Problem 09709/11/O/N/23/ Q1

(a) Expand $(1 + 3x)^6$ in ascending powers of x up to, and including, the term in x^2. [2]

(b) Hence find the coefficient of x^2 in the expansion of $(1 - 7x + x^2)(1 + 3x)^6$. [2]

Sol (a)

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2\cdot 1}x^2$$

$$(1+3x)^6 = 1 + 6(3x) + \frac{\cancel{6}^{3} \times 5}{\cancel{2}}(3x)^2$$

$$= 1 + 18x + 135x^2$$

(b)

$$(1 - 7x + x^2)(1 + 18x + 135x^2)$$

$$1 - 126 + 135$$

$$10$$

Problem 09709/11/O/N/23/ Q2

A line has equation $y = 2cx + 3$ and a curve has equation $y = cx^2 + 3x - c$, where c is a constant.

Showing all necessary working, determine which of the following statements is correct.

A The line and curve intersect only for a particular set of values of c.

B The line and curve intersect for all values of c.

C The line and curve do not intersect for any values of c. [4]

Sol

$cx^2 + 3x - c = 2cx + 3$

$cx^2 + (3-2c)x + (-c-3) = 0$

$b^2 - 4ac$

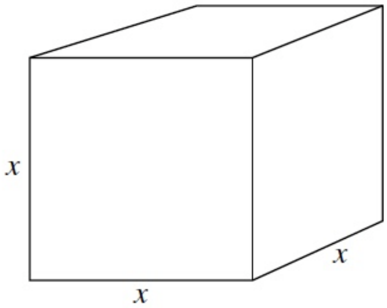
$(3-2c)^2 - 4c(-c-3)$

$9 + 4c^2 - \cancel{12c} + 4c^2 + \cancel{12c}$

$8c^2 + 9 > 0$

hence statement B is correct.

Problem 09709/11/O/N/23/ Q3



The diagram shows a cubical closed container made of a thin elastic material which is filled with water and frozen. During the freezing process the length, x cm, of each edge of the container increases at the constant rate of 0.01 cm per minute. The volume of the container at time t minutes is V cm^3.

Find the rate of increase of V when $x = 20$. [3]

Sol

$$V = x^3$$

$$\frac{dV}{dx} = 3x^2$$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

$$\frac{dV}{dt} = 3x^2 \times 0.01$$

$x = 20$

$$\frac{dV}{dt} = 3 \times (20)^2 \times 0.01 = 12$$

Problem 09709/11/O/N/23/ Q4

The transformation R denotes a reflection in the x-axis and the transformation T denotes a translation of $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

(a) Find the equation, $y = g(x)$, of the curve with equation $y = x^2$ after it has been transformed by the sequence of transformations R followed by T. [2]

(b) Find the equation, $y = h(x)$, of the curve with equation $y = x^2$ after it has been transformed by the sequence of transformations T followed by R. [2]

(c) State fully the transformation that maps the curve $y = g(x)$ onto the curve $y = h(x)$. [2]

Sol(a)

$$y = x^2$$
$$y = -x^2$$
$$= -(x-3)^2 - 1$$

(b)

$$y = x^2$$
$$y = (x-3)^2 - 1$$
$$= -(x-3)^2 + 1$$

(c)

translation by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Problem 09709/11/O/N/23/ Q5

(a) Show that the equation

$$4\sin x + \frac{5}{\tan x} + \frac{2}{\sin x} = 0$$

may be expressed in the form $a\cos^2 x + b\cos x + c = 0$, where a, b and c are integers to be found. [3]

(b) Hence solve the equation $4\sin x + \frac{5}{\tan x} + \frac{2}{\sin x} = 0$ for $0° \leqslant x \leqslant 360°$. [3]

