Problem : 09709/31/M/J/24/Q1

Expand $(3+x)(1-2x)^{\frac{1}{2}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

Sol $(3+x)(1-2x)^{1/2}$

$(3+x)\left(1+\frac{1}{2}(-2x)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(-2x)^2 \ldots\right)$

$(3+x)\left(1-x+-\frac{1}{8}\times 4x^2 --\right)$

$(3+x)\left(1-x-\frac{1}{2}x^2 \ldots\right)$

$3-3x-\frac{3}{2}x^2+x-x^2$

$3-2x-\frac{5}{2}x^2$

Problem : 09709/31/M/J/24/Q2

Solve the equation $\ln(x-5) = 7 - \ln x$. Give your answer correct to 2 decimal places. [4]

Sol

$$\ln(x-5) + \ln x = 7$$

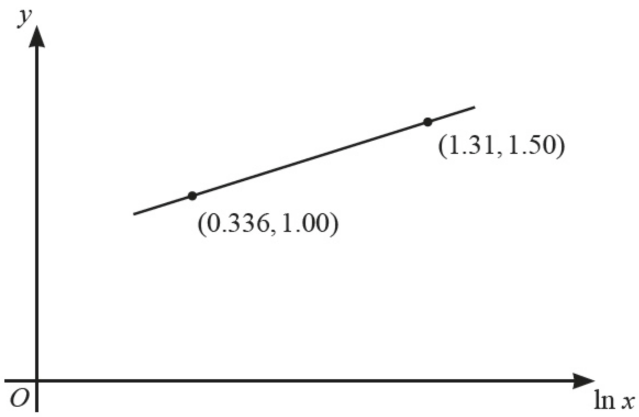
$$\ln x(x-5) = 7$$

$$x^2 - 5x = e^7 = 1096.6$$

$$x^2 - 5x - 1096.6 = 0$$

$$\underline{x = 35.7} \quad x = -30.71 \; \times$$

Problem : 09709/31/M/J/24/Q3



The variables x and y satisfy the equation $a^y = bx$, where a and b are constants. The graph of y against $\ln x$ is a straight line passing through the points $(0.336, 1.00)$ and $(1.31, 1.50)$, as shown in the diagram.

Find the values of a and b. Give each value correct to the nearest integer. [4]

Sol

$$a^y = bx$$

$$y \ln a = \ln b + \ln x$$

$$1 . \ln a = \cancel{\ln b} + 0.336 \checkmark$$

$$1.5 \ln a = \cancel{\ln b} + 1.31 \checkmark$$

$$-0.5 \ln a = -0.974$$

$$\ln a = 1.948$$

$$a = e^{1.948} = 7.01 \approx \boxed{7}$$

$$1.948 - 0.336 = \ln b$$

$$\ln b = 1.612$$

$$b = e^{1.612} = 5.01$$

$$\boxed{b = 5}$$

Problem : 09709/31/M/J/24/Q4

The complex number u is given by $u = -1 - \mathrm{i}\sqrt{3}$.

(a) Express u in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ. [2]

The complex number v is given by $v = 5\left(\cos\frac{1}{6}\pi + \mathrm{i}\sin\frac{1}{6}\pi\right)$. $= 5e^{i\frac{\pi}{6}}$

(b) Express the complex number $\frac{v}{u}$ in the form $re^{\mathrm{i}\theta}$ where $r > 0$ and $-\pi < \theta \leqslant \pi$. [2]

Sol (a) $u = -1 - i\sqrt{3}$

$$r = \sqrt{(-1)^2 + (-\sqrt{3})^2}$$

$$= \sqrt{1+3} = 2$$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right)$$

$$\theta = \frac{1}{3}\pi$$

$$\theta = -\pi + \frac{1}{3}\pi = -\frac{2}{3}\pi$$

(b)
$$\frac{v}{u} = \frac{5e^{i\frac{\pi}{6}}}{2e^{i(-\frac{2}{3}\pi)}}$$

$$= \frac{5}{2}e^{i\left[\frac{\pi}{6} + \frac{2}{3}\pi\right]}$$

$$= \frac{5}{2}e^{i\left(\frac{5}{6}\pi\right)}$$

Problem : 09709/31/M/J/24/Q5

The equation of a curve is $y = \dfrac{e^{\sin x}}{\cos^2 x}$ for $0 \leqslant x \leqslant 2\pi$.

