Problem : 09709/12/M/J/24/Q1

The coefficient of x^2 in the expansion of $(1-4x)^6$ is 12 times the coefficient of x^2 in the expansion of $(2+ax)^5$.

Find the value of the positive constant a. [3]

Sol

$(1-4x)^6 \quad x^2$

${}^6C_r (1)^{6-r}(-4x)^r \quad x^2$

$r=2$

${}^6C_2 (-4x)^2$

$15 \times 16 = 240$

$(2+ax)^5 \quad x^2$

${}^5C_r (2)^{5-r}(ax)^r$

$r=2$

${}^5C_2 \, 2^3 \, a^2$

$10 \times 8 \times a^2$

$80a^2$

$240 = 12 \times 80 \times a^2$

$a^2 = \frac{1}{4}$

$a = \frac{1}{2} = 0.5$

Problem : 09709/12/M/J/24/Q2

The curve $y = x^2$ is transformed to the curve $y = 4(x-3)^2 - 8$.

Describe fully a sequence of transformations that have been combined, making clear the order in which the transformations have been applied. [5]

Sol (i) stretch parallel to y-axis by scale factor 4.

(ii) translation right side on x-axis by 3 unit

(iii) translation down on y-axis by 2 unit

$\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Problem : 09709/12/M/J/24/Q3

(a) Show that the equation $\dfrac{7\tan\theta}{\cos\theta} + 12 = 0$ can be expressed as

$$12\sin^2\theta - 7\sin\theta - 12 = 0. \qquad [3]$$

(b) Hence solve the equation $\dfrac{7\tan\theta}{\cos\theta} + 12 = 0$ for $0^\circ \leqslant \theta \leqslant 360^\circ$. [3]

Sol (a)

$$\frac{7\sin\theta}{\cos^2\theta} + 12 = 0$$

$$\frac{7\sin\theta}{1-\sin^2\theta} = -12$$

$$7\sin\theta = -12 + 12\sin^2\theta$$

$$12\sin^2\theta - 7\sin\theta - 12 = 0$$

(b) let $\sin\theta = t$

$$12t^2 - 7t - 12 = 0$$

$$12t^2 - 16t + 9t - 12 = 0$$

$$4t(3t-4) + 3(3t-4) = 0$$

$$(4t+3)(3t-4) = 0$$

$t = -3/4 \qquad t = 4/3$

$\sin\theta = -3/4 \qquad \sin\theta = \frac{4}{3}$ ×

Consider $\sin\theta = 3/4 \qquad \theta = \sin^{-1} 3/4$

$= 48.59$

$= 48.6$

$\theta = 180 + 48.6$

$= 228.6$

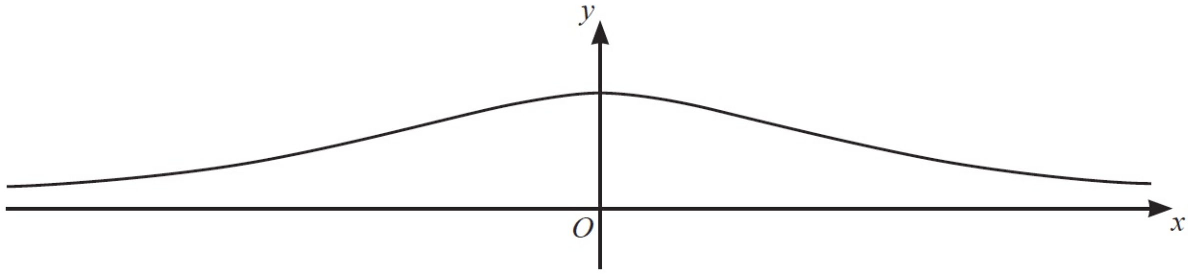
$\theta = 360 - 48.6 = 311.4$

Problem : 09709/12/M/J/24/Q4

The function f is defined as follows:

$$f(x) = \sqrt{x} - 1 \text{ for } x > 1.$$

(a) Find an expression for $f^{-1}(x)$. [1]



The diagram shows the graph of $y = g(x)$ where $g(x) = \dfrac{1}{x^2+2}$ for $x \in \mathbb{R}$.

