Problem : 09709/11/M/J/24/Q1

(a) Express $3y^2 - 12y - 15$ in the form $3(y+a)^2 + b$, where a and b are constants. [2]

(b) Hence find the exact solutions of the equation $3x^4 - 12x^2 - 15 = 0$. [3]

Sol (a) $3y^2 - 12y - 15$

$3(y^2 - 4y) - 15$

$3(y^2 - 4y + 4 - 4) - 15$

$3(y-2)^2 - 27$

$a = -2 \quad b = -27$

(b) $3x^4 - 12x^2 - 15 = 0$

let $x^2 = y$

$3y^2 - 12y - 15 = 0$

$3(y-2)^2 - 27 = 0$

$(y-2)^2 = \frac{27}{3} = 9$

$y - 2 = \pm 3$

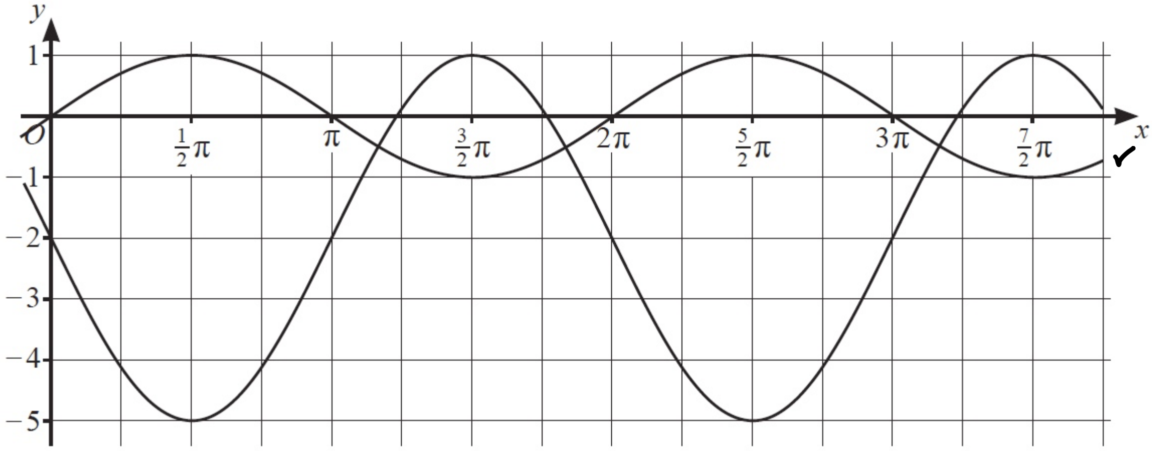
$y = 2 + 3 \qquad y = 2 - 3 = -1$

$y = 5 \qquad x^2 = -1$ ✗

$x^2 = 5$

$x = \pm\sqrt{5}$

Problem : 09709/11/M/J/24/Q2



The diagram shows two curves. One curve has equation $y = \sin x$ and the other curve has equation $y = f(x)$.

(a) In order to transform the curve $y = \sin x$ to the curve $y = f(x)$, the curve $y = \sin x$ is first reflected in the x-axis.

Describe fully a sequence of two further transformations which are required. [4]

(b) Find $f(x)$ in terms of $\sin x$. [2]

Sol (a) stretch parallel to y-axis by scale factor $\frac{\text{Max Value} - \text{Min Value}}{2}$

$= \frac{1-(-5)}{2} = 3$

vertical translation by

$= \frac{\text{Max Value} + \text{Min Value}}{2}$

$= \frac{1-5}{2} = -2$

$\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

(b) $f(x) = -3\sin x - 2$

Problem : 09709/11/M/J/24/Q3

The coefficient of x^3 in the expansion of $(3+ax)^6$ is 160.

(a) Find the value of the constant a. [2]

(b) Hence find the coefficient of x^3 in the expansion of $(3+ax)^6(1-2x)$. [3]

Sol (a) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$

$(3+ax)^6 = 3^6\left(1+\frac{ax}{3}\right)^6$

$160 = \frac{3^6 \times 6 \times 5 \times 4}{3!}\,\frac{a^3}{27}$

$a^3 = \frac{160 \times 6 \times 27}{3^6 \times 6 \times 5 \times 4} = \frac{8}{27}$

$a = \frac{2}{3}$

(b) $(1-2x)\left(3+\frac{2}{3}x\right)^6$ $\quad x^3$

$x^3 - 160$

$x^2 - ?$

$\left(3+\frac{2}{3}x\right)^6 \quad x^2$

${}^6C_r\, 3^{6-r}\left(\frac{2}{3}x\right)^r \quad x^2$

$r=2$

${}^6C_2\, 3^4 \left(\frac{2}{3}\right)^2 x^2$

$540x^2$

$(1-2x)(540x^2 + 160x^3)$

$160 - 1080 = -920$

Problem : 09709/11/M/J/24/Q4

The equation of a curve is $y = f(x)$, where $f(x) = (2x-1)\sqrt{3x-2} - 2$. The following points lie on the curve. Non-exact values have been given correct to 5 decimal places.