Find $\dfrac{dy}{dx}$ and hence find the x-coordinates of the stationary points of the curve. [7]

Sol By quotient rule.

$$\frac{dy}{dx} = \frac{\cos^2 x . e^{\sin x} . \cos x - e^{\sin x} . 2\cos x \times -\sin x}{(\cos^2 x)^2}$$

$$\frac{dy}{dx} = \frac{e^{\sin x}(\cos^{\cancel{3}\,2} x + 2\sin x . \cancel{\cos x})}{(\cos x)^{\cancel{4}\,3}}$$

$$\frac{dy}{dx} = \frac{e^{\sin x}(\cos^2 x + 2\sin x)}{(\cos x)^3}$$

As stationary point coordinates $\frac{dy}{dx} = 0$

$\cos^2 x + 2\sin x = 0$

$1 - \sin^2 x + 2\sin x = 0$

$\sin^2 x - 2\sin x - 1 = 0$

let $\sin x = t$

$t^2 - 2t - 1 = 0$

$t = 2.414 \quad t = -0.414$

$\sin x = 2.414$ is out range so solution not possible

$\sin x = -0.414$

Consider $\sin x = 0.414$

$x = \sin^{-1}(0.414)$

$$x = 0.4268$$

$$x = \pi + 0.4268 = 3.57$$

$$x = 2\pi - 0.4268 = 5.86$$

Problem : 09709/31/M/J/24/Q6

(a) By sketching a suitable pair of graphs, show that the equation $\operatorname{cosec}\frac{1}{2}x = e^x - 3$ has exactly one root, denoted by α, in the interval $0 < x < \pi$. [2]

(b) Verify by calculation that α lies between 1 and 2. [2]

(c) Show that if a sequence of values in the interval $0 < x < \pi$ given by the iterative formula

$$x_{n+1} = \ln(\operatorname{cosec}\tfrac{1}{2}x_n + 3)$$

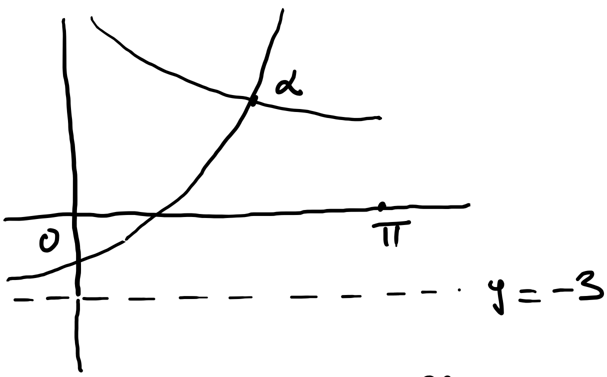
converges, then it converges to α. [1]

(d) Use this iterative formula with an initial value of 1.4 to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(e) State the minimum number of calculated iterations needed with this initial value to determine α correct to 2 decimal places. [1]

Sol (a) $y = e^x - 3$

$y = \dfrac{1}{\sin\frac{1}{2}x}$



(b) $\operatorname{cosec}\frac{1}{2}x - e^x + 3$

$x = 1$ $\operatorname{cosec}\frac{1}{2}\times 1 - e^1 + 3$

$2.3675 > 0$

$x = 2$ $\operatorname{cosec}\frac{1}{2}\times 2 - e^2 + 3$

$-3.2006 < 0$

hence root lies between 1 and 2.

(c) $\operatorname{cosec}\frac{1}{2}x = e^x - 3$

$e^x = \operatorname{cosec}\frac{1}{2}x + 3$

$$x = \ln\left(\operatorname{cosec}\tfrac{1}{2}x + 3\right)$$

$$x_{n+1} = \ln\left(\operatorname{cosec}\tfrac{1}{2}x_n + 3\right)$$

(d)

$$x_0 = 1.4$$

$$x_1 = \ln\left(\operatorname{cosec}\tfrac{1}{2}(1.4) + 3\right) = 1.5156$$

$$x_2 = \ln\left(\operatorname{cosec}\tfrac{1}{2}(1.5156) + 3\right) = 1.4940$$

$$x_3 = \ln\left(\operatorname{cosec}\tfrac{1}{2}(1.4940) + 3\right) = 1.4977$$

$$x_4 = \ln\left(\operatorname{cosec}\tfrac{1}{2}(1.4977) + 3\right) = 1.4971$$

$$x = 1.50$$

(e) 4

Problem : 09709/31/M/J/24/Q7

(a) On a single Argand diagram sketch the loci given by the equations $|z-3+2i|=2$ and $|w-3+2i|=|w+3-4i|$ where z and w are complex numbers. [4]