(b) State the range of g and explain whether g^{-1} exists. [2]

The function h is defined by $h(x) = \dfrac{1}{x^2+2}$ for $x \geqslant 0$.

(c) Solve the equation $hf(x) = f\left(\dfrac{25}{16}\right)$. Give your answer in the form $a + b\sqrt{c}$, where a, b and c are integers. [4]

Sol (a) $f(x) = \sqrt{x} - 1$

let $y = \sqrt{x} - 1$

$y + 1 = \sqrt{x}$

$x = (y+1)^2$

$f^{-1}(x) = (x+1)^2$

(b) $x = 0$

$g(x) = \frac{1}{2}$

hence

$0 < g(x) \leq \frac{1}{2}$

g^{-1} does not exists as function is not one-one.

(c) $hf(x) = \frac{1}{(\sqrt{x}-1)^2+2}$

$f\left(\frac{25}{16}\right) = \sqrt{\frac{25}{16}} - 1$

$= \frac{5}{4} - 1 = 1/4$

$\frac{1}{(\sqrt{x}-1)^2+2} = \frac{1}{4}$

$4 = (\sqrt{x}-1)^2 + 2$

$2 = (\sqrt{x}-1)^2$

$\pm\sqrt{2} = \sqrt{x} - 1$

$\sqrt{x} = 1 + \sqrt{2}$

$x = (1+\sqrt{2})^2$

$= 1 + 2 + 2\sqrt{2}$

$= 3 + 2\sqrt{2}$

Problem : 09709/12/M/J/24/Q5

The first and second terms of an arithmetic progression are $\tan\theta$ and $\sin\theta$ respectively, where $0 < \theta < \frac{1}{2}\pi$.

(a) Given that $\theta = \frac{1}{4}\pi$, find the exact sum of the first 40 terms of the progression. [4]

The first and second terms of a geometric progression are $\tan\theta$ and $\sin\theta$ respectively, where $0 < \theta < \frac{1}{2}\pi$.

(b) **(i)** Find the sum to infinity of the progression in terms of θ. [2]

(ii) Given that $\theta = \frac{1}{3}\pi$, find the sum of the first 10 terms of the progression. Give your answer correct to 3 significant figures. [3]

Sol (a)

$$a = \tan\theta$$
$$a + d = \sin\theta$$
$$d = \sin\theta - \tan\theta$$
$$\theta = \frac{1}{4}\pi$$
$$a = \tan\frac{1}{4}\pi = 1$$
$$d = \sin\frac{1}{4}\pi - \tan\frac{1}{4}\pi$$
$$= \frac{1}{\sqrt{2}} - 1$$
$$S_{40} = \frac{40}{2}\left[2\times1 + 39\times\left(\frac{1}{\sqrt{2}} - 1\right)\right]$$
$$= 20\left[2 + \frac{39}{\sqrt{2}} - 39\right]$$
$$= 20\left[\frac{39}{\sqrt{2}} - 37\right]$$
$$= 20\left[\frac{39\sqrt{2}}{2} - 37\right]$$
$$= 390\sqrt{2} - 740$$

(b) (i) $a = \tan\theta$

$ar = \sin\theta$

$r = \frac{\sin\theta}{\tan\theta} = \frac{\sin\theta}{\sin\theta} \times \cos\theta$

$= \cos\theta$

$S_\infty = \frac{\tan\theta}{1-\cos\theta}$

(ii) $\theta = \frac{1}{3}\pi$

$a = \tan\frac{1}{3}\pi = \sqrt{3}$

$r = \cos\frac{1}{3}\pi = \frac{1}{2}$

$S_{10} = \frac{\sqrt{3}\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}}$