$A(2, 4)$, $B(2.0001, k)$, $C(2.001, 4.00625)$, $D(2.01, 4.06261)$, $E(2.1, 4.63566)$, $F(3, 11.22876)$

(a) Find the value of k. Give your answer correct to 5 decimal places. [1]

The table shows the gradients of the chords AB, AC, AD and AF.

Chord	AB	AC	AD	AE	AF
Gradient of chord	6.2501	6.2511	6.2608	6.3566	7.2288

(b) Find the gradient of the chord AE. Give your answer correct to 4 decimal places. [1]

(c) Deduce the value of $f'(2)$ using the values in the table. [1]

Sol (a) $f(2.0001) = k = (2 \times 2.0001 - 1)\sqrt{3 \times 2.0001 - 2} - 2$

$k = 4.00062501$

$= 4.00063$

(b) gradient of AE

$$= \frac{4.63566 - 4}{2.1 - 2} = 6.3566$$

(c) 6.25

Problem : 09709/11/M/J/24/Q5

(a) Prove the identity $\dfrac{\sin^2 x - \cos x - 1}{1 + \cos x} \equiv -\cos x$. [3]

(b) Hence solve the equation $\dfrac{\sin^2 x - \cos x - 1}{2 + 2\cos x} = \dfrac{1}{4}$ for $0° \leqslant x \leqslant 360°$. [3]

Sol (a) LHS

$$\frac{\sin^2 x - \cos x - 1}{1 + \cos x}$$

$$\frac{\cancel{1} - \cos^2 x - \cos x - \cancel{1}}{1 + \cos x}$$

$$\frac{-\cos x\,(\cancel{\cos x + 1})}{\cancel{1 + \cos x}}$$

$$-\cos x \quad (\text{RHS})$$

(b) $$\frac{\sin^2 x - \cos x - 1}{2 + 2\cos x} = \frac{1}{4}$$

$$\frac{1}{2}\left[\frac{\sin^2 x - \cos x - 1}{1 + \cos x}\right] = \frac{1}{4}$$

$$\frac{1}{2} \times -\cos x = \frac{1}{4}$$

$$\cos x = -\frac{1}{2}$$

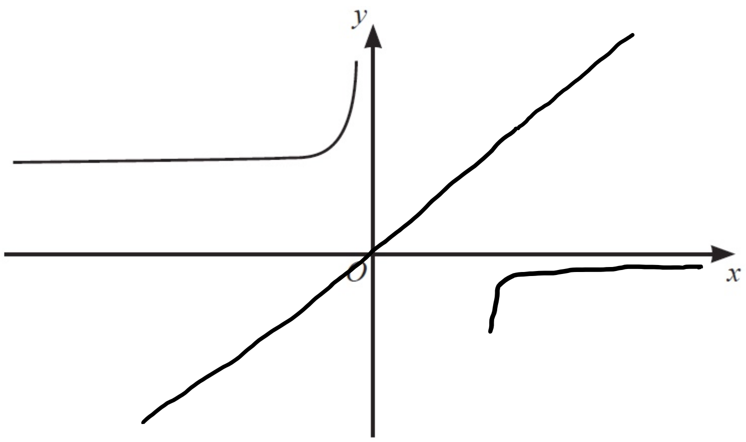
consider $\cos x = \frac{1}{2}$

$x = 60°$

$x = 180 - 60° = 120°$

$x = 180 + 60° = 240°$

Problem : 09709/11/M/J/24/Q6



The function f is defined by $f(x) = \frac{2}{x^2} + 4$ for $x < 0$. The diagram shows the graph of $y = f(x)$.

(a) On this diagram, sketch the graph of $y = f^{-1}(x)$. Show any relevant mirror line. [2]

(b) Find an expression for $f^{-1}(x)$. [3]

(c) Solve the equation $f(x) = 4.5$. [1]

(d) Explain why the equation $f^{-1}(x) = f(x)$ has no solution. [1]

Sol (a) on graph.