(b) Hence find the least value of $|z-w|$ for points on these loci. Give your answer in an exact form. [2]

Sol (a) $(x-3)^2+(y+2)^2=2^2$

which is equation of circle with centre $(3,-2)$ and radius 2

$(x-3)^2+(y+2)^2=(x+3)^2+(y-4)^2$

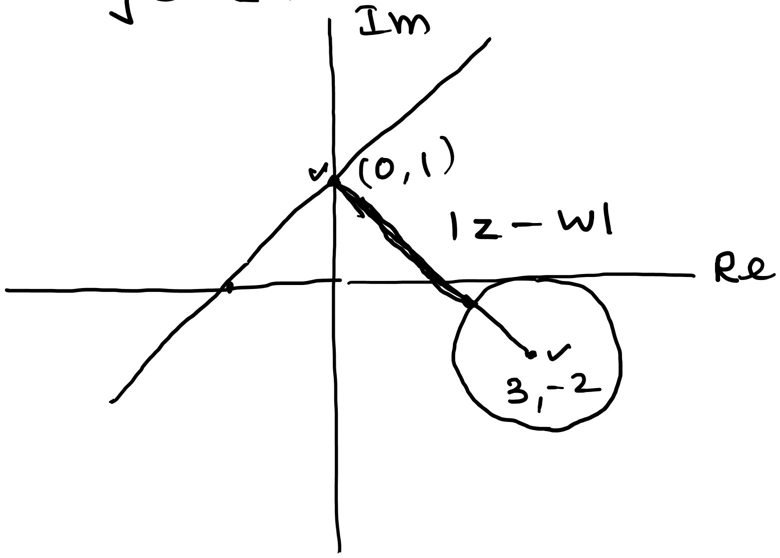
$\cancel{x^2}-6x+9+\cancel{y^2}+4y+4=\cancel{x^2}+6x+9+\cancel{y^2}-8y+16$

$-6x+4y+13=6x-8y+25$

$12x-12y+12=0$

$x-y=-1$

$y=x+1$



(b) least value of $|z-w|$

$=\sqrt{(3-0)^2+(-2-1)^2}$

$=\sqrt{9+9}=\sqrt{18}$

least value of $|z-w|$

$= \sqrt{18} - 2$

Problem : 09709/31/M/J/24/Q8

Use the substitution $u = 1 - \sin x$ to find the exact value of

$$\int_{\pi}^{\frac{3}{2}\pi} \frac{\sin 2x}{\sqrt{1-\sin x}}\, dx.$$

Give your answer in the form $a + b\sqrt{2}$ where a and b are rational numbers to be determined. [7]

Sol

$u = 1 - \sin x$

$\sin x = 1 - u$

$du = -\cos x \, dx$

$x = \pi \quad u = 1 - \sin(\pi) = 1 - 0 = 1$

$x = \frac{3}{2}\pi \quad u = 1 - \sin\left(\frac{3}{2}\pi\right) = 1 - (-1) = 2$

$$\int_1^2 \frac{2 \sin x \cos x \, dx}{\sqrt{u}}$$

$$\int_1^2 \frac{2(1-u)}{\sqrt{u}} (-du)$$

$$-2 \int_1^2 (u^{-1/2} - u^{1/2})\, du$$

$$-2 \left[\frac{u^{1/2}}{1/2} - \frac{u^{3/2}}{3/2} \right]_1^2$$

$$-4 \left[u^{1/2} - \frac{u^{3/2}}{3} \right]_1^2$$

$$-4 \left[\sqrt{2} - \frac{2^{3/2}}{3} - 1 + \frac{1}{3} \right]$$

$$-4 \left[\sqrt{2} - \frac{2\sqrt{2}}{3} - \frac{2}{3} \right]$$

$$-4\left[\frac{\sqrt{2}}{3}-\frac{2}{3}\right]$$

$$\frac{8}{3}-\frac{4}{3}\sqrt{2}$$

Problem : 09709/31/M/J/24/Q9

The equations of two straight lines l_1 and l_2 are

$$l_1: \quad \mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + a\mathbf{k}) \quad \text{and} \quad l_2: \quad \mathbf{r} = -\mathbf{i} - \mathbf{j} - \mathbf{k} + \mu(3\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}),$$

where a is a constant.