$= 2\sqrt{3}\left(1-\left(\frac{1}{2}\right)^{10}\right)$

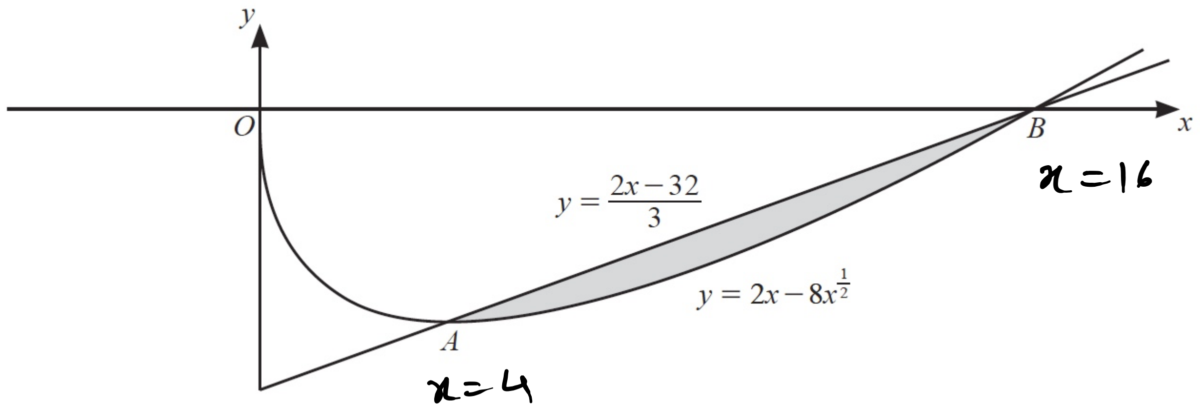
$= 3.46$

Problem : 09709/12/M/J/24/Q6

The curve with equation $y = 2x - 8x^{\frac{1}{2}}$ has a minimum point at A and intersects the positive x-axis at B.

(a) Find the coordinates of A and B. [4]

(b)



The diagram shows the curve with equation $y = 2x - 8x^{\frac{1}{2}}$ and the line AB. It is given that the equation of AB is $y = \frac{2x-32}{3}$.

Find the area of the shaded region between the curve and the line. [5]

Sol (a) $y = 2x - 8x^{1/2}$

$\frac{dy}{dx} = 2 - 8 \times \frac{1}{2} x^{-1/2}$

As A is stationary point hence $\frac{dy}{dx} = 0$

$2 - 4x^{-1/2} = 0$

$4x^{-1/2} = 2$

$x^{-1/2} = \frac{1}{2}$

$x^{1/2} = 2$

$x = 4$ ✓

$y = 2 \times 4 - 8(4)^{1/2} = 8 - 16 = -8$

therefore $A(4, -8)$

At B curve intersect positive x-axis

therefore $y = 0$

$$0 = 2x - 8x^{1/2}$$

let $x^{1/2} = t$

$$0 = 2t^2 - 8t$$

$$2t(t-4) = 0$$

$$t = 0 \quad t = 4$$

$$x = 0 \quad x = 16$$

$B(16, 0)$

(b) $$\text{Area} = \int_4^{16} \left[\frac{2x-32}{3} - 2x + 8x^{1/2}\right] dx$$

$$= \left[\frac{2}{3}\frac{x^2}{2} - \frac{32}{3}x - 2\frac{x^2}{2} + 8\frac{x^{3/2}}{3/2}\right]_4^{16}$$

$$= \left[\frac{x^2}{3} - \frac{32}{3}x - x^2 + \frac{16}{3}x^{3/2}\right]_4^{16}$$

$$= \left[\frac{16^2}{3} - \frac{32}{3}\times 16 - 16^2 + \frac{16}{3}(16)^{3/2}\right]$$

$$- \left[\frac{4^2}{3} - \frac{32}{3}(4) - (4)^2 + \frac{16}{3}(4)^{3/2}\right]$$

$$= \frac{32}{3}$$

Problem : 09709/12/M/J/24/Q7

The equation of a circle is $(x-6)^2+(y+a)^2=18$. The line with equation $y=2a-x$ is a tangent to the circle.

