
$(x-3)^2 + (y-1)^2 \leq 3$

$\cancel{x^2} + y^2 \geq \cancel{x^2} + (y-4)^2$

$\cancel{y^2} \geq \cancel{y^2} + 16 - 8y$

$8y \geq 16$

$y \geq 2$



Problem : 09709/33/M/J/23/Q4

The parametric equations of a curve are

$$x = \frac{\cos\theta}{2 - \sin\theta}, \qquad y = \theta + 2\cos\theta.$$

Show that $\frac{dy}{dx} = (2 - \sin\theta)^2$. [5]

Sol

$$\frac{dx}{d\theta} = \frac{(2-\sin\theta)(-\sin\theta) - \cos\theta(-\cos\theta)}{(2-\sin\theta)^2}$$

$$= \frac{-2\sin\theta + \sin^2\theta + \cos^2\theta}{(2-\sin\theta)^2}$$

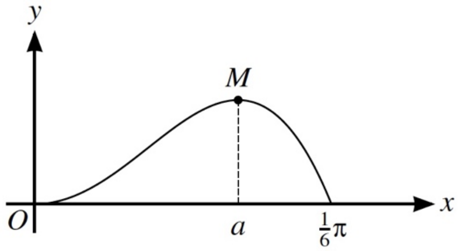
$$= \frac{1 - 2\sin\theta}{(2-\sin\theta)^2}$$

$$\frac{dy}{d\theta} = 1 - 2\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cancel{1-2\sin\theta}}{\cancel{1-2\sin\theta}} \times (2-\sin\theta)^2$$

$$\frac{dy}{dx} = (2-\sin\theta)^2$$

Problem : 09709/33/M/J/23/Q5



The diagram shows the part of the curve $y = x^2 \cos 3x$ for $0 \leqslant x \leqslant \frac{1}{6}\pi$, and its maximum point M, where $x = a$.

(a) Show that a satisfies the equation $a = \frac{1}{3}\tan^{-1}\left(\frac{2}{3a}\right)$. [3]

(b) Use an iterative formula based on the equation in **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Sol (a) $y = x^2 \cos 3x$

$$\frac{dy}{dx} = x^2 \times -\sin 3x \times 3 + 2x\cos 3x$$

As M is stationary point hence $\frac{dy}{dx} = 0$

$$0 = -3x^2 \sin 3x + 2x\cos 3x$$

$$3x^2 \sin 3x = 2x \cos 3x$$

$$\frac{\sin 3x}{\cos 3x} = \frac{2\cancel{x}}{3x^{\cancel{2}}}$$

$$\tan 3x = \frac{2}{3x}$$

$$3x = \tan^{-1}\frac{2}{3x}$$

$$x = \frac{1}{3}\tan^{-1}\frac{2}{3x}$$

$x = a$ $\quad a = \frac{1}{3}\tan^{-1}\frac{2}{3a}$

(b) $a_{n+1} = \frac{1}{3}\tan^{-1}\frac{2}{3a_n}$

$a_1 = 0.5$ $\quad a_2 = \frac{1}{3}\tan^{-1}\frac{2}{3\times 0.5} = 0.3090$

$a_3 = \frac{1}{3}\tan^{-1}\frac{2}{3\times 0.3090} = 0.3788$

$a_4 = 0.3513$ $\quad a_5 = 0.3619$

$a_6 = 0.3578$ $\quad a_7 = 0.3594$

$a_8 = 0.3587$ $\quad a = 0.36$

Problem : 09709/33/M/J/23/Q6

(a) Express $3\cos x + 2\cos(x - 60°)$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. State the exact value of R and give α correct to 2 decimal places. [4]

(b) Hence solve the equation

$$3\cos 2\theta + 2\cos(2\theta - 60°) = 2.5$$

for $0° < \theta < 180°$. [4]