Sol (a)

$$4\sin x + 5\frac{\cos x}{\sin x} + \frac{2}{\sin x} = 0$$

$$4\sin^2 x + 5\cos x + 2 = 0$$

$$4(1-\cos^2 x) + 5\cos x + 2 = 0$$

$$4 - 4\cos^2 x + 5\cos x + 2 = 0$$

$$-4\cos^2 x + 5\cos x + 6 = 0$$

$$4\cos^2 x - 5\cos x - 6 = 0$$

(b)

let $\cos x = t$

$$4t^2 - 5t - 6 = 0$$

$t = 2$ $\qquad t = -\frac{3}{4}$

$\cos x = 2$ ✗ $\qquad \cos x = -\frac{3}{4}$ ✓

$$\cos x = 3/4$$

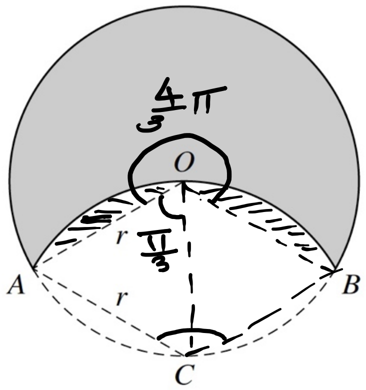
$$x = \cos^{-1} 3/4$$

$$= 41.41$$

$$x = 180 - 41.41 = 138.6$$

$$x = 180 + 41.41 = 221.4$$

Problem 09709/11/O/N/23/ Q6



$2\pi - \frac{2}{3}\pi = \frac{4}{3}\pi$

The diagram shows a motif formed by the major arc AB of a circle with radius r and centre O, and the minor arc AOB of a circle, also with radius r but with centre C. The point C lies on the circle with centre O.

(a) Given that angle $ACB = k\pi$ radians, state the value of the fraction k. [1]

(b) State the perimeter of the shaded motif in terms of π and r. [1]

(c) Find the area of the shaded motif, giving your answer in terms of π, r and $\sqrt{3}$. [5]

Sol (a) $\triangle AOC$ all sides are equal $= r$

$\angle ACB = \frac{2}{3}\pi$

$K\pi = \frac{2}{3}\pi$

$K = \frac{2}{3}$

(b) Arc length of Major Arc

$= r\frac{4}{3}\pi$

Arc length of Arc AOB

$= r\frac{2}{3}\pi$

Perimeter $= \frac{4}{3}r\pi + \frac{2}{3}r\pi$

$= 2\pi r$

(c) Area of Major sector

$$= \frac{1}{2} r^2 \frac{4}{3}\pi$$

Area of sector CAO

$$= \frac{1}{2} r^2 \frac{\pi}{3}$$

Area of $\triangle$ CAO

$$= \frac{1}{2} r^2 \sin\frac{\pi}{3}$$

$$= \frac{1}{2} r^2 \frac{\sqrt{3}}{2} = \frac{\sqrt{3} r^2}{4}$$

Area of region OA + OB

$$= 2\left[\frac{1}{2} r^2 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right]$$

$$= r^2 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

Area of shaded region

$$= \frac{1}{\cancel{2}} r^2 \frac{\overset{2}{\cancel{4}}}{3}\pi - r^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$= \frac{2}{3} r^2 \pi - \frac{\pi}{3} r^2 + \frac{\sqrt{3}}{2} r^2$$

$$= r^2\left[\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right]$$

Problem 09709/11/O/N/23/ Q7

The sum of the first two terms of a geometric progression is 15 and the sum to infinity is $\frac{125}{7}$. The common ratio of the progression is negative.

Find the third term of the progression. [7]

Sol

$a + ar = 15 \qquad a(1+r) = 15$

$$\frac{a}{1-r} = \frac{125}{7}$$

$$\frac{15}{(1-r)(1+r)} = \frac{125}{7}$$

$$\frac{15}{1-r^2} = \frac{125}{7}$$

$$\frac{15 \times 7}{125} = 1 - r^2$$

$$r^2 = 1 - \frac{105}{125} = \frac{20}{125}$$

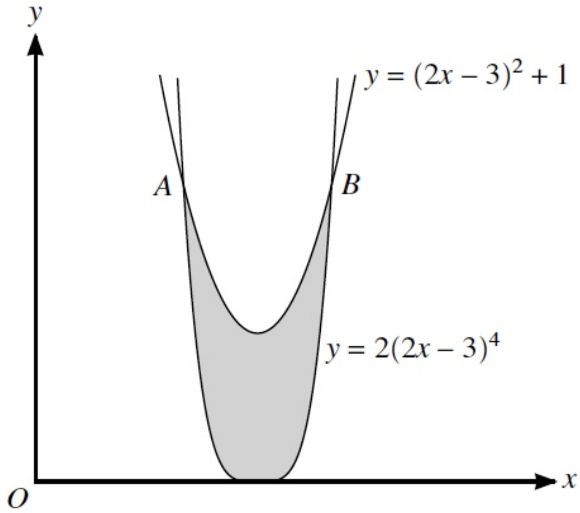
$$r = -\sqrt{\frac{\cancel{20}\,4}{\cancel{125}\,25}} = -\sqrt{\frac{4}{25}} = -\frac{2}{5}$$