(b) $f(x) = \frac{2}{x^2} + 4$

let $y = \frac{2}{x^2} + 4$

$y - 4 = \frac{2}{x^2}$

$x^2 = \frac{2}{y-4}$

$x = \pm\sqrt{\frac{2}{y-4}}$

$f^{-1}(x) = -\sqrt{\frac{2}{x-4}}$

(c) $f(x) = 4\cdot5$

$\frac{2}{x^2} + 4 = 4\cdot5$

$$\frac{2}{x^2} = 4{\cdot}5 - 4 = {\cdot}5$$

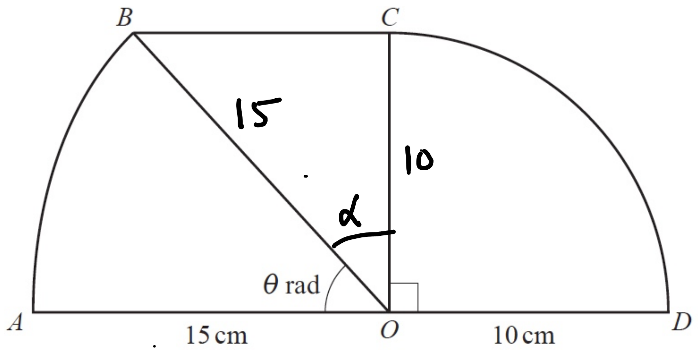
$$x^2 = \frac{2}{{\cdot}5} = 4$$

$$x = \pm 2$$

$$x = -2$$

(d) Since $f^{-1}(x)$ is negative and $f(x)$ is positive, hence No solution in $f^{-1}(x) = f(x)$

Problem : 09709/11/M/J/24/Q7



In the diagram, AOD and BC are two parallel straight lines. Arc AB is part of a circle with centre O and radius 15 cm. Angle $BOA = \theta$ radians. Arc CD is part of a circle with centre O and radius 10 cm. Angle $COD = \frac{1}{2}\pi$ radians.

(a) Show that $\theta = 0.7297$, correct to 4 decimal places. [1]

(b) Find the perimeter and the area of the shape $ABCD$. Give your answers correct to 3 significant figures. [7]

Sol (a) $\theta = \frac{\pi}{2} - \alpha$

$= \frac{\pi}{2} - \cos^{-1}\frac{10}{15} = 0.7297$

(b) Perimeter

$= OA + \text{Arc}AB + BC + CD + DO$

$= 15 + 15 \times 0.7297 + \sqrt{15^2 - 10^2} + 10 \times \frac{\pi}{2} + 10$

$= 62.8$ cm

Area

$= \frac{1}{2} 15^2 \times 0.7297 + \frac{1}{2} \times 10 \times \sqrt{15^2 - 10^2} + \frac{1}{2} \times 10^2 \times \frac{\pi}{4}$

$= 217$

Problem : 09709/11/M/J/24/Q8

(a) The first three terms of an arithmetic progression are 25, $4p-1$ and $13-p$, where p is a constant.

Find the value of the tenth term of the progression. [4]

(b) The first three terms of a geometric progression are 25, $4q-1$ and $13-q$, where q is a positive constant.

Find the sum to infinity of the progression. [4]