Sol (a) $3\cos x + 2(\cos x \cos 60 + \sin x . \sin 60)$

$3\cos x + \cos x + \sqrt{3}\sin x$

$4\cos x + \sqrt{3}\sin x$

$R = \sqrt{a^2+b^2} = \sqrt{4^2 + (\sqrt{3})^2} = \sqrt{19}$

$\alpha = \tan^{-1}\frac{\sqrt{3}}{4} = 23.41$

(b) $\sqrt{19}\cos(2\theta - 23.41) = 2.5$

$\cos(2\theta - 23.41) = \frac{2.5}{\sqrt{19}}$

$2\theta - 23.41 = \cos^{-1}\left(\frac{2.5}{\sqrt{19}}\right) = 55.0$

$2\theta = 55 + 23.41$

$\theta = 39.2°$

$2\theta - 23.41 = 360 - 55$

$\theta = \frac{305 + 23.41}{2}$

$\theta = 164.2°$

Problem : 09709/33/M/J/23/Q7

(a) Use the substitution $u = \cos x$ to show that

$$\int_0^{\pi} \sin 2x\, e^{2\cos x}\, dx = \int_{-1}^{1} 2u e^{2u}\, du. \qquad [4]$$

(b) Hence find the exact value of $\int_0^{\pi} \sin 2x\, e^{2\cos x}\, dx$. [4]

Sol (a) $u = \cos x$

$du = -\sin x\, dx$

$x = 0 \quad u = \cos 0 = 1$

$x = \pi \quad u = -1$

$$\int_0^{\pi} \sin 2x\, e^{2\cos x}\, dx$$

$$\int_1^{-1} 2\sin x \cos x\, e^{2\cos} dx$$

$$\int_1^{-1} 2u e^{2u} \cancel{\sin x} \frac{-du}{\cancel{\sin x}}$$

$$\int_{-1}^{1} 2u e^{2u} du$$

(b) $$2\int_{-1}^{1} u \cdot e^{2u}\, du$$

by integration by parts

$$2\left[\frac{u \cdot e^{2u}}{2}\right]_{-1}^{1} - \left[\int_{-1}^{1} 1\frac{e^{2u}}{2}\, du\right]$$

$$\left[u \cdot e^{2u}\right]_{-1}^{1} - \left[\frac{e^{2u}}{2}\right]_{-1}^{1}$$

$$e^2 + e^{-2} - \left[\frac{e^2}{2} - \frac{e^{-2}}{2}\right]$$

$$\frac{e^2}{2} + \frac{3}{2} e^{-2}$$

Problem : 09709/33/M/J/23/Q8

The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{y^2+4}{x(y+4)}$$

for $x > 0$. It is given that $x = 4$ when $y = 2\sqrt{3}$.

Solve the differential equation to obtain the value of x when $y = 2$. [8]

Sol By Variable seperable method.

$$\int \frac{y+4}{y^2+4}\,dy = \int \frac{1}{x}\,dx$$

$$\int \frac{y}{y^2+4}\,dy + \int \frac{4}{4+y^2}\,dy = \int \frac{1}{x}\,dx$$

let $y^2+4 = u$

$2y\,dy = du$

$y\,dy = \frac{du}{2}$

$$\int \frac{1}{u}\,\frac{du}{2} + 4\int \frac{1}{4+y^2}\,dy = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}\ln(y^2+4) + 2\tan^{-1}\frac{y}{2} = \ln x + c$$

$x = 4 \quad y = 2\sqrt{3}$

$$\frac{1}{2}\ln(16) + 2\frac{\pi}{3} = \ln 4 + c$$

$$\cancel{\ln 4} + \frac{2\pi}{3} = \cancel{\ln 4} + c \qquad \therefore c = \frac{2\pi}{3}$$

$$\frac{1}{2}\ln(y^2+4) + 2\tan^{-1}\frac{y}{2} = \ln x + \frac{2\pi}{3}$$

$y = 2$

$$\ln\sqrt{8} + \frac{\pi}{2} = \ln x + \frac{2\pi}{3}$$

$$\ln\frac{\sqrt{8}}{x} = \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$$

$$\frac{\sqrt{8}}{x} = e^{\frac{\pi}{6}}$$

$$x = \sqrt{8}\, e^{-\frac{\pi}{6}}$$

Problem : 09709/33/M/J/23/Q9

The lines l and m have equations

$$l: \quad \mathbf{r} = a\mathbf{i} + 3\mathbf{j} + b\mathbf{k} + \lambda(c\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}),$$
$$m: \quad \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}).$$

Relative to the origin O, the position vector of the point P is $4\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$.