$$a\left(1 - \frac{2}{5}\right) = 15$$

$$a\left(\frac{3}{5}\right) = 15 \qquad a = 25$$

$$T_3 = ar^2 = \cancel{25} \times \frac{4}{\cancel{25}} = 4$$

Problem 09709/11/O/N/23/ Q8



The diagram shows the curves with equations $y = 2(2x - 3)^4$ and $y = (2x - 3)^2 + 1$ meeting at points A and B.

(a) By using the substitution $u = 2x - 3$ find, by calculation, the coordinates of A and B. [4]

(b) Find the exact area of the shaded region. [5]

Sol (a) $u = 2x - 3$

$2(2x-3)^4 = (2x-3)^2 + 1$

$2u^4 = u^2 + 1$

$2u^4 - u^2 - 1 = 0$

let $u^2 = t$

$2t^2 - t - 1 = 0$

$t = 1 \qquad t = -\frac{1}{2}$

$u^2 = 1 \qquad u^2 = -\frac{1}{2}$ x

$u = \pm 1$

$2x - 3 = 1 \qquad 2x - 3 = -1$

$2x = -1 + 3 = 2$

$2x = 4$

$x = 2 \qquad x = 1$

$y = 2(2x-3)^4$

$x = 2 \quad y = 2(2 \times 2 - 3)^4 = 2 \qquad (2, 2)$

$x = 1 \quad y = 2(2 \times 1 - 3)^4 = 2 \qquad (1, 2)$

(b) $$\int_1^2 \left[(2x-3)^2+1-2(2x-3)^4\right] dx$$

$$\left[\frac{(2x-3)^3}{3}\times\frac{1}{2}+x-2\frac{(2x-3)^5}{5}\times\frac{1}{2}\right]_1^2$$

$$\left[\frac{(2x-3)^3}{6}+x-\frac{(2x-3)^5}{5}\right]_1^2$$

$$\left[\left(\frac{2\times2-3)^3}{6}+2-\frac{(2\times2-3)^5}{5}\right)-\left(\frac{(2\times1-3)^3}{6}+1-\frac{(2\times1-3)^5}{5}\right]\right.$$

$$=\frac{14}{15}$$

Problem 09709/11/O/N/23/ Q9

(a) Express $4x^2 - 12x + 13$ in the form $(2x + a)^2 + b$, where a and b are constants. [2]

The function f is defined by $f(x) = 4x^2 - 12x + 13$ for $p < x < q$, where p and q are constants. The function g is defined by $g(x) = 3x + 1$ for $x < 8$.

(b) Given that it is possible to form the composite function gf, find the least possible value of p and the greatest possible value of q. [3]

(c) Find an expression for $gf(x)$. [1]

The function h is defined by $h(x) = 4x^2 - 12x + 13$ for $x < 0$.

(d) Find an expression for $h^{-1}(x)$. [3]

Sol (a) $4x^2 - 12x + 13$

$4x^2 - 12x + 9 - 9 + 13$

$(2x - 3)^2 + 4$

(b) $4x^2 - 12x + 13 < 8$

$4x^2 - 12x + 5 < 0$

$(2x - 5)(2x - 1) < 0$

$\frac{1}{2} < x < \frac{5}{2}$

(c) $3(4x^2 - 12x + 13) + 1$

$12x^2 - 36x + 40$

(d) $h(x) = 4x^2 - 12x + 13 \quad x < 0$

let $y = (2x - 3)^2 + 4$

$\pm\sqrt{y - 4} = 2x - 3$

$2x = 3 \pm \sqrt{y - 4}$

$x = \frac{3}{2} \pm \frac{\sqrt{y - 4}}{2}$

$h^{-1}(x) = \frac{3}{2} - \frac{\sqrt{x - 4}}{2}$

Problem 09709/11/O/N/23/ Q10

A curve has a stationary point at $(2, -10)$ and is such that $\frac{d^2y}{dx^2} = 6x$.