The lines l_1 and l_2 are perpendicular.

(a) Show that $a = 4$. [1]

The lines l_1 and l_2 also intersect.

(b) Find the position vector of the point of intersection. [4]

The point A has position vector $-5\mathbf{i} + \mathbf{j} - 9\mathbf{k}$.

(c) Show that A lies on l_1. [2]

The point B is the image of A after a reflection in the line l_2.

(d) Find the position vector of B. [2]

Sol (a) dot product of direction vector of lines l_1 and l_2 will be zero.

$(2i - j + ak) \cdot (3i - 2j - 2k) = 0$

$2 \times 3 + (-1) \times (-2) + a \times -2 = 0$

$6 + 2 - 2a = 0$

$2a = 8$

$a = 4$

(b) l_1 $\left.\begin{aligned} x &= 1 + 2\lambda \\ y &= -2 - \lambda \\ z &= 3 + 4\lambda \end{aligned}\right\}$

l_2 $\begin{aligned} x &= -1 + 3\mu \\ y &= -1 - 2\mu \\ z &= -1 - 2\mu \end{aligned}$

$1 + 2\lambda = -1 + 3\mu$

$2\lambda - 3\mu = -2$ — (i)

$-2 - \lambda = -1 - 2\mu$

$-\lambda + 2\mu = 1$ — (ii) ✓

$\mu = 0 \quad \lambda = -1$

$l_2 \quad \mu = 0$

$-i - j - k$

$l_1 \quad \lambda = -1$

$-i - j - k$

hence position vector would be

$-i - j - k$

(c) $x = 1 + 2\lambda$

$y = -2 - \lambda$

$z = 3 + 4\lambda$

$-5 = 1 + 2\lambda$

$\lambda = -3$

$1 = -2 - \lambda$

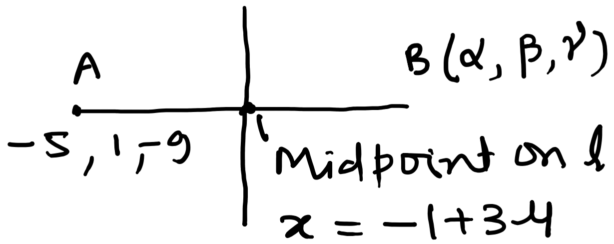
$\lambda = -3$

$-9 = 3 + 4\lambda$

$\lambda = -3$

Since value of $\lambda = -3$ in all three components hence point A lies on line l_1

(d)



$\frac{-5 + \alpha}{2} = -1$

$\alpha = -2 + 5 = 3$

$\frac{1 + \beta}{2} = -1$

$\beta = -2 - 1 = -3$

$\frac{-9 + \gamma}{2} = -1$

$\gamma = -2 + 9 = 7$

$3i - 3j + 7k$

Midpoint on l_2

$x = -1 + 3\mu$

$y = -1 - 2\mu$

$z = -1 - 2\mu$

$\mu = 0$

$(-1, -1, -1)$

Problem : 09709/31/M/J/24/Q10

(a) Given that $2x = \tan y$, show that $\frac{dy}{dx} = \frac{2}{1+4x^2}$. [3]

(b) Hence find the exact value of $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} x\tan^{-1}(2x)\,dx$. [7]

<u>Sol</u> (a) $2x = \tan y$

$y = \tan^{-1} 2x$

$\frac{dy}{dx} = \frac{1}{1+(2x)^2} \times 2$

$= \frac{2}{1+4x^2}$

(b) $\int_{1/2}^{\sqrt{3}/2} x \tan^{-1}(2x)\,dx$

$u = \tan^{-1} 2x \quad v' = x$

$du = \frac{2}{1+4x^2} \quad v = \frac{x^2}{2}$

using integration By parts method

$\left[\tan^{-1} 2x \cdot \frac{x^2}{2}\right]_{1/2}^{\sqrt{3}/2} - \int_{1/2}^{\sqrt{3}/2} \frac{\cancel{2}}{1+4x^2} \cdot \frac{x^2}{\cancel{2}}\,dx$

$\left[\frac{x^2}{2} \tan^{-1} 2x\right]_{1/2}^{\sqrt{3}/2} - \int_{1/2}^{\sqrt{3}/2} \frac{x^2}{1+4x^2}\,dx$