(a) Given that l is perpendicular to m and that P lies on l, find the values of the constants a, b and c. [4]

(b) The perpendicular from P meets line m at Q. The point R lies on PQ extended, with $PQ : QR = 2 : 3$.

Find the position vector of R. [6]

Sol (a) $l \perp m$ hence scalar product of line l and m will be zero.

$c \times 2 + 6 + 4 = 0 \quad \therefore c = -5$

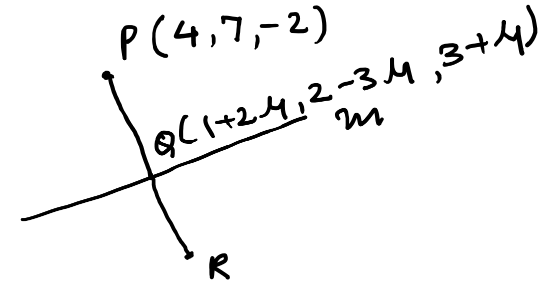
$4i + 7j - 2k = (a - 5\lambda)i + (3 - 2\lambda)j + (b + 4\lambda)k$

$4 = a - 5\lambda \qquad 7 = 3 - 2\lambda \qquad -2 = b + 4\lambda$

$4 = a + 10 \qquad 4 = -2\lambda \qquad -2 = b - 8$

$a = -6 \qquad \lambda = -2 \qquad b = 6$

(b)



$P(4, 7, -2) \quad Q(1 + 2\mu, 2 - 3\mu, 3 + \mu)$

$\vec{PQ} = \vec{OQ} - \vec{OP}$

$= \begin{pmatrix} -3 + 2\mu \\ -5 - 3\mu \\ 5 + \mu \end{pmatrix}$

$\vec{PQ} \perp$ direction vector of line m.

$(-3 + 2\mu)2 + (-5 - 3\mu) \times -3 + (5 + \mu) \times 1 = 0$

$-6 + 4\mu + 15 + 9\mu + 5 + \mu = 0$

$14\mu = -14$

$\mu = -1$

$$\vec{PQ} = \begin{pmatrix} -5 \\ -2 \\ 4 \end{pmatrix}$$

$$\frac{\vec{PQ}}{\vec{QR}} = \frac{2}{3} \quad \therefore \vec{QR} = \frac{3}{2}\vec{PQ}$$

$$\vec{PR} = \vec{PQ} + \vec{QR}$$

$$= \vec{PQ} + \frac{3}{2}PQ = \frac{5}{2}\vec{PQ}$$

$$\vec{OR} - \vec{OP} = \frac{5}{2}\vec{PQ}$$

$$\vec{OR} = \vec{OP} + \frac{5}{2}\vec{PQ}$$

$$= 4i + 7j - 2k + \frac{5}{2}(-5i - 2j + 4k)$$

$$= -\frac{17}{2}i + 2j + 8k$$

Problem : 09709/33/M/J/23/Q10

Let $f(x) = \dfrac{21 - 8x - 2x^2}{(1 + 2x)(3 - x)^2}$.