(a) Find $\frac{dy}{dx}$. [3]

(b) Find the equation of the curve. [3]

(c) Find the coordinates of the other stationary point and determine its nature. [3]

(d) Find the equation of the tangent to the curve at the point where the curve crosses the y-axis. [2]

Sol (a) $\int \frac{d^2y}{dx^2} = \int 6x$

$\frac{dy}{dx} = \frac{6x^2}{2} + c$

$\frac{dy}{dx} = 3x^2 + c$

$x = 2$ $\frac{dy}{dx} = 0$

$0 = 3 \times 4 + c$

$c = -12$

$\frac{dy}{dx} = 3x^2 - 12$ ✓

(b) $\int dy = \int (3x^2 - 12)\, dx$

$y = \frac{3x^3}{3} - 12x + c$

$y = x^3 - 12x + c$

$(2, -10)$ $-10 = 2^3 - 12 \times 2 + c$

$-10 = 8 - 24 + c$

$c = -10 + 24 - 8 = 6$

$y = x^3 - 12x + 6$

(c) $3x^2 - 12 = 0$

$x^2 = 12/3 = 4$

$\underline{x = 2}$ $\quad x = -2$ ✓

$x = -2 \quad y = (-2)^3 - 12 \times -2 + 6$

$= 22$

$(-2, 22)$

$\frac{d^2y}{dx^2} = 6x - 2 < 0$

hence stationary point would be maximum.

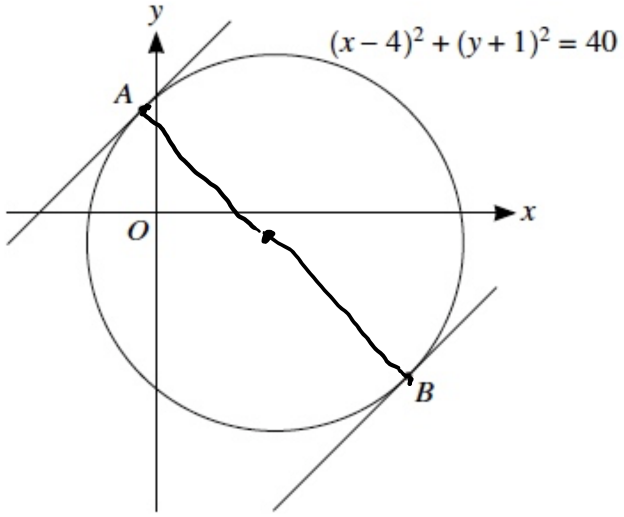
(d) $x = 0 \quad y = 0^3 - 12 \times 0 + 6 = 6$

$(0, 6) \quad \frac{dy}{dx} = 3(0)^2 - 12 = -12$

$y - 6 = -12(x - 0)$

$y = -12x + 6$

Problem 09709/11/O/N/23/ Q11



The diagram shows the circle with equation $(x-4)^2+(y+1)^2=40$. Parallel tangents, each with gradient 1, touch the circle at points A and B.

(a) Find the equation of the line AB, giving the answer in the form $y = mx + c$. [3]

(b) Find the coordinates of A, giving each coordinate in surd form. [4]

(c) Find the equation of the tangent at A, giving the answer in the form $y = mx + c$, where c is in surd form. [2]

Sol (a) centre of circle $(4,-1)$

the gradient of line $AB = -1$

the equation of line AB

$$y-(-1) = -(x-4)$$

$$y+1 = -x+4$$

$$y = -x+3$$

(b)

$$(x-4)^2+(-x+3+1)^2=40$$

$$(x-4)^2+(-x+4)^2=40$$

$$(x-4)^2+(-1)^2(x-4)^2=40$$

$$2(x-4)^2=40$$

$$(x-4)^2=20$$

$$x - 4 = \pm\sqrt{20}$$

$$x = 4 \pm \sqrt{20}$$

$$x = 4 - \sqrt{20}$$

$$y = -(4 - \sqrt{20}) + 3$$

$$= -4 + \sqrt{20} + 3$$

$$= -1 + \sqrt{20}$$

$$A(4 - \sqrt{20}, -1 + \sqrt{20})$$

(c) $A(4 - \sqrt{20}, -1 + \sqrt{20})$

gradient of tangent = 1

Equation of tangent

$$y - (-1 + \sqrt{20}) = 1(x - 4 + \sqrt{20})$$

$$y = x - 4 + \sqrt{20} - 1 + \sqrt{20}$$

$$y = x - 5 + 2\sqrt{20}$$