$1+4x^2 \,)\, \cancel{x^2} \,(\, \frac{1}{4}$

$\quad \cancel{x^2} + \frac{1}{4}$

$\quad - \quad -\frac{1}{4}$

$\frac{1}{4} - \frac{\frac{1}{4}}{1+4x^2}$

$\left[\frac{x^2}{2} \tan^{-1} 2x\right]_{1/2}^{\sqrt{3}/2} - \int_{1/2}^{\sqrt{3}/2} \frac{1}{4} - \frac{1/4}{1+4x^2}\,dx$

$\left[\frac{x^2}{2} \tan^{-1} 2x\right]_{1/2}^{\sqrt{3}/2} - \left[\frac{1}{4}x - \frac{1}{8}\tan^{-1} 2x\right]_{1/2}^{\sqrt{3}/2}$

$$\left[\frac{3/4}{2}\tan^{-1}\cancel{2}\times\frac{\sqrt{3}}{\cancel{2}} - \frac{1/4}{2}\tan^{-1}\cancel{2}\times\frac{1}{\cancel{2}}\right] - \left[\frac{1}{4}\frac{\sqrt{3}}{2} - \frac{1}{8}\tan^{-1}\cancel{2}\times\frac{\sqrt{3}}{\cancel{2}}\right.$$

$$\left. - \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{8}\tan^{-1}2\times\frac{1}{2}\right]$$

$$\frac{\cancel{3}}{8}\cdot\frac{\pi}{\cancel{3}} - \frac{1}{8}\frac{\pi}{4} - \frac{\sqrt{3}}{8} + \frac{1}{8}\frac{\pi}{3} + \frac{1}{8} - \frac{\pi}{32}$$

$$\frac{\pi}{8} - \frac{\pi}{32} - \frac{\sqrt{3}}{8} + \frac{\pi}{24} + \frac{1}{8} - \frac{\pi}{32}$$

$$\frac{5}{48}\pi - \frac{\sqrt{3}}{8} + \frac{1}{8}$$

Problem : 09709/31/M/J/24/Q11

In a field there are 300 plants of a certain species, all of which can be infected by a particular disease. At time t after the first plant is infected there are x infected plants. The rate of change of x is proportional to the product of the number of plants infected and the number of plants that are **not** yet infected. The variables x and t are treated as continuous, and it is given that $\frac{dx}{dt} = 0.2$ and $x = 1$ when $t = 0$.

(a) Show that x and t satisfy the differential equation

$$1495\frac{dx}{dt} = x(300 - x).$$ [2]

(b) Using partial fractions, solve the differential equation and obtain an expression for t in terms of a single logarithm involving x. [9]

Sol (a) $\frac{dx}{dt} \propto x(300-x)$

$$\frac{dx}{dt} = k\,x(300-x)$$

$$0.2 = k \times 1 \times (299$$

$$k = \frac{0.2}{299} = \frac{1}{1495}$$

$$\frac{dx}{dt} = \frac{1}{1495}x(300-x)$$

$$1495\frac{dx}{dt} = x(300-x)$$

(b) By variable seperable method

$$\int \frac{dx}{x(300-x)} = \int \frac{1}{1495}dt$$

LHS $\frac{1}{x(300-x)} = \frac{A}{x} + \frac{B}{300-x}$

$$1 = A(300-x) + Bx$$

$x = 0 \quad A = \frac{1}{300}$

$x = 300 \quad B = \frac{1}{300}$

$$= \frac{1}{300}\left[\frac{1}{x} + \frac{1}{300-x}\right]$$

$$\frac{1}{300}\int\left(\frac{1}{x}+\frac{1}{300-x}\right)dx=\int\frac{1}{1495}\,dt$$

$$\frac{1}{300}\left[\ln x-\ln(300-x)\right]=\frac{1}{1495}t+C$$

$t=0 \quad x=1$

$$\frac{1}{300}\left[\ln 1-\ln 299\right]=C$$

$$C=-\frac{1}{300}\ln 299$$

$$\frac{1}{300}\left[\ln\frac{x}{300-x}\right]=\frac{1}{1495}t-\frac{1}{300}\ln 299$$

$$\ln\frac{x}{300-x}+\ln 299=\frac{300}{1495}t$$

$$\ln\frac{299x}{300-x}=\frac{60}{299}t$$

$$t=\frac{299}{60}\ln\frac{299x}{300-x}$$