Sol (a) $13-p-4p+1 = 4p-1-25$

$14-5p = 4p-26$

$14+26 = 9p$

$40 = 9p$

$p = \frac{40}{9}$

$d = 4\times\frac{40}{9} - 26 = -\frac{74}{9}$

10th term $= 25 + (10-1)\times -\frac{74}{9}$

$= 25 - \frac{\cancel{9}\times 74}{\cancel{9}}$

$= -49$

(b) $\frac{13-q}{4q-1} = \frac{4q-1}{25}$

$(4q-1)^2 = 25(13-q)$

$16q^2 - 8q + 1 = 325 - 25q$

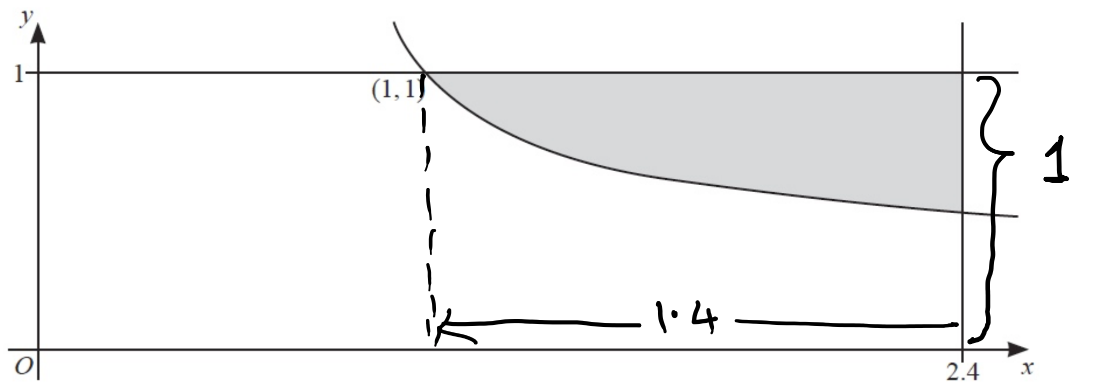
$16q^2 + 17q - 324 = 0$

$\underline{\underline{q=4}} \quad q = -\frac{81}{16}$

Common ratio $r = \frac{13-4}{4\times4-1} = \frac{9}{15}$

$$S_\infty = \frac{25}{1-\frac{9}{15}} = \frac{125}{2}$$

Problem : 09709/11/M/J/24/Q9



The diagram shows part of the curve with equation $y = \frac{1}{(5x-4)^{\frac{1}{3}}}$ and the lines $x = 2.4$ and $y = 1$. The curve intersects the line $y = 1$ at the point $(1, 1)$.

Find the exact volume of the solid generated when the shaded region is rotated through 360° about the x-axis. [6]

Sol Volume of cylinder

$$= \pi 1^2 \times 1\cdot4$$

$$= \frac{7}{5}\pi$$

Volume under the curve

$$= \pi \int_1^{2\cdot4} \frac{1}{(5x-4)^{2/3}} dx$$

$$= \pi \left[\frac{(5x-4)^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} \times \frac{1}{5} \right]_1^{2\cdot4}$$

$$= \frac{3}{5}\pi \left[(5x-4)^{1/3} \right]_1^{2\cdot4}$$

$$= \frac{3}{5}\pi \left[(5\times2\cdot4-4)^{1/3} - (5\times1-4)^{1/3} \right]$$

$$= 3/5\pi$$

Volume of shaded region $= \frac{7}{5}\pi - \frac{3}{5}\pi$

$$= 4/5\pi$$

Problem : 09709/11/M/J/24/Q10

The equation of a circle is $(x-3)^2+y^2=18$. The line with equation $y=mx+c$ passes through the point $(0,-9)$ and is a tangent to the circle.

Find the two possible values of m and, for each value of m, find the coordinates of the point at which the tangent touches the circle. [8]