(a) Express f(x) in partial fractions. [5]

(b) Hence obtain the expansion of f(x) in ascending powers of x, up to and including the term in x^2. [5]

Sol (a)

$$\frac{21-8x-2x^2}{(1+2x)(3-x)^2} = \frac{A}{(1+2x)} + \frac{B}{(3-x)} + \frac{C}{(3-x)^2}$$

$$21-8x-2x^2 = A(3-x)^2 + B(1+2x)(3-x) + C(1+2x)$$

Substitute $x=3$

$$21 - 8\times3 - 2(3)^2 = C(1+2\times3)$$

$$-21 = C(7)$$

$$C = -3$$

Substitute $x = -\frac{1}{2}$

$$21 - 8\times -\tfrac{1}{2} - 2\left(-\tfrac{1}{2}\right)^2 = A\left(3+\tfrac{1}{2}\right)^2$$

$$\frac{49}{2} = A\left(\frac{49}{4}\right)$$

$$A = 2$$

by constant term

$$21 = 9A + 3B + C$$

$$21 = 18 + 3B - 3$$

$$6 = 3B \quad B = 2$$

$$\frac{2}{(1+2x)} + \frac{2}{(3-x)} - \frac{3}{(3-x)^2}$$

(b) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2$

$2(1+2x)^{-1} = 2\left(1 - 2x + \frac{-1(-1-1)}{2!}(2x)^2\right)$

$= 2(1 - 2x + 4x^2)$

$= 2 - 4x + 8x^2$ ✓

$2(3-x)^{-1} = 2\,3^{-1}(1 - \frac{x}{3})^{-1}$

$= \frac{2}{3}\left(1 + \frac{x}{3} + \frac{-1(-1-1)}{2!}\left(-\frac{x}{3}\right)^2\right)$

$= \frac{2}{3}\left(1 + \frac{x}{3} + \frac{x^2}{9}\right)$

$= \frac{2}{3} + \frac{2x}{9} + \frac{2x^2}{27}$ ✓

$-3(3-x)^{-2} = -3 \cdot 3^{-2}(1 - \frac{x}{3})^{-2}$

$= -\frac{1}{3}\left(1 + \frac{2x}{3} + \frac{-2(-2-1)}{2!}\left(-\frac{x}{3}\right)^2\right)$

$= -\frac{1}{3}\left(1 + \frac{2}{3}x + \frac{3x^2}{9}\right)$

$= -\frac{1}{3} - \frac{2}{9}x - \frac{x^2}{9}$ ✓

$= \frac{7}{3} - 4x + \frac{215}{27}x^2$

Problem : 09709/33/M/J/23/Q11

The complex number z is defined by $z = \dfrac{5a - 2\mathrm{i}}{3 + a\mathrm{i}}$, where a is an integer. It is given that $\arg z = -\frac{1}{4}\pi$.

(a) Find the value of a and hence express z in the form $x + \mathrm{i}y$, where x and y are real. [6]

(b) Express z^3 in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the simplified exact values of r and θ. [3]

Sol (a)

$$\frac{5a-2i}{3+ai} \times \frac{3-ai}{3-ai}$$

$$\frac{15a - 5a^2i - 6i - 3a}{9+a^2}$$

$$\frac{13a - i(5a^2+6)}{9+a^2}$$

$$\mathrm{Arg}(z) = -\frac{\pi}{4}$$

$$-\frac{5a^2+6}{13a} = \tan -\frac{\pi}{4} = -1$$

$$5a^2+6 = 13a$$

$$5a^2 - 13a + 6 = 0$$

$$\boxed{a = 2} \quad a = \frac{3}{5}$$

(b)

$$z = \frac{13 \times 2 - i(5 \times 2^2 + 6)}{9+4}$$

$$= \frac{26 - 26i}{13}$$

$$= 2 - 2i$$

$$r = |z| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$\theta = \tan^{-1} -1 = -\frac{\pi}{4}$$

$$z = 2\sqrt{2}\, e^{-i\frac{\pi}{4}}$$

$$z^3 = 2^3(\sqrt{2})^3 e^{-\frac{3\pi}{4}i}$$

$$= 16\sqrt{2}\, e^{-\frac{3}{4}\pi i}$$