(a) Find the two possible values of the constant a. [5]

(b) For the greater value of a, find the equation of the diameter which is perpendicular to the given tangent. [3]

Sol (a) $y = 2a - x$

$(x-6)^2 + (y+a)^2 = 18$

$(x-6)^2 + (2a-x+a)^2 = 18$

$(x-6)^2 + (3a-x)^2 = 18$

$x^2 - 12x + 36 + 9a^2 - 6ax + x^2 = 18$

$2x^2 - (12+6a)x + 18 + 9a^2 = 0$

As line tangent to the circle

hence $b^2 - 4ac = 0$

$[-(12+6a)]^2 - 4 \times 2 \times (18+9a^2) = 0$

$(12+6a)^2 = 144 + 72a^2$

$\cancel{144} + 144a + 36a^2 - \cancel{144} - 72a^2 = 0$

$144a - 36a^2 = 0$

$4a - a^2 = 0$

$a(4-a) = 0$

$a = 0 \quad a = 4$

(b) $a = 4 \qquad y = -x + 8$

centre $(6, -4)$

As diameter is perpendicular to tangent hence gradient of diameter will be 1

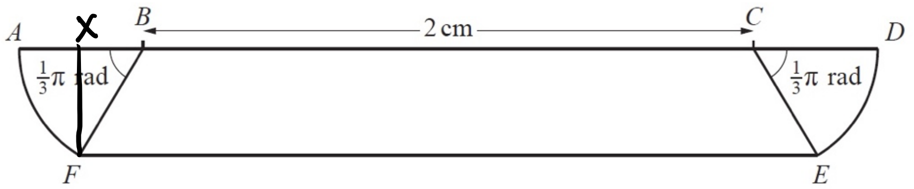
therefore equation of diameter is

$$y-(-4) = 1(x-6$$

$$y+4 = x-6$$

$$y = x-10$$

Problem : 09709/12/M/J/24/Q8



The diagram shows a symmetrical plate $ABCDEF$. The line $ABCD$ is straight and the length of BC is 2 cm. Each of the two sectors ABF and DCE is of radius r cm and each of the angles ABF and DCE is equal to $\frac{1}{3}\pi$ radians.

(a) It is given that $r = 0.4$ cm.

(i) Show that the length $EF = 2.4$ cm. [2]

(ii) Find the area of the plate. Give your answer correct to 3 significant figures. [4]

(b) It is given instead that the perimeter of the plate is 6 cm.

Find the value of r. Give your answer correct to 3 significant figures. [4]

Sol (a)(i) $EF = 2 \times BX + BC \qquad BF = 0{\cdot}4$

$$BX = 0{\cdot}4 \cos\frac{\pi}{3} = 0{\cdot}2$$

$$EF = 2 \times 0{\cdot}2 + 2 = 2{\cdot}4$$

(ii) Area of sector $= \frac{1}{2} \times 0{\cdot}4^2 \times \frac{\pi}{3}$

$$= \frac{2}{75}\pi$$

$$FX = 0{\cdot}4 \sin\frac{\pi}{3} = \frac{\sqrt{3}}{5}$$

Area of plate = Area of sector × 2 + Area of

$$= 2 \times \frac{2}{75}\pi + \frac{1}{2} \times \frac{\sqrt{3}}{5} \times (2 + 2{\cdot}4)$$

$$= 0{\cdot}9296$$

$$= 0{\cdot}930$$

(b) Perimeter of plate = 6

$6 = EF + \text{Arc } FA + AB + BC + CD + \text{Arc } DE$

$$6 = 2 + 2r\cos\frac{\pi}{3} + r\frac{\pi}{3} + r + 2 + r + r\frac{\pi}{3}$$

$$2 = 2r\cos\frac{\pi}{3} + 2r + \frac{2r\pi}{3}$$

$$2 = 3r + \frac{2}{3}r\pi$$

$$2 = r\left(3 + \frac{2}{3}\pi\right)$$

$$r = \frac{2}{3 + \frac{2}{3}\pi} = 0\cdot393$$

Problem : 09709/12/M/J/24/Q9

A function f is such that $f'(x) = 6(2x-3)^2 - 6x$ for $x \in \mathbb{R}$.

