Problem : 09709/12/F/M/24/Q1

Find the exact value of $\int_3^{\infty} \frac{2}{x^2}\,\mathrm{d}x$. [3]

Sol

$$2\int_3^{\infty} x^{-2}\,dx$$

$$2\left[\frac{x^{-2+1}}{-2+1}\right]_3^{\infty}$$

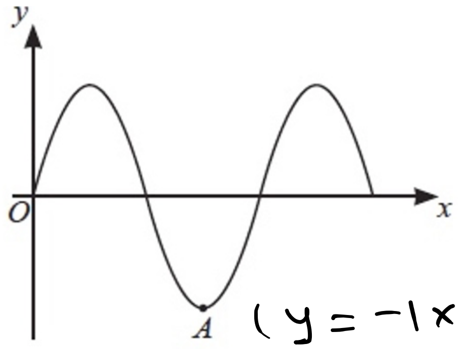
$$-2\left[\frac{1}{x}\right]_3^{\infty}$$

$$-2\left[\frac{1}{\infty}-\frac{1}{3}\right]$$

$$-2\left[0-\frac{1}{3}\right]$$

$$\frac{2}{3}$$

Problem : 09709/12/F/M/24/Q2



$(y = -1 \times k)$ $\theta = \frac{3\pi}{2}$

The diagram shows part of the curve with equation $y = k \sin \frac{1}{2}x$, where k is a positive constant and x is measured in radians. The curve has a minimum point A.

(a) State the coordinates of A. [1]

(b) A sequence of transformations is applied to the curve in the following order.

Translation of 2 units in the negative y-direction

Reflection in the x-axis

Find the equation of the new curve and determine the coordinates of the point on the new curve corresponding to A. [3]

Sol (a) $A(3\pi, -k)$

(b) $y = c \pm k\sin\frac{1}{2}x$ where c is nonzero constt

$y = 2 - k\sin\frac{1}{2}x$

$A(3\pi, 2+k)$

Problem : 09709/12/F/M/24/Q3

A curve is such that $\frac{dy}{dx} = 3(4x+5)^{\frac{1}{2}}$. It is given that the points $(1,9)$ and $(5,a)$ lie on the curve.

Find the value of a. [5]

Sol

$$\int dy = 3\int (4x+5)^{1/2}\,dx$$

$$y = \cancel{3}\,\frac{(4x+5)^{3/2}}{\frac{\cancel{3}}{\cancel{2}}} \times \frac{1}{\cancel{4}\,2} + c$$

$$y = \frac{1}{2}(4x+5)^{3/2} + c$$

$(1,9)$ $$9 = \frac{1}{2}(9)^{3/2} + c$$

$$9 = \frac{1}{2} \times 27 + c$$

$$c = 9 - \frac{27}{2} = -\frac{9}{2}$$

$$y = \frac{1}{2}(4x+5)^{3/2} - \frac{9}{2}$$

$(5,a)$ $$a = \frac{1}{2}(4\times 5+5)^{3/2} - \frac{9}{2}$$

$$= \frac{1}{2} \times 125 - \frac{9}{2}$$

$$= 58$$

Problem : 09709/12/F/M/24/Q4

(a) Prove that $\dfrac{(\sin\theta+\cos\theta)^2-1}{\cos^2\theta} \equiv 2\tan\theta$. [3]

(b) Hence solve the equation $\dfrac{(\sin\theta+\cos\theta)^2-1}{\cos^2\theta} = 5\tan^3\theta$ for $-90° < \theta < 90°$. [3]

Sol (a) LHS

$$\frac{(\sin\theta+\cos\theta)^2-1}{\cos^2\theta}$$

$$\frac{\underline{\sin^2\theta+\cos^2\theta}+2\sin\theta\cos\theta-1}{\cos^2\theta}$$

$$\frac{2\sin\theta\cancel{\cos\theta}}{\cos^{\cancel{2}}\theta}$$

$2\tan\theta$ (RHS)

(b) $2\tan\theta = 5\tan^3\theta$

$2\tan\theta - 5\tan^3\theta = 0$

$\tan\theta(2-5\tan^2\theta)=0$

$\tan\theta=0$ $\quad$ $\tan^2\theta = \frac{2}{5}$

$\tan\theta = \pm\sqrt{\frac{2}{5}}$

$\theta = \tan^{-1}\sqrt{\frac{2}{5}}$

$= 32.3°$

$\theta = \tan^{-1} -\sqrt{\frac{2}{5}}$

$= -32.3°$

Problem : 09709/12/F/M/24/Q5

A curve has the equation $y = \dfrac{3}{2x^2 - 5}$.