Sol $(x-3)^2+y^2=18$

$y=mx+c$

Since line passes through the point $(0,-9)$ therefore

$-9=m(0)+c \quad \therefore c=-9$

hence $y=mx-9$ ✓

$(x-3)^2+(mx-9)^2=18$

$x^2-6x+9+m^2x^2-18mx+81-18=0$

$(1+m^2)x^2-(6+18m)x+72=0$ ✓

Since line is tangent to the circle

$b^2-4ac=0$

$[-(6+18m)]^2-4(1+m^2)\times 72=0$

$(6+18m)^2-288(1+m^2)=0$

$36+324m^2+216m-288-288m^2=0$

$36m^2+216m-252=0$

$m=1 \quad m=-7$

$m=1 \quad 2x^2-24x+72=0$

$x=6 \quad y=1\times 6-9=-3 \quad (6,-3)$

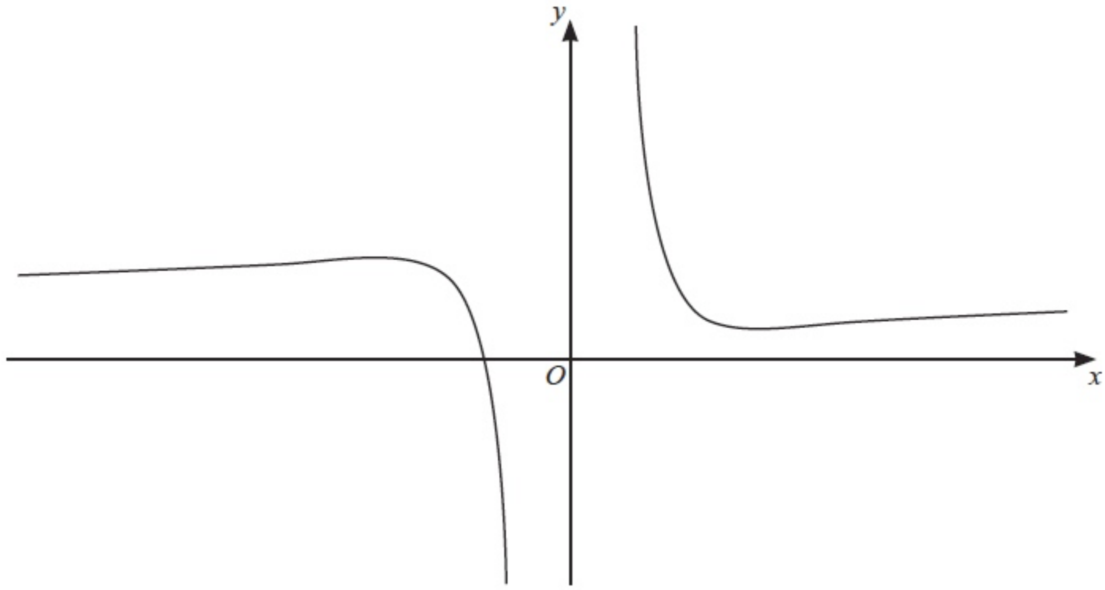
$m=-7 \quad 50x^2+120x+72=0$

$x=-6/5 \quad y=-7\times\frac{-6}{5}-9$

$y=-3/5$

$(6,-3) \ (-6/5, -3/5)$

Problem : 09709/11/M/J/24/Q11



A function is defined by $f(x) = \frac{4}{x^3} - \frac{3}{x} + 2$ for $x \neq 0$. The graph of $y = f(x)$ is shown in the diagram.

(a) Find the set of values of x for which $f(x)$ is decreasing. [5]

(b) A triangle is bounded by the y-axis, the normal to the curve at the point where $x = 1$ and the tangent to the curve at the point where $x = -1$.

Find the area of the triangle. Give your answer correct to 3 significant figures. [8]

Sol (a) $f'(x) < 0$

$$f'(x) = -\frac{12}{x^4} + \frac{3}{x^2} + 0$$

$$-\frac{12}{x^4} + \frac{3}{x^2} < 0$$

$$-12 + 3x^2 < 0$$

$$3x^2 - 12 < 0$$

$$3(x^2 - 4) < 0$$

$$(x^2 - 4) < 0$$

$$(x+2)(x-2) < 0$$

$$-2 < x < 2 \quad , x \neq 0$$

(b) Eq of Normal

$$f'(x) = -\frac{12}{x^4} + \frac{3}{x^2}$$

$$f'(1) = -12 + 3 = -9$$

gradient of Normal $= \frac{1}{9}$

Now $f(1) = \frac{4}{(1)^3} - \frac{3}{1} + 2 = 4 - 3 + 2 = 3$

$$y - 3 = \frac{1}{9}(x - 1)$$

$$y = \frac{1}{9}x - \frac{1}{9} + 3 = \frac{1}{9}x + \frac{26}{9}$$

Equation of tangent

gradient at $x = -1$

$$f'(-1) = \frac{-12}{(-1)^4} + \frac{3}{(-1)^2} = -12 + 9 = -9$$

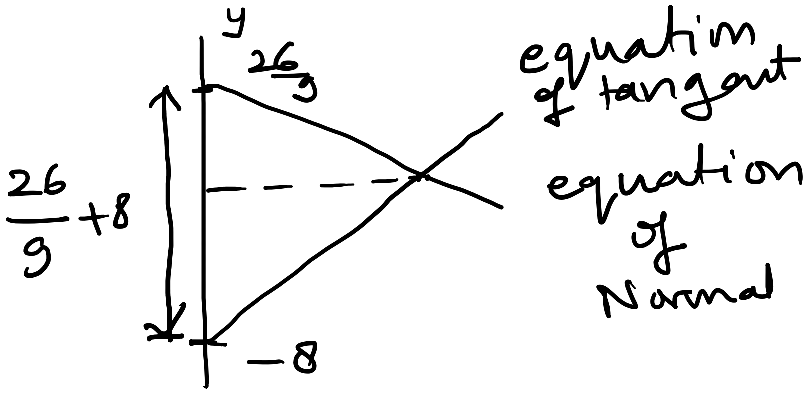
$$f(-1) = \frac{4}{(-1)^3} - \frac{3}{(-1)} + 2 = -4 + 3 + 2 = 1$$

$$y - 1 = -9(x + 1)$$

$$y = -9x - 9 + 1 = -9x - 8$$

$$-9x - 8 = \frac{1}{9}x + \frac{26}{9}$$

$$\underline{\underline{x = -1.195}}$$



Area of triangle

$$= \frac{1}{2} \times 1.195 \times \left(\frac{26}{9} + 8\right)$$

$$= 6.51$$