(a) Determine the set of values of x for which $f(x)$ is decreasing. [4]

(b) Given that $f(1) = -1$, find $f(x)$. [4]

Sol (a)

$$f'(x) < 0$$

$$6(2x-3)^2 - 6x < 0$$

$$(2x-3)^2 - x < 0$$

$$4x^2 - 12x + 9 - x < 0$$

$$4x^2 - 13x + 9 < 0$$

$$4x^2 - 9x - 4x + 9 < 0$$

$$x(4x-9) - 1(4x-9) < 0$$

$$(4x-9)(x-1) < 0$$

$$1 < x < 9/4$$

(b)

$$f'(x) = 6(2x-3)^2 - 6x$$

$$\int f'(x)\,dx = \int \left(6(2x-3)^2 - 6x\right)dx$$

$$f(x) = 6\,\frac{(2x-3)^3}{3} \times \frac{1}{2} - 6\frac{x^2}{2} + c$$

$$f(x) = (2x-3)^3 - 3x^2 + c$$

$$f(1) = -1$$

$$-1 = (2\times 1-3)^3 - 3(1)^2 + c$$

$$-1 = -1 - 3 + c$$

$$c = 3$$

$$f(x) = (2x-3)^3 - 3x^2 + 3$$

Problem : 09709/12/M/J/24/Q10

The equation of a curve is $y = (5-2x)^{\frac{3}{2}} + 5$ for $x < \frac{5}{2}$.

(a) A point P is moving along the curve in such a way that the y-coordinate of point P is decreasing at 5 units per second.

Find the rate at which the x-coordinate of point P is increasing when $y = 32$. [4]

(b) Point A on the curve has y-coordinate 32. Point B on the curve is such that the gradient of the curve at B is -3.

Find the equation of the perpendicular bisector of AB. Give your answer in the form $ax+by+c=0$, where a, b and c are integers. [6]

Sol (a) $y = (5-2x)^{3/2} + 5 \qquad \frac{dy}{dt} = 5 \qquad \frac{dx}{dt} = ?$

$$\frac{dy}{dx} = \frac{3}{\cancel{2}}(5-2x)^{1/2} \times -\cancel{2} + 0$$

$$\frac{dy}{dx} = -3(5-2x)^{1/2}$$

$$32 = (5-2x)^{3/2} + 5$$

$$27 = (5-2x)^{3/2}$$

$$5-2x = 27^{2/3} = 9$$

$$-2x = 4$$

$$x = -2$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$5 = -3(5-2\times-2)^{1/2} \times \frac{dx}{dt}$$

$$5 = -9 \times \frac{dx}{dt}$$

$$\frac{dx}{dt} = -\frac{5}{9}$$

(b) $A(-2, -32)$ and At B $\frac{dy}{dx} = -3$

$$-3 = -3(5-2x)^{1/3}$$

$$1 = (5-2x)^{1/3}$$

$1 = 5 - 2x$

$2x = 4$

$\underline{\underline{x = 2}}$

$y = (5 - 2\times 2)^{3/2} + 5$

$= 6$

$B(2, 6) \quad A(-2, 32)$

mid point $(0, 19)$

gradient of AB $= \frac{32-6}{-2-2} = \frac{26}{-4} = -\frac{13}{2}$

gradient of $\perp$ to AB $= \frac{2}{13}$

Equation of perpendicular bisector

$y - 19 = \frac{2}{13}(x - 0)$

$y - 19 = \frac{2}{13}x$

$13y - 247 = 2x$

$2x - 13y + 247 = 0$