Find the equation of the normal to the curve at the point $(2, 1)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. [6]

Sol

$$\frac{dy}{dx} = \frac{3 \times -1}{(2x^2-5)^2} \times 4x$$

$$x=2 \quad \frac{dy}{dx} = \frac{-\cancel{3}}{\cancel{9}\ 3} \times 4 \times 2$$

$$= -\frac{8}{3}$$

gradient of normal $= \dfrac{3}{8}$

$$y - 1 = \frac{3}{8}(x-2)$$

$$8y - 8 = 3x - 6$$

$$3x - 8y + 2 = 0$$

Problem : 09709/12/F/M/24/Q6

It is given that the coefficient of x^3 in the expansion of

$$(2+ax)^4(5-ax)$$

is 432.

x^0 x^1

Find the value of the constant a. [5]

Sol $(2+ax)^4$

$^4C_r(2)^{4-r}(ax)^r$ $\quad x^3, x^2$

$x^r = x^3$

$r = 3$ ✓

$x^r = x^2$

$r = 2$ ✓

$r=3$ $\quad {}^4C_3(2)^1 a^3 x^3$

$= 4 \times 2 \times a^3 \times x^3$

$= 8a^3x^3$

$r = 2$ $\quad {}^4C_2(2)^2 a^2 x^2$

$= 6 \times 4 \times a^2 x^2$

$= 24a^2x^2$

$(\underline{8a^3x^3} + 24a^2x^2)(\underline{5} - ax)$

$40a^3x^3 - 24a^3x^3$

$40a^3 - 24a^3 = 432$

$16a^3 = 432$

$a^3 = 27$

$\underline{\underline{a = 3}}$

Problem : 09709/12/F/M/24/Q7

The straight line $y = x + 5$ meets the curve $2x^2 + 3y^2 = k$ at a single point P.

(a) Find the value of the constant k. [4]

(b) Find the coordinates of P. [2]

Sol (a)

$$2x^2 + 3(x+5)^2 = k$$

$$2x^2 + 3(x^2 + 10x + 25) = k$$

$$2x^2 + 3x^2 + 30x + 75 - k = 0$$

$$5x^2 + 30x + 75 - k = 0 \quad \checkmark$$

$$b^2 - 4ac = 0$$

$$(30)^2 - 4 \times 5\,(75 - k) = 0$$

$$90\cancel{0} = 2\cancel{0}\,(75 - k)$$

$$45 = 75 - k$$

$$k = 75 - 45 = 30$$

(b)

$$5x^2 + 30x + 45 = 0$$

$$x^2 + 6x + 9 = 0$$

$$(x+3)^2 = 0$$

$$x = -3$$

$$y = (-3) + 5 = 2$$

$$P(-3, 2)$$

Problem : 09709/12/F/M/24/Q8

(a) An arithmetic progression is such that its first term is 6 and its tenth term is 19.5 .

Find the sum of the first 100 terms of this arithmetic progression. [4]

(b) A geometric progression $a_1, a_2, a_3, \ldots$ is such that $a_1 = 24$ and the common ratio is $\frac{1}{2}$.

The sum to infinity of this geometric progression is denoted by S. The sum to infinity of the even-numbered terms (i.e. $a_2, a_4, a_6, \ldots$) is denoted by S_E.

Find the values of S and S_E. [4]

Sol (a) $a = 6$

$$19.5 = a + (10-1)d$$
$$19.5 = 6 + 9d$$
$$d = \frac{19.5 - 6}{9}$$
$$= \frac{3}{2}$$

$$S_{100} = \frac{100}{2}\left[2 \times 6 + (100-1)\frac{3}{2}\right]$$
$$= 8025$$

(b) $$S_\infty = \frac{24}{1-\frac{1}{2}} = \frac{24}{\frac{1}{2}} = 48$$

$a_2 = 12 \quad a_4 = 3 \quad a_6 = \frac{3}{4}$

$$r = \frac{3}{12} = \frac{1}{4}$$

$$S_E = \frac{12}{1-\frac{1}{4}} = \frac{12}{\frac{3}{4}} = 16$$

Problem : 09709/12/F/M/24/Q9

The functions f and g are defined for all real values of x by

$$f(x) = (3x-2)^2 + k \quad \text{and} \quad g(x) = 5x - 1,$$

where k is a constant.

(a) Given that the range of the function gf is $gf(x) \geqslant 39$, find the value of k. [4]

(b) For this value of k, determine the range of the function fg. [2]

(c) The function h is defined for all real values of x and is such that $gh(x) = 35x + 19$.

Find an expression for $g^{-1}(x)$ and hence, or otherwise, find an expression for $h(x)$. [3]

Sol (a) $gf(x) = 5\left((3x-2)^2 + k\right) - 1$

$= 5(3x-2)^2 + 5k - 1$

$5k - 1 = 39$ or $5k - 1 \geqslant 39$

$5k = 40$

$k = 8$

(b) $f(x) = (3x-2)^2 + 8$

$g(x) = 5x - 1$

$fg(x) = (3(5x-1)-2)^2 + 8$ ✓

$fg(x) \geqslant 8$

(c) $gh(x) = 35x + 19$ ✓

$g(x) = 5x - 1$

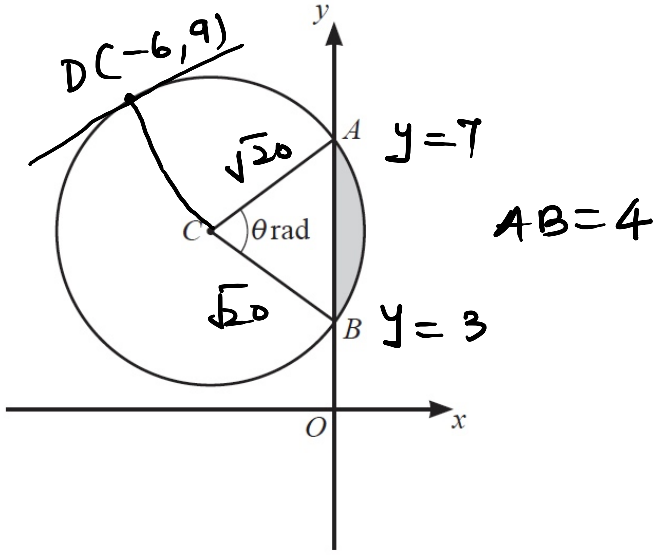
$g^{-1}(x) = \frac{1}{5}(x+1)$

$h(x) = g^{-1}(35x+19)$

$= \frac{1}{5}(35x + 19 + 1)$

$$h(x) = 7x + 4$$

Problem : 09709/12/F/M/24/Q10



The diagram shows the circle with centre $C(-4, 5)$ and radius $\sqrt{20}$ units. The circle intersects the y-axis at the points A and B. The size of angle ACB is θ radians.

(a) Find the equation of the tangent to the circle at the point $(-6, 9)$. [3]

(b) Find the equation of the circle in the form $x^2 + y^2 + ax + by + c = 0$. [2]

(c) Find the value of θ correct to 4 significant figures. [3]

(d) Find the perimeter and area of the segment shaded in the diagram. [4]

Sol (a) gradient of DC $= \dfrac{9-5}{-6+4} = \dfrac{4}{-2} = -2$

gradient of tangent $= \dfrac{1}{2}$

equation of tangent

$$y - 9 = \frac{1}{2}(x+6)$$

$$y = \frac{1}{2}x + 3 + 9$$

$$y = \frac{1}{2}x + 12$$

(b) $(x-h)^2 + (y-k)^2 = a^2$

$(x+4)^2 + (y-5)^2 = 20$

$x^2 + 8x + 16 + y^2 - 10y + 25 = 20$

$$x^2+y^2+8x-10y+21=0$$

(c) $$0^2+y^2+0-10y+21=0$$

$$y^2-10y+21=0$$

$$y^2-7y-3y+21=0$$

$$y(y-7)-3(y-7)=0$$

$$(y-3)(y-7)=0$$

$$y=3,7$$

By cosine rule

$$\cos\theta = \frac{20+20-16}{2\times\sqrt{20}\times\sqrt{20}} = \frac{24}{40}$$

$$\theta = \cos^{-1}\frac{24}{40} = 0.9273.$$

(d) $$\text{perimeter} = AB + r\theta$$

$$= 4+\sqrt{20}\times 0.9273$$

$$= 8.1470$$

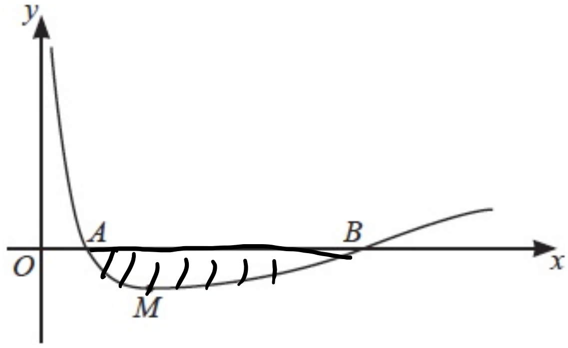
$$= 8.15$$

$$\text{Area} = \frac{1}{2}r^2\theta - \frac{1}{2}ab\sin\theta$$

$$= \frac{1}{2}(\sqrt{20})^2\times 0.9273 - \frac{1}{2}\sqrt{20}\sqrt{20}\sin 0.9273$$

$$= 1.27$$

Problem : 09709/12/F/M/24/Q11



The diagram shows the curve with equation $y = 2x^{-\frac{2}{3}} - 3x^{-\frac{1}{3}} + 1$ for $x > 0$. The curve crosses the x-axis at points A and B and has a minimum point M.

(a) Find the exact coordinates of M. [4]

(b) Find the area of the region bounded by the curve and the line segment AB. [7]

Sol(a) $y = 2x^{-\frac{2}{3}} - 3x^{-\frac{1}{3}} + 1$

$$\frac{dy}{dx} = 2 \times -\frac{2}{3} x^{-\frac{2}{3}-1} - 3 \times -\frac{1}{3} x^{-\frac{1}{3}-1}$$

$$\frac{dy}{dx} = -\frac{4}{3} x^{-5/3} + x^{-4/3}$$

As M is stationary point $\frac{dy}{dx} = 0$

$$0 = x^{-4/3}\left[-\frac{4}{3}x^{-1/3} + 1\right]$$

$$+\frac{4}{3}x^{-1/3} = +1$$

$$\frac{4}{3} = \frac{1}{x^{-1/3}}$$

$$\frac{4}{3} = x^{1/3}$$

$$x = \left(\frac{4}{3}\right)^3 = \frac{64}{27}$$

$$y = 2 \times \left(\frac{64}{27}\right)^{-2/3} - 3\left(\frac{64}{27}\right)^{-1/3} + 1$$

$$= 2 \times \frac{9}{16} - 3 \times \frac{3}{4} + 1$$

$$= \frac{18}{16} - \frac{9}{4} + 1$$

$$= -\frac{1}{8}$$

$$M\left(\frac{64}{27}, -\frac{1}{8}\right)$$

(b)

$$0 = 2x^{-2/3} - 3x^{-1/3} + 1$$

$$0 = 2\left(\frac{1}{x^{1/3}}\right)^2 - 3\left(\frac{1}{x^{1/3}}\right) + 1$$

let $\frac{1}{x^{1/3}} = t$

$$2t^2 - 3t + 1 = 0$$

$$2t^2 - 2t - t + 1 = 0$$

$$2t(t-1) - 1(t-1) = 0$$

$$(2t-1)(t-1) = 0$$

$$t = \frac{1}{2} \qquad t = 1$$

$$\frac{1}{x^{1/3}} = \frac{1}{2} \qquad \frac{1}{x^{1/3}} = 1$$

$$x = 8 \qquad x = 1$$

$$\text{Area} = \int_1^8 \left(2x^{-2/3} - 3x^{-1/3} + 1\right) dx$$

$$= \left[2 \times \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} - 3\,\frac{x^{-1/3+1}}{-1/3+1} + x \right]_1^8$$

$$= \left[2 \times 3 \times x^{1/3} - 3 \times \frac{3}{2}\, x^{2/3} + x \right]_1^8$$

$$= \left[6x^{1/3} - \frac{9}{2}\, x^{2/3} + x \right]_1^8$$

$$= 6(8)^{1/3} - \frac{9}{2}(8)^{2/3} + 8 - 6(1)^{1/3} + \frac{9}{2}(1)^{2/3} - 1$$

$$= -1/2$$

Since area is positive

hence area $= \frac{1